Last time:

(G is asymptotically commutative if $\exists$ an injective sequence $(c_n)_{n=1}^{\infty}$ ($c : \mathbb{N} \to G$ is injective) s.t. $\forall g \in G$ commutes with c_n for all but finitely many n.)

Prop 5 Let $1 \to N \to G \to K \to 1$ (exact)

(i) If N is inner amenable, K is amenable $\} \Rightarrow G$ is inner amenable

(ii) If G is inner-amenable then either N or K is inner amenable.

<Pf> (i) Fix $m_N \in M(N)$ N-conj invariant atomless.
$\subseteq M(G)$.

$$G \overset{\text{conj}}{\curvearrowright} N \rightsquigarrow G \overset{\text{conj}}{\curvearrowright} M(N)$$

N acts trivially on the orbit $\{g m_N g^{-1}\}_{g\in G}$ of m_N.

$G \overset{\text{conj}}{\curvearrowright} \{g m_N g^{-1}\}_{g\in G}$ descends to an action of K ($k g m_N g^{-1} k^{-1}$)

Let $m := \int_K k m_N k^{-1} \, dm_K(k)$

m is atomless, since m_N is, Also letting $\varphi : G \to K$ denote the map above.

$$g m g^{-1} = \int_K \varphi(g) k m_N \underbrace{k^{-1}\varphi(g)^{-1}}_{(\varphi(g)k)^{-1}} dm_K(k)$$

$$= \int_K k m_N k^{-1} \, dm_K(k)$$

(ii) <u>Key Observation</u>

If $m, n \in M(G)$ are both G-conj invariant and atomless, then so is $m * n$,

Since, $g(m*n)g^{-1} = (gmg^{-1}) * (gng^{-1})$
$= m*n$. (also atomless)

Fix $m \in M(G)$ conj-inv, atomless.

$\Rightarrow \varphi(m)$ is conj-inv.

So K is inner-amenable if $\varphi(m)$ is atomless.

If $m(N) > 0$, then N is inner-amenable.

Suppose $\varphi(m)$ has an atom, i.e., there is some $g_0 \in G$ with $m(Ng_0) > 0$.

Consider $m * m^{-1}$, which is conj-inv and atomless.

(recall $m^{-1}(A) := m(A^{-1})$)

$$(m*m^{-1})(N) = \int_G (gm^{-1})(N)\, dm(g) = \int_G m(Ng)\, dm(g)$$

$$\geq \int_{g \in Ng_0} m(Ng_0)\, dm(g) = m(Ng_0)^2 > 0$$

$\Rightarrow$ N is inner-amenable.

<u>Corollary</u>

(Finite index subgroup of an inner-amenable group is inner-amenable)

<pf>

Fix $H \underset{\text{finite index}}{\leq} G$ ← inner-amenable. $\qquad N = \bigcap_g gHg^{-1} \leq H.$

Let N be normal, finite index in G contained in H, then we can apply

Suffices to show : N is inner-amenable

(i) to $1 \to N \to H \to H/N \to 1$

$$\begin{array}{ccccccccc} 1 & \to & N & \to & H & \to & H/N & \to & 1 \\ & & \uparrow & & & & \uparrow & & \\ & & \text{inner amen} & & & & \text{amen.} & & \end{array}$$

Now we have

$$\begin{array}{ccccccccc} 1 & \to & N & \to & G & \to & G/N & \to & 1 \\ & & & & \uparrow & & \uparrow & & \\ & & & & \text{inner amen} & & \text{finite} \Rightarrow \text{inner-amen} & & \end{array}$$

So by (ii) N is inner-amenable.

Prop 6 (Stalder '06)

Baumslag – Solitar groups
$BS(m,n) = \langle t, a \mid t a^m t^{-1} = a^n \rangle$
are inner-amenable.

<Pf> Let $G := BS(m,n)$, $N := \{ g \in G \mid g$ commutes with a finite index subgroup of $\langle a \rangle$ (i.e. with a power of a) $\}$

N is a subgroup: if g_0, g_1 commutes with $H_0 \leq_{fi} \langle a \rangle$, $H_1 \leq_{fi} \langle a \rangle$

then $g_0 g_1$ commutes with $H_0 \cap H_1$

so $g_0 g_1 \in N$.

N is normal in G

$a \in N$ so $a N a^{-1} = N$.

If $g \in N$ commutes with a^k, then g commutes with $a^{mk} = (a^m)^k$

So $t g t^{-1}$ commutes with $t (a^m)^k t^{-1} = (t a^m t^{-1})^k = (a^n)^k$

$= a^{nk}$ So $t g t^{-1} \in N$.

Similarly $t^{-1} g t \in N$.

So $N \lhd G$.

N is asymptotically commutative, as witnessed by $c_n = a^{n!}$

So N is inner-amenable.

$$1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1$$

N: inner amen

G/N: cyclic (generated by image of t) hence amenable.

Hence G is inner-amenable. $\square$

Prop 7

(F_2 (free group) is not inner-amenable.)

<pf> Recall Nielsen-Schreier Thm.

(i) Subgroups of free groups are free.

(ii) If $g_1, \dots, g_n \in F_n$ generating F_n, then they freely generate.

(iii) F_n cannot be generated by fewer than n elements.

Claim If $g \in F_2 - \{1\}$

(Then $C_{F_2}(g)$ (centralizer) is cyclic.)

< Proof > $C_{F_2}(g)$ is a free group. (by above (i))

Show it's abelian.

If $h_0, h_1 \in C_{F_2}(g)$, then $\langle h_0, g\rangle$, $\langle h_1, g\rangle$

are abelian hence cyclic.

Then $\langle h_0, g\rangle = \langle {}^{\exists} k_0\rangle$ $\langle h_1, g\rangle = \langle {}^{\exists} k_1\rangle$

Then $g \in \langle k_0\rangle \cap \langle k_1\rangle$

So $\langle k_0, k_1\rangle$ cannot generate free group of rank 2

(g is a nonzero power of k_0, k_1)

So $\langle k_0, k_1\rangle$ is cyclic.

Since $h_0, h_1 \in \langle k_0, k_1\rangle$, they commute.

Thus $C_{F_2}(g)$ is abelian, hence cyclic.

Proof of Prop 7 Suppose F_2 is inner-amen.

Then $F_2 \overset{\text{conj}}{\curvearrowright} F_2 - \{1\}$ is amenable.

All the stabilizers are cyclic by above, hence amenable.

By Important Lemma, this implies F_2 is amenable,

a contradiction. $\square$

The AC-center
and inner-radical.

$$C_G(A) = \{ g \in G \mid g a g^{-1} = a \text{ for all } a \in A \}$$

If N is normal in G, then $C_G(N)$ is normal as well,

since $g\, C_G(N)\, g^{-1} = C_G(g N g^{-1}) = C_G(N)$.

The <u>AC-center</u> of G is the subgroup

$$AC(G) = \Big\langle \underbrace{\{ N \leq G \mid N \text{ normal in } G \ \& \ G/C_G(N) \text{ is amenable} \}}_{=:\, P} \Big\rangle$$

<u>Prp 8</u>

($AC(G)$ is an amenable <u>normal</u> subgroup of G.
(in fact <u>characteristic</u>)
invariant under $\mathrm{Aut}(G)$)

<pf> (i) Each $N \in P$ is amenable.
(ii) If $N_0, N_1 \in P$ then $N_0 N_1 \in P$

[$AC(G)$ = direct union of such N's.
So this will imply $AC(G)$ is amenable.]

(i) $1 \longrightarrow N \cap C_G(N) \hookrightarrow N \longrightarrow N/N \cap C_G(N) \longrightarrow 1$

$N \cap C_G(N) = Z(N)$ abelian so amenable

$N/N \cap C_G(N) \cong \dfrac{N \cdot C_G(N)}{C_G(N)} \leq \dfrac{G}{C_G(N)}$: amenable

$\dfrac{N \cdot C_G(N)}{C_G(N)}$ ⇓ amenable

Then N is amenable.