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<u>Thm</u> G: discrete. TFAE

1. G is C^*-simple.
2. For ~~some~~ all G-boundaries X, $C(X) \rtimes_r G$ is simple.
3. G acts topologically freely on some G-boundary.
4. G acts ~~top~~ freely on $\partial_F G$. (G-boundary)

$\partial_F G$: extremally disconnected.

<Pf>(Sketch)

(1) $\Rightarrow$ (2) If X is a G-boundary, the inclusion

$$C_r^*(G) \subseteq C(X) \rtimes_r G \quad \text{is essential.}$$

So follows by result from last time.

(3) $\Rightarrow$ (2) this follows from Archbold – Spielberg. (Kishimoto / Tomiyama)

(3) $\Rightarrow$ (4) clear, plus fact about extremally disconnected G-spaces.
(i.e. $U \overset{\text{open}}{\subset} X \Rightarrow \overline{U}$: open & closed)

(2) $\Rightarrow$ (1) Suppose $C(X) \rtimes_r G$ is simple, X: G-boundary.
want to show : $C_r^*(G)$ is simple.
Fix $J \trianglelefteq_{\neq} C_r^*(G)$. Let $\pi : C_r^*(G) \twoheadrightarrow C_r^*(G)/J$.

G-alg $\qquad\qquad$ G-alg

G-action $\qquad\qquad$ G-action

$\lambda_g \cdot \lambda_g^*$ — $\qquad$ $\pi(\lambda_g) \cdot \pi(\lambda_g)^*$

π is G-equivariant, hence by G-injectivity (in G-Op Sys)
and the fact $C_r^*(G) \subseteq C(X) \rtimes_r G$.

Hence can extend π to a G equivariant map

$$\hat{\pi} : C(X) \rtimes_r G \longrightarrow I_G(C_r^*(G)/J)$$

$I_G(\cdot)$ injective envelope in G-Op Sys.

Suppose $\hat{\pi}$ was a $*$-hom, then $\underset{\underset{J.}{\cup\!\!\!\mid}}{\ker(\hat{\pi})} \lhd C(X) \rtimes_r G$

By simplicity $J = \{0\}$.

We would be done.

Unfortunately $\hat{\pi}$ is probably not a $*$-hom.

However can perturb $\hat{\pi}$ to a $*$-hom. to get a $*$-hom.

$$\rho : C(X) \rtimes_r G \longrightarrow I_G(C_r^*(G)/J)^{**}$$

while the image of $\hat{\pi}$ is not near a C^*-alg.

but take C^*-alg this generates in $I_G(C_r^*(G)/J)^{**}$

contains a $*$-hom image, this lets us construct

π.

(2) $\Rightarrow$ (4) $C(\partial_F G) \rtimes_r G$ simple $\Rightarrow$ $G \curvearrowright \partial_F G$ free.

Need to use the fact that for $x \in \partial_F G$ the stabilizer Stab_x

$= \{ s \in G \mid s.x = x \}$ is amenable.

$$\lambda_{G/\mathrm{Stab}_x} \prec \lambda \Rightarrow \exists \, *\text{-hom } C_r^*(G) \longrightarrow C^*(\lambda_{G/\mathrm{Stab}_x}(G))$$

Fact (Day)

Every group has a largest amenable normal subgroup

$\mathrm{Ra}(G)$ (amenable radical)

Fact If X is a G-boundary, then $\{ s \in G \mid sx = x \} \supseteq \mathrm{Ra}(G)$

If $X = \partial_F G$ $\mathrm{Stab}_x = \mathrm{Ra}(G)$.

Cor If G is C^*-simple (so $G \curvearrowright \partial_F G$ is free)

then $\mathrm{Ra}(G) = \{1\}$.

Open If $\mathrm{Ra}(G) = \{1\}$ is G C^*-simple ?

<u>Example</u>

$\mathbb{F}_n$ is C^*-simple. $(n \geq 2)$

$\partial \mathbb{F}_n = \{$ infinite words $\}$ is a $\mathbb{F}_n$-boundary.

<u>claim</u> $\mathbb{F}_n \curvearrowright \partial \mathbb{F}_n$ is <u>top</u> free

Note: the action isn't free.

Suppose $\mathbb{F}_2 = \langle a, b \rangle$ Then $a \cdot a^{\infty} = a^{\infty}$

Take $x \in \mathbb{F}_n$ $x \neq e$. Consider, for

$$w = w_1 w_2 \cdots\cdots \quad = \text{word in } \{a, b\}$$

Suppose $xw = w$ $\quad x = x_1 \cdots x_k$

<u>Case 1</u> xw is a reduced word.

$$xw = w \Rightarrow x_1 \cdots x_k \, w_1 w_2 \cdots\cdots$$

$$= w_1 \cdots w_k w_{k+1} w_{k+2} \cdots$$

$$\Rightarrow w = x x x \cdots\cdots$$

<u>Case 2</u> xw not reduced, but

write $w = uv$ $\quad |u| = |x|$ then

one of $u^{-1} x u v$ or $u^{-1} x^{-1} u v$ is reduced.

Finite number of ways this can happen.

<u>Conclusion</u>

Fixed points of $x \curvearrowright \partial \mathbb{F}_n$ is a finite set.

so has trivial interior, hence top-free.

Even though non-amenable G has many boundaries, we can't "identify" them in general.