Tucker-Drob 4

Last time : $BS(m,n)$ inner-amenable
F_n not inner-amenable. $(n \geq 2)$

$Z(G)$ infinite
$H \times K^{\mathbb{N}}$ K infinite amenable } obviously inner-amenable

AC-center of G

$$AC(G) = \langle \underbrace{\{ \underset{\text{normal}}{N \triangleleft G} \mid G/C_G(N) \text{ amenable} \}}_{P} \rangle$$

Prop 8 $AC(G)$ is amenable, characteristic in G.

<Pf> (i) Each $N \in P$ is amenable.
(ii) If $N_0, N_1 \in P$ then $N_0 N_1 \in P$

(i) Done last time.
(ii) Let $M_i = C_G(N_i)$.

Then $C_G(N_0 N_1) = C_G(N_0) \cap C_G(N_1) = M_0 \cap M_1$

$$1 \longrightarrow M_0/M_0 \cap M_1 \hookrightarrow G/M_0 \cap M_1 \longrightarrow \underset{\text{amenable}}{G/M_0} \longrightarrow 1$$

$$\simeq \underset{\text{amenable}}{M_0 M_1/M_1} \leq \underset{\text{amenable}}{G/M_1}$$

$\Longrightarrow$ $G/M_0 \cap M_1$ is amenable.
$\Longrightarrow$ $AC(G)$ is amenable.

Prop 9 (If $AC(G)$ is infinite, then G is inner-amenable.)

<Pf> postponed.

Annoyance : It is NOT in general true that $AC(G/AC(G))$ is trivial.

If $N \triangleleft G$ then $G \overset{\text{conj}}{\curvearrowright} N$ $\quad N \rtimes G \curvearrowright N$ $\quad (h,g)\cdot k = hgkg^{-1}$
$(g \in G,\ h,k \in N)$

The <u>inner-radical</u> of G is

$$I(G) = \left\langle \underbrace{\{ N \lhd G \mid \underbrace{N \rtimes G \curvearrowright N \text{ is an amenable action}} \}}_{Q} \right\rangle$$

There is a mean $m \in \mathcal{M}(N)$ which is simultaneously invariant under conjugation by G and left translation by N

<u>Prop 10</u> $I(G)$ is an amenable characteristic subgroup of G
(Moreover $I(G)$ is the unique maximal subgroup in Q)

<Proof> (i) Each $N \in Q$ is amenable

(ii) If $N_0, N_1 \in Q$ then $N_0 N_1 \in Q$

(iii) If $Q_0 \subseteq Q$ directed by inclusion, then $M_0 = \bigcup Q_0 \in Q$

Fix $m_i \in M(N_i)$ $N_i \rtimes G \curvearrowright N_i$ – invariant.

Then $m_0 * m_1 \in M(N_0 N_1)$ is $N_0 N_1 \rtimes G$ – inv.

$m_0 * m_1$ is G – conj inv

If $h_0 \in N_0$ then $h_0 (m_0 * m_1) = (h_0 m_0) * m_1 = m_0 * m_1$

if $h_1 \in N_1$ then $h_1 (m_0 * m_1) = h_1 m_0 h_1^{-1} * h_1 m_1$

$= m_0 * m_1$

(iii) For each $N \in Q_0$ fix $m_N \in M(N) \subseteq M(M_0)$ $N \rtimes G$ – inv

Then any weak*-cluster point $m \in M(M_0)$ witnesses that $M_0 \in Q$.

<u>Prop 11</u>
($AC(G) \leq I(G)$)

<pf> By (iii) in previous proof suffices to show that $P \subseteq Q$.

If $N \in P$ then $1 \rtimes C_G(N) \subseteq N \rtimes G$. acts trivially in the action $N \rtimes G \curvearrowright N$

So this action descends to an action of the group

$N \rtimes G / C_G(N)$, which is amenable, hence $N \rtimes G \curvearrowright N$ is amenable.

i.e. $N \in Q$.

Prop 12

(If $I(G)$ is infinite, then G is inner amenable)

Fact $I(G/I(G)) = \{1\}$

Linear inner-amenable groups

Thm (T-D) Let G be a linear group

Then $AC(G) = I(G)$.

Moreover, the following are equivalent

(1) G is innear-amenable

(2) $AC(G) = I(G)$ is infinite

(3) There exists a short exact sequence

$$1 \to N \to G \to K \to 1$$

where K is amenable and either

(i) N has infinite center

or (ii) $N = LM$ where L, M are pairwise commuting normal subgroups of G, with $L \cap M$: finite

and N is infinite amenable.

(3) ⇒ (2) ✓

(2) ⇒ (1) ✓

(2) ⇒ (3)

Dani's forgotten Lemma (1985)

Let $G \curvearrowright X$ be amenable with invariant mean m.

For $B \subseteq X$ Let $G_B = \{ g \in G \mid gx = x, \forall x \in B \}$

Let $G_0 = \{ g \in G \mid m(\mathrm{fix}_X(g)) = 1 \}$

Then $G_0 \lhd G$.

Lem (Dani, 1985)

(Suppose $\{ G_B \mid B \overset{\text{Borel}}{\subseteq} X \} =: \mathcal{C}$ satisfies descending chain condition D.C.C.

every nonempty subset of $\mathcal{C}$ has a minimal element.

Then G/G_0 is amenable.)

Important Lemma : If $G \curvearrowright X$ amenable and G_x is amenable $\forall x \in X$ then G is amenable.

<u>Improvement</u> Let $G \curvearrowright X$ $G \curvearrowright Y$ If $G \curvearrowright X$ is amenable, and if $G_x \curvearrowright Y$ is amenable $\forall x \in X$ then $G \curvearrowright Y$ is amenable.

<u>pf of Dani's Lemma</u>

$G \curvearrowright X$ is amenable

Want : $G \curvearrowright G/G_0$ is amenable.

Enough : $G_x \curvearrowright G/G_0$ is amenable.

Suppose not, so that

$$\{ G_B \mid B \overset{\text{Borel}}{\subseteq} X,\ \underset{\text{non-amenable}}{G_B \curvearrowright G/G_0} \} = \mathcal{C}_0$$

is non-empty. $(G_x \in \mathcal{C}_0)$

Let $\underset{G_{B_0}}{\overset{\shortparallel}{L}} \in \mathcal{C}_0$ be minimal.

Then $L \curvearrowright G/G_0$ is non-amenable.

So $L \not\subseteq G_0$, $m(\mathrm{fix}_X(L)) < 1$

$L \curvearrowright \overbrace{X - \mathrm{fix}_X(L)}^{\text{positive measure}}$ & for each $x \in X - \mathrm{fix}_X(L)$

$\underset{G_{B \cup \{x\}}}{\overset{\shortparallel}{L_x}}$ is a proper subgroup contained in L.

$G_{B \cup \{x\}} \in \mathcal{C}$.

Hence $L_x \curvearrowright \overset{\shortparallel\, Y}{G/G_0}$ is amenable (minimality of G_{B_0})

By Improved Important Lemma, $L \curvearrowright G/G_0$ is amenable, a contradiction $\square$

$AC(G) \leqslant I(G) \leqslant \mathrm{Rad}(G)$.