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Simple case

Definition / Theorem: A a C^*-algebra (simple) $A \neq \mathbb{C}$. TFAE

(i) $\forall B \underset{\text{her}}{\subseteq} A$, B contains an infinite projection
(ii) $\forall a, b \in A \setminus \{0\}$ $\exists x, y \in A$ $b = xay$
(iii) $RR(A) = 0$ and all projections in A are properly infinite ($p \oplus p \precsim p$)
(iv) $W(A) = Cu(A) \cong \{0, \infty\}$

Example $\mathcal{O}_n$, $2 \leq n < \infty$

$$\mathcal{O}_n = C^*(s_1, s_2, \dots, s_n \mid s_j^* s_j = 1 = \textstyle\sum s_j s_j^*)$$

$\mathcal{O}_n$ properly infinite and simple

$n = \infty$, $\mathcal{O}_\infty = C^*(s_1, s_2, \dots \mid s_j^* s_j = 1,\ i \neq j = s_i s_i^* \perp s_j s_j^*)$

$$K_0(\mathcal{O}_n) = \begin{cases} \mathbb{Z}/(n-1) & n < \infty \\ \mathbb{Z} & n = \infty \end{cases} \qquad K_1(\mathcal{O}_n) = 0.$$

A properly infinite $\Rightarrow T(A) = \emptyset$

Exhausting (K_0, K_1) by purely infinite and simple C^*-algebras

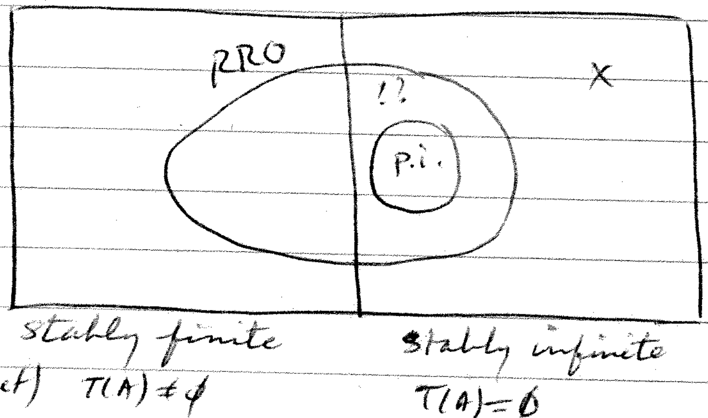
i) $A = \varinjlim \bigoplus_{j=1}^{r} M_{r_j}(\mathcal{O}_{n_j}) \otimes C(\mathbb{T})$ [simple $\Rightarrow$ p.i.]

(ii) (simple stable $A\mathbb{T}$-alg) $\rtimes_\alpha \mathbb{Z}$

Theorem: A simple, separable, exact $T(A) \neq \emptyset$, $A \otimes \mathcal{Z} = A$, stable $\alpha \in Aut(A)$. Then $A \rtimes_\alpha \mathbb{Z}$ purely infinite simple $\Leftrightarrow$ A has no α-invariant traces.

Definition: A Kirchberg algebra if A is purely infinite, simple, separable, nuclear.

Simple C^*-algebras



A stably infinite $\Leftrightarrow$ $A \otimes \mathcal{K}$ contains infinite proj.

$\exists$ stably infinite simple C^*-algebra that does not have real rank zero, hence not purely infinite.

Question: A simple, $RR(A) = 0$, A stably infinite $\Rightarrow$ A purely infinite?

Question: Suppose A is simple, all projections are infinite, $P(A) \neq 0$ (ie has non trivial projections) $\Rightarrow$ A purely infinite?

Question: A stably infinite $\Rightarrow$ A has property (SP)?

Theorem (Kirchberg): A, B simple, not type I (ie not matrix algebras, not $\mathcal{K}$).

A	B	$A \otimes_{min} B$
stably finite	stably finite	?? [But A, B exact $\Rightarrow$ stably finite]
stably finite	stably infinite	~~properly~~ purely infinite
stably infinite	stably infinite	purely infinite

(A, B simple $\Rightarrow A \otimes_{min} B$ simple)

K1 A simple, separable, nuclear. Then A is purely infinite $\Leftrightarrow A \cong A \otimes \mathcal{O}_\infty$ $[K_*(\mathcal{O}_\infty) \cong K_*(\mathbb{C})]$

K2 A simple, unital, separable, nuclear $\Leftrightarrow A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ $[K_*(\mathcal{O}_2) = 0]$

K3 A separable, exact $\Leftrightarrow A \hookrightarrow \mathcal{O}_2$

Theorem (Kirchberg, Phillips): Let A, B be Kirchberg algebras

1) $A \otimes \mathcal{K} \cong B \otimes \mathcal{K} \Leftrightarrow A \sim_{KK} B$
2) If A, B has UCT then $(A \otimes \mathcal{K} \cong B \otimes \mathcal{K}) \Leftrightarrow (K_0(A), K_1(A)) \cong (K_0(B), K_1(B))$

Non-simple case

Definition / Theorem: A C^*-algebra with no non-zero abelian quotients. TFAE

i) $\forall a, b \in A_+$, $a \in \overline{AbA} \Leftrightarrow a \precsim b$
ii) $\forall a \in A_+$, a is properly infinite $[a \oplus a \precsim a]$

$x \in W(A)$, x properly infinite $\overset{def.}{\Leftrightarrow} 2x \leq x$
$\Leftrightarrow 2x = x$

x infinite if $\exists y \neq 0$ $y \in W(A)$ st $x + y \underset{(=)}{\leq} x$

A is purely infinite $\iff$ $W(A)$ is purely infinite
$W(A)$ purely infinite $\iff$ $\forall x \in W(A)$ x is properly infinite

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A seperable
$W(A) \longrightarrow \mathrm{Ideal}(A)$, $\langle a \rangle \longmapsto \overline{AaA}$
A is purely infinite $\iff$ this map is 1-1
$\iff$ this map is an order ~~isomop~~ isomorphism

Example: Cuntz-Krieger algebras.
$A \in M_n(\{0,1\})$ non-degenerate
$\rightsquigarrow \mathcal{O}_A = C^*\left(s_1, s_2, \ldots, s_n \;\middle|\; \sum_{j=1}^{n} s_j s_j^* = 1,\ s_j^* s_j = \sum_{i=1}^{n} A(j,i)\, s_i s_i^* \right)$

Example: A a C^*-algebra, then $A \otimes \mathcal{O}_\infty$ is purely infinite
eg $C_0(\mathbb{R}) \otimes \mathcal{O}_\infty$ is purely infinite
Note these need not have projections.

Permanence properties:
i) $0 \to I \to A \to B \to 0$ ~~with~~ Then I, B are purely infinite $\iff$ A is purely infinite
ii) $A = \varinjlim A_n$, if A_n is purely infinite $\forall n$ then A is purely infinite
(iii) A, B purely infinite, A exact $\Rightarrow$ $A \otimes_{\min} B$ purely infinite

Question: A or B purely infinite
$\overset{?}{\Rightarrow}$ $A \otimes_{\min} B$ purely infinite
A p.i. $\overset{?}{\Rightarrow}$ $A \otimes C([0,1])$ p.i.

Theorem (Kirchberg, Rørdam): A a C^*-algebra, exact, separable

(i) $A \cong A \otimes \mathcal{O}_\infty$

(ii) A strongly purely infinite

(i) $\Rightarrow$ (ii) and if A is separable and nuclear then (ii) $\Rightarrow$ (i)

(ii) $\not\Rightarrow$ (i) in general

Definition: A is strongly purely infinite if $\forall \begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \in M_2(A)_+$ $\forall \varepsilon > 0$ $\exists d_1, d_2 \in A$ s.t

$$\left\| \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}^* \begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\| < \varepsilon$$

$$a \sim \begin{pmatrix} a & a \\ a & a \end{pmatrix}$$

(iii) A is purely infinite

(iv) A is weakly purely infinite

(v) $\ell^\infty(A)/C_0(A)$ ~~weakly purely infinite~~ traceless

(vi) A traceless

(ii) $\Rightarrow$ (iii) and if A simple or $RR(A) = 0$ or $A \cong A \otimes \mathcal{Z}$ then (iii) $\Rightarrow$ (ii)

(iii) $\Rightarrow$ (iv) $\Leftrightarrow$ (v) $\Rightarrow$ (vi) and if A simple or $RR(A) = 0$ or $A \cong A \otimes \mathcal{Z}$ then (iv) $\Rightarrow$ (iii). In general (vi) $\not\Rightarrow$ (v) But if $A \cong A \otimes \mathcal{Z}$ then (vi) $\Rightarrow$ (v)

Remark: $W(A)$ has no dimension function $\Leftrightarrow$ $\forall x \in W(A)$ $\exists k \in \mathbb{N}$ such that kx is properly infinite.

Definition: A is weakly purely infinite if and only if $\exists k \in \mathbb{N}$ $\forall x \in W(A)$ kx is properly infinite

Theorem (Kirchberg): A, B separable, nuclear C^*-algebras with $\mathrm{Prim}(A) \cong \mathrm{Prim}(B)$ $(= X)$

$A \otimes \mathcal{O}_\infty \otimes \mathcal{K} \cong B \otimes \mathcal{O}_\infty \otimes \mathcal{K} \iff A \sim_{KK_X} B$

If A, B strongly purely infinite then

$A \otimes \mathcal{K} \cong B \otimes \mathcal{K} \iff A \sim_{KK_X} B$

Primitive ideal space:

$\mathrm{Prim}\, C(X) = X$ (X compact, Hausdorff)

$\mathrm{Prim}(A) = \{\ker \pi \mid \pi \text{ irreducible representation}\}$

$\overset{\text{thm}}{=} \{I \in \mathrm{Ideal}(A) \mid I \text{ prime}\}$

Jacobson topology, T_0-space

Corollary: if A, B are separable and nuclear then

$A \otimes \mathcal{O}_2 \otimes \mathcal{K} \cong B \otimes \mathcal{O}_2 \otimes \mathcal{K}$ if and only if

$\mathrm{Prim}(A) \cong \mathrm{Prim}(B) \longleftrightarrow \mathrm{Ideal}(A) \cong \mathrm{Ideal}(B)$

Question: Which T_0-spaces arise as $\mathrm{Prim}(A)$ for A separable C^*-algebras (nuclear)

Example: $(t_n)_{n=1}^\infty \overset{\text{dense}}{\subseteq} (0,1)$

AH_0-alg

$$C_0((0,1]) \xrightarrow{\varphi_1} M_2(C_0((0,1])) \xrightarrow{\varphi_2} M_4(C_0((0,1])) \to \cdots \to A$$

$$\varphi_n(f)(t) = \begin{pmatrix} f(t) & 0 \\ 0 & f(t \wedge t_n) \end{pmatrix}$$

$\mathrm{Ideal}(A) \cong [0,1)$ totally ordered

Fact: A has no projections and $\forall I \triangleleft A$: A/I has no projections

$A \cong A \otimes M_{2^\infty} \Rightarrow A \cong A \otimes \mathcal{Z}$ (since $M_{2^\infty} \otimes \mathcal{Z} \cong M_{2^\infty}$)

$A \cong A \otimes \mathcal{O}_\infty \iff A$ traceless
and this is indeed the case!

$A \sim_{h,\text{ideal}} 0$, ie. $\exists$ $*$-homomorphisms $\varphi_t : A \to A$, $t \in [0,1]$, $\varphi_1 = id$, $\varphi_0 = 0$ and $\forall J \lhd A$ $\varphi_t(J) \subseteq J$

A strongly purely infinite, separable, nuclear
$A \sim_{h,\text{ideal}} 0 \Rightarrow A = A \otimes \mathcal{O}_2$

<u>Theorem</u>: A a nuclear, separable, stable, strongly purely infinite C^*-algebra. Then
$A \sim_{h,\text{ideal}} 0 \Rightarrow A \cong A \otimes \mathcal{O}_2$ and A is an AH_0-algebra

<u>Remark</u>: A as in the above example.
A AH_0-alg $\Rightarrow A \hookrightarrow AF$-algebra, A quasidiagonal
$A \cong A \otimes \mathcal{O}_2 \Rightarrow C\mathcal{O}_2 \hookrightarrow A \hookrightarrow AF$

cone

B-separable, exact, CB

cone