Tucker-Prob 1

1) Amenable actions & groups.
2) Inner amenable groups.
3) Linear groups.
4) Cost, stability (Jones - Schmidt)

<u>Def</u> A mean on a set X is a finitely additive probability measure m on X. (defined on all subsets of X)

$$\left\{\begin{matrix}\text{means}\\ \text{on}\\ X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{positive linear}\\ \text{functionals}\\ \text{on } \ell^\infty(X)\end{matrix}\right\}$$
with $\varphi(1_X)=1$

$$m \longmapsto \int_X \cdot \, dm$$

$f \in \ell^\infty(X)$ $\quad m(f) = \int_X f(x)\,dm(x)$

$\exists (f_n)_{n\in\mathbb{N}}$ simple. $\|f-f_n\|_\infty \to 0$

Then $m(f) = \lim_n m(f_n)$

$M(X)$ = set of all means on X. $M(X)$ is weak$*$-compact.

$\subseteq \ell^\infty(X)^*_1 \qquad |\int_X f\,dm| \leq \|f\|_\infty$

(<u>Recall</u> $\varphi_i \xrightarrow{w*} \varphi$ iff $\forall f \in \ell^\infty(X)$ $\varphi_i(f) \to \varphi(f)$.)

<u>Def</u> (von Neumann '29)

$G \curvearrowright X$ (action on a set) is <u>amenable</u> if $\exists\, m$: a mean $\in M(X)$ which is G-invariant.

i.e. m is a fixed pt of $G \curvearrowright M(X)$ $\quad (g\cdot m)(A) = m(g^{-1}A)$.

The group G is <u>amenable</u> if $G \curvearrowright G$ (left multiplication) is amenable.

<u>Prop 1</u>

$\left.\begin{matrix}\text{Finite groups,}\\ \mathbb{Z}\end{matrix}\right\}$ are amenable.

<pf> For each $n \geq 1$, let m_n be the normalized counting measure on $\{-n, \cdots, n\}$

Then let $m \in M(\mathbb{Z})$ be a weak*-cluster pt of $(m_n)_{n=1}^\infty$

If $A \subseteq \mathbb{Z}$ and $k \in \mathbb{Z}$ then $|(k \cdot m)(A) - m(A)|$

$$\le \limsup_{n\to\infty} |(k\cdot m_n)(A) - m_n(A)| = 0.$$

$$\le \|k\cdot m_n - m_n\| = \frac{2k}{2n+1}$$

FACT: If G is amenable, then G admits a 2-sided invariant mean, m, i.e., $g\cdot m = m = m\cdot g \quad \forall g \in G$

$$(m\cdot g)(A) = m(Ag^{-1})$$

If $m, n \in M(G)$ then $m * n = \int_G g\cdot n \, dm(g)$

i.e. $m*n(A) = \int_G g\cdot n(A)\, dm(g)$

Fix now $n \in M(G)$ which is left-invariant and define

$m := n * n^{-1} \quad (n^{-1})(A) = n(A^{-1})$

$$g\cdot m = g\cdot(n*n^{-1}) = (gn)*n^{-1} = n*n^{-1} = m$$

$$m\cdot g = (n*n^{-1})\cdot g = n*\underbrace{(n^{-1}\cdot g)}_{\substack{\| \\ (g^{-1}n)^{-1} = n^{-1}}} = n*n^{-1} = m$$

<u>Closure properties</u>

1) Directed unions of amenable groups are amenable.

$G = \varinjlim G_n \qquad G_n \subseteq G_{n+1} \subseteq \cdots$

all amenable

Let $m_i \in M(G_i) \subseteq M(G)$ be G_i-invariant.

Let $m =$ weak*-cluster pt of $(m_i)_{i=1}^{\infty}$

Then m is G-invariant.

2) Subgroups of amenable groups are amenable.

Given $H \le G$, G: amenable.

$T \subseteq G$. consists of 1 point from each right coset of H

If $m_G \in M(G)$ is G-invariant,

Then $m_H(A) = m_G(AT)$ is an H-invariant mean.

3) Quotients of amenable groups are amenable.

If m_G : inv-mean on G,

$\varphi : G \twoheadrightarrow K$ quotient.

$m_K := \varphi_*(m_G)$ $\qquad \varphi_*(m_G)(A) := m_G(\varphi^{-1}(A))$

$$A \subseteq K$$

Given $k \in K$ Let $g \in \varphi^{-1}(k)$

Then $\varphi^{-1}(k^{-1}A) = \{ h \in G \mid \varphi(h) \in k^{-1}A = \varphi(g)^{-1}A \} = g^{-1}\varphi^{-1}(A)$

$$\Updownarrow$$

$$\varphi(gh) \in A \iff h \in g^{-1}\varphi^{-1}(A)$$

So m_K is K-invariant.

4) If $1 \to N \to G \xrightarrow{\varphi} K \to 1$ (exact)

Then G amenable $\iff$ N, K amenable.

$(\Rightarrow)$ ✓

$(\Leftarrow)$ Consider the action $G \overset{\varphi}{\curvearrowright} K$ $\quad g \cdot k = \varphi(g)k$ is an <u>amenable</u> action (K is amenable)

$\mathrm{Stab}_G(k) = N$ is <u>amenable</u>. by assumption

<u>Important Lemma</u>

If $G \curvearrowright X$ is amenable.
and every stalizer $G_x = \{ g \in G \mid g \cdot x = x \}$ is amenable
then G is amenable.

<pf> Let $X_0 \subseteq X$ which contains one element from each orbit

For $x_0 \in X_0$ let $m_{x_0} \in M(G_{x_0}) \subseteq M(G)$. be an G_{x_0}-invariant mean. Extend the assignment $x \mapsto m_x$ for all $x \in X$

by taking $m_{g \cdot x_0} = g \cdot m_{x_0}$ for $g \in G$, $x_0 \in X_0$

(check: this is well-defined)

$$g x_0 = h x_0 \iff g^{-1}h \in G_{x_0}$$

$$\text{so} \quad g^{-1}h \cdot m_{x_0} = m_{x_0} \quad \text{i.e.} \quad g\, m_{x_0} = h\, m_{x_0}$$

So it's well-defined!

Note: for $x \in X$, $g \in G$, $m_{g \cdot x} = g \cdot m_x$.

Therefore letting $m_X \in M(X)$ be G-invariant, the mean

$$m = \int_X m_x \, dm_X(x) \quad \text{is } G\text{-invariant.}$$

Since $g \cdot m = \int_X g \cdot m_x \, dm_X(x) = \int_X m_{gx} \, dm_X(x)$

$$= \int_X m_x \, dm_X(g^{-1}x) = \int_X m_x \, dm_X(x) = m. \quad \square$$

<u>Prop 2</u>

(i) Abelian groups are amenable.

Suffices to show all finitely generated subgroups are amenable
(then take directed union)

$$\left.\begin{array}{l} \text{Since } \mathbb{Z} \text{ is amenable} \\ \quad \text{finite groups are amenable} \end{array}\right\} \overset{\text{extension (iv)}}{\Rightarrow} \begin{array}{l} \text{finitely generated abelian} \\ \text{groups are amenable.} \end{array}$$

(ii) <u>Solvable</u> groups are amenable.

i.e., $G \rhd N_1 \rhd N_2 \rhd \cdots \rhd N_k$, $G = N_0$

N_i / N_{i+1} : abelian $\forall i = 0, \dots, k-1$

(Follows from (i) and closure property (iv))

<u>Prop</u>

(F_2 is non-amenable.)

More generally if G contains a copy of F_2, then G is non-amenable.

(cf: von Neumann – Day problem)

Inner Amenability

Def (Effros '75) A group G is inner-amenable if the conjugation (action $G \overset{\text{Conj}}{\curvearrowright} G$ is amenable with atomless mean.

$\forall g \in G, \quad m(\{g\}) = 0$

!)

Note (finite groups are not inner-amenable)

Prop 3. The following groups are inner amenable.

i) infinite amenable groups.

$\exists m$: two-sided invariant mean.
Necessarily atomless for infinite groups.

ii) Groups with infinite center $Z(G)$.

(Any atomless mean on $Z(G)$ works)

iii) $H \times K$ where K is inner-amenable

If $m_K \in M(K)$ conjugation-invariant mean then

view m_K as a mean on $\{1_H\} \times K$

and check that m_K is K-invariant & H-invariant

hence $H \times K$ - (conjugation) invariant

$$\widetilde{m_K}(A) = m_K(A \cap (\{1_H\} \times K))$$

$$k \xmapsto{\varphi} (1_H, k) \qquad \widetilde{m_K} = \varphi_*(m_K).$$

iv) Asymptotically commutative groups. G

i.e. $\exists$ injective sequence $(c_n)_{n=1}^{\infty}$ in G s.t. $\forall g \; \exists n_0 \; \forall n \geq n_0$ $\; c_n g = g c_n$

Ex. $\bigcup_n SL_n(\mathbb{Z})$

$$c_n = \begin{pmatrix} 1_n & \\ & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$$

<pf>

Let $m \in M(G)$ be any cluster pt of $(\delta_{c_n})_{n=1}^{\infty}$

Alternatively let m be any atomless mean on $\{c_n\}_{n=1}^{\infty}$

Then $gAg^{-1} \triangle A$ is finite for any $A \subseteq \{c_n\}$

So $m(gAg^{-1} \triangle A) = 0$, hence $m(gAg^{-1}) = m(A)$.

(e.g. $H_n \neq 1 \Rightarrow \bigoplus_n H_n$ is asymptotically commutative.)

Suppose $G \curvearrowright X$ is amenable. with an atomless mean.

(e.g. if all orbits are infinite)

Fix any group $H \neq 1$.

$H \wr_X G$ is inner-amenable.

$G \curvearrowright \bigoplus_X H$ by automorphism. via $(g \cdot f)(x) = f(g^{-1}x)$

$H \wr_X G := \bigoplus_X H \rtimes G$

<u>Proof</u> Fix m_X on X atomless & G-inv.

Fix $h \in H \setminus \{1\}$

Define $\varphi(x) \in \bigoplus_X H$ by $\varphi(x)(y) = \begin{cases} h & (y = x) \\ 1 & (y \neq x) \end{cases}$

Then $\varphi(gx) = g \cdot \varphi(x)$

So $\varphi_*(m_X)$ is invariant under conj by G.

atomless, since φ injective.

In fact $\varphi_*(m_X)(H_x) = m_X(\{x\}) = 0$

(H_x = copy of H in $\bigoplus_X H$)

Thus for any $x \in X$, $h \in H_x$

$\varphi_*(m_X)(\{f \in \bigoplus_X H \,;\, hfh^{-1} = f\}) = 1$

So $\varphi_*(m_X)(hAh^{-1}) = \varphi_*(m_X)(A)$

$\Rightarrow$ $\varphi_*(m_X)$ is $H \curvearrowright_X G$ $\overset{\text{conj}}{-}$invariant $\square$

(Exercise
Every action of an amenable group is amenable.)

Prop 4

Let $1 \to N \to G \to K \to 1$
be a short exact sequence of groups.

(i) if N is inner-amenable,
K is amenable, (fine even if K finite)

then G is innear-amenable.

(ii) If G is inner-amenable then either N is inner-amenable
or K is inner-amenable.