Kennedy VI

G : discrete

X : G-boundary means $\forall \nu \in P(X)$

$\forall x \in X \ \exists t_n \in G \quad t_n \cdot \nu \xrightarrow{\text{weak}^*} \delta_x$

<u>Def</u> $G \curvearrowright X$ is proximal if $\forall x_1, \dots, \forall x_m \in X$, $\forall y \in X$
minimal

$$\exists t_n \in G \quad \text{s.t.} \quad t_n x_k \xrightarrow{n \to \infty} y \quad (k = 1, \dots, m)$$

<u>Exercise</u> If X is a G-boundary, then X is minimal + proximal.
[Use
$\nu = \frac{1}{m}(\delta_{x_1} + \dots + \delta_{x_m})$]

<u>Prop</u> Let X be a G-boundary. If $G \curvearrowright X$ is not topologically free then for $\forall x \in X$, $\forall t_1, \dots, \forall t_n \in G$. Suppose G has no finite normal subgroups $\neq \{1\}$

$$G_x^{t_1} \cap \dots \cap G_x^{t_n} \neq \{1\} \quad \text{is } \underline{\underline{\text{infinite}}}$$

$G_x = \{ s \in G \mid s \cdot x = x \} \quad G_x^s = s G_x s^{-1} \quad (= G_{s \cdot x})$
("Stabilizers are almost normal")

<pf> Not top-free $\Rightarrow \exists s \in G \setminus \{1\}$ s.t. Fix(s) has non-empty interior. Fix $x \in X$, $t_1, \dots, t_n \in G$

<u>claim</u>

$\exists s' \in G$ s.t. $x \in \mathrm{Fix}(s')$

By minimality $\exists u \in G \quad ux \in \mathrm{int}(\mathrm{Fix}(s))$ ← minimal + non-$\emptyset$ interior

$\Rightarrow sux = ux \Rightarrow u^{-1}sux = x$

$$\Rightarrow x \in \mathrm{int}(\mathrm{Fix}(\underbrace{u^{-1}su}_{= s'}))$$

By proximality $\exists r \in G$ s.t.

$$r t_i x \in \mathrm{Fix}(s') \quad \forall i$$

$$\Rightarrow \quad s' r t_i x = r t_i x \quad \forall i$$

$$\Rightarrow \quad r^{-1} s' r t_i x = t_i x \quad \forall i$$

$$\Rightarrow \quad t_i x \in \mathrm{Fix}(r^{-1} s' r) \quad \forall i$$

$$\Rightarrow \quad r^{-1} s' r \in \bigcap_{i=1}^{n} G_{t_i x} = \underbrace{\bigcap_{i=1}^{n} G_x^{t_i}}_{\neq \emptyset}.$$

If $\bigcap_{i=1}^{n} G_x^{t_i}$ is finite, then by FIP (finite intersection property)

$$\bigcap_{t \in G} G_{tx} = \bigcap_{t \in G} G_x^t \neq \{1\} \leftarrow \text{normal in } G, \neq \{1\}$$

Contradicts the assumption that G has no finite normal subgroup

so $\bigcap_{i=1}^{n} G_x^{t_i}$ is infinite. $\square$

<u>Def</u> G: discrete. $H \leqslant G$ is <u>normalish</u> if $\forall n \geq 1$ $t_1, \cdots, t_n \in G$ the intersection $\bigcap_{i=1}^{n} H^{t_i}$ is infinite.

<u>Thm</u> If G has no non-trivial finite normal subgroups (e.g. if $R_a(G) = \{1\}$) and no normalish amenable subgroups then G is C^*-simple.

<pf> Enough by last time to show $G \curvearrowright \partial_F G$ is free.

If not, stabilizers are amenable and normalish, by previous result.

(<u>Thm</u> If G has countably many amenable subgroups, then it is C^*-simple iff $R_a(G) = \{1\}$.)

The Tarski monster group is a discrete simple finitely generated group with all proper subgroups cyclic of order p for some fixed prime p. ($p > 10^{75}$)

<u>Cor</u> Tarski monster groups are C^*-simple.

<pf> Every subgroup corresponds to an element of G
Only countably many elements, hence countably many amenable subgroups. $\square$

(<u>Cor</u> The free Burnside group $B(m,n)$ is C^*-simple for $m \geq 2$ n odd $\gg 1$.)

<Pf> For these m, n, it is known that subgroups are either cyclic or contain $\mathbb{F}_2$ (hence non-amenable) $\square$

<u>Proof of Thm</u>

Suppose $R_a(G) = \{1\}$ Suppose G is not C^*-simple.
Then
$G \curvearrowright \partial_F G$ is not free. ($\partial_F G$ is Stonean)
top free = free

Fix $s \in G$ $\mathrm{Fix}(s) \neq \emptyset$. Fix $x \in \mathrm{Fix}(s)$

Let $F_x = \bigcap_{t \in \mathrm{Stab}_x} \mathrm{Fix}(t)$ F_x is closed and contains x.

Consider $\bigcup_{x \in \mathrm{Fix}(s)} F_x =: F$

Notice $\mathrm{Fix}(s) \subseteq F$ and $\mathrm{Fix}(s) \neq \emptyset$ and it is clopen.

Hence F has nonempty interior. Also, can only be countably many distinct stabilizers, since they are amenable.

Hence $\exists$ c'ble sequence $x_1, x_2, \dots$

$$F = \bigcup_{n=1}^{\infty} F_{x_n}$$

Baire category $\Rightarrow$ Some F_{x_k} must have non-empty interior, say $\mathcal{U}$.

By compactness $\exists s_1, \dots, \exists s_m \in G$; $\partial_F G = \bigcup_{j=1}^{m} s_j \mathcal{U}$

Now $\mathcal{U} \subseteq F_x$ everything in Stab_x fixes $\mathcal{U}$

hence $s_j (\mathrm{Stab}_x) s_j^{-1} = \mathrm{Stab}_{s_j \cdot x}$ fixes $s_j \mathcal{U}$

Hence $\bigcap_{j=1}^{m} s_j (\mathrm{Stab}\, x) s_j^{-1}$ fixes $\partial_F G = \bigcup_{j=1}^{m} s_j \mathcal{U}$.

But Stab_x is normalish, this is non-trivial

By last time these elements belong to $R_a(G)$, a contradiction. $\square$

unique trace $\Longleftrightarrow$ $R_a(G) = \{1\}$

$\Uparrow$? $\Downarrow$ $\qquad$ $\Uparrow$

C^*-simple