

# Factorizable maps and traces on the universal free product of matrix algebras

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## Abstract

We relate factorizable quantum channels on  $M_n(\mathbb{C})$ , for  $n \geq 2$ , via their Choi matrix, to certain matrices of correlations, which, in turn, are shown to be parametrized by traces on the unital free product  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$ . Factorizable maps that admit a finite dimensional ancilla are parametrized by finite dimensional traces on  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$ , and factorizable maps that approximately factor through finite dimensional  $C^*$ -algebras are parametrized by traces in the closure of the finite dimensional ones. The latter set of traces is shown to be equal to the set of hyperlinear traces on  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$ . We finally show that each metrizable Choquet simplex is a face of the simplex of tracial states on  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$ .

## 1 Introduction

Factorizable maps were introduced by C. Anantharaman-Delaroche in [2] in her study of non-commutative analogues of classical ergodic theory results. This notion has lately found interesting applications in quantum information theory, e.g., in solving in the negative the asymptotic quantum Birkoff conjecture, [15]. A factorizable channel  $T$  on  $M_n(\mathbb{C})$  is a unital completely positive trace-preserving map  $M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  that *factors* through a finite tracial von Neumann algebra  $(M, \tau_M)$  via two unital  $*$ -homomorphisms  $M_n(\mathbb{C}) \rightarrow M$  (see more details in Section 3). Factorizable maps were in [15] equivalently characterized as arising from an ancillary tracial von Neumann algebra  $(N, \tau_N)$  and a unitary  $u$  in  $M_n(\mathbb{C}) \otimes N$  such that  $T(x) = (\text{id}_n \otimes \tau_N)(u(x \otimes 1_N)u^*)$ , for all  $x \in M_n(\mathbb{C})$ . It was recently shown in [23] that the ancilla  $N$  cannot always be taken to be finite dimensional (or even of type I). U. Haagerup and the first named author proved in [16, Theorem 3.7] that each factorizable channel can be approximated by factorizable ones possessing a finite dimensional ancilla if and only if the Connes Embedding Problem has an affirmative answer.

In this paper we present a different viewpoint on factorizable channels, that bears resemblance to the description of quantum correlation matrices arising in Tsirelson's conjecture. In Section 3, we establish when a linear map on  $M_n(\mathbb{C})$  is factorizable in terms of certain properties of its Choi matrix. Fritz, [13], and, independently, Junge et al., [19], expressed the quantum correlation matrices appearing in Tsirelson's conjecture in terms of states on the minimal, respectively, the maximal tensor product of the full group  $C^*$ -algebra associated

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with the free product of finitely many copies of a finite cyclic group. (This characterization, in turn, was the bridge needed to prove the equivalence between Tsirelson’s conjecture and the Connes Embedding Problem, with the finishing touch provided by Ozawa, [25].) In a similar spirit, we recast the description of factorizable maps on  $M_n(\mathbb{C})$  in terms of traces on the unital universal free product  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$ . We show that factorizable channels with finite dimensional ancilla are parametrized by finite dimensional traces on  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$ , and factorizable channels that can be approximated by ones possessing a finite dimensional ancilla are parametrized by traces in the closure of the finite dimensional ones.

This new viewpoint on factorizable channels led us to further analyze the trace simplex of the unital universal free product  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$ . This  $C^*$ -algebra is known to be residually finite dimensional, [12], and semiprojective, [4]. As explained in Section 2, it is not the case that the set of finite dimensional traces on a residually finite dimensional  $C^*$ -algebra necessarily is weak\*-dense. In this section, we further review known facts and establish new results on the closure of finite dimensional traces on general (residually finite dimensional) unital  $C^*$ -algebras, and we relate this set to the convex compact sets of quasi-diagonal, amenable, respectively, hyperlinear traces. For the particular  $C^*$ -algebra  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$ , we show in Theorem 2.10 that the closure of the finite dimensional traces is equal to the set of hyperlinear traces. By the aforementioned result, [16, Theorem 3.7], by Haagerup and the first named author, if the finite dimensional tracial states on  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$  are weak\*-dense in the set of all tracial states, then the Connes Embedding Problem has an affirmative answer. Theorem 2.10 implies that the converse also holds, so these two statements are equivalent.

We further show that whenever  $\mathcal{A}$  is a unital  $C^*$ -algebra generated by  $n - 1$  unitaries, then  $M_n(\mathbb{C}) \otimes \mathcal{A}$  is a quotient of  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$ , and therefore generated by two copies of  $M_n(\mathbb{C})$ . For  $n \geq 3$ , the class of  $C^*$ -algebras  $\mathcal{A}$  as above includes all singly generated  $C^*$ -algebras (and hence, for example, all finite dimensional  $C^*$ -algebras). As an interesting application, we show in Theorem 3.9 that the Poulsen simplex is a face of the trace simplex of  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$ , whenever  $n \geq 3$ . We leave open if the trace simplex of  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$  itself is the Poulsen simplex. We recommend Alfsen’s book, [1], as an excellent reference for Choquet theory.

## 2 Finite dimensional traces and their convex structure

Let  $\mathcal{A}$  be a unital  $C^*$ -algebra, and denote by  $T(\mathcal{A})$  the simplex of tracial states on  $\mathcal{A}$ . For each  $\tau \in T(\mathcal{A})$  consider the closed two-sided ideal

$$I_\tau = \{a \in \mathcal{A} : \tau(a^*a) = 0\} \tag{2.1}$$

in  $\mathcal{A}$ . A tracial state  $\tau$  on  $\mathcal{A}$  is said to *factor through* another unital  $C^*$ -algebra  $\mathcal{B}$ , if  $\tau = \tau' \circ \varphi$ , for some unital  $*$ -homomorphism  $\varphi: \mathcal{A} \rightarrow \mathcal{B}$  and some tracial state  $\tau'$  on  $\mathcal{B}$ . If  $\varphi$  is surjective, we say that  $\tau$  *factors surjectively through*  $\mathcal{B}$ . Furthermore,  $\tau$  is said to be *finite dimensional* if it factors through a finite dimensional  $C^*$ -algebra. Equivalently,  $\tau$  is finite dimensional if and only if  $\mathcal{A}/I_\tau$  is finite dimensional. This, again, is equivalent to the enveloping von Neumann algebra  $\pi_\tau(\mathcal{A})''$  of the GNS representation  $\pi_\tau$  arising from  $\tau$  being finite dimensional. (Note that  $\pi_\tau(\mathcal{A}) \cong \mathcal{A}/I_\tau$ .) The set of finite dimensional tracial states on  $\mathcal{A}$  is denoted by  $T_{\text{fin}}(\mathcal{A})$ . Clearly,  $T_{\text{fin}}(\mathcal{A})$  is non-empty precisely when  $\mathcal{A}$  admits at least one finite dimensional representation.

**Proposition 2.1.** *Let  $\mathcal{A}$  be a unital  $C^*$ -algebra, and assume that  $T_{\text{fin}}(\mathcal{A})$  is non-empty. Then:*

- (i)  $T_{\text{fin}}(\mathcal{A})$  is a (convex) face of  $T(\mathcal{A})$ , and its closure  $\overline{T_{\text{fin}}(\mathcal{A})}$  is a closed face of  $T(\mathcal{A})$ .
- (ii)  $T_{\text{fin}}(\mathcal{A}) = \text{conv}(\partial_e T(\mathcal{A}) \cap T_{\text{fin}}(\mathcal{A}))$ , and  $\partial_e T(\mathcal{A}) \cap T_{\text{fin}}(\mathcal{A})$  consists of those tracial states on  $\mathcal{A}$  that factor surjectively through  $M_k(\mathbb{C})$ , for some  $k \geq 1$ . (Here  $\partial_e T(\mathcal{A})$  denotes the set of extreme points of  $T(\mathcal{A})$ .)

*Proof.* (i). Let  $\tau_1, \tau_2$  belong to  $T_{\text{fin}}(\mathcal{A})$ , witnessed by finite dimensional  $C^*$ -algebras  $\mathcal{B}_1$  and  $\mathcal{B}_2$ , unital  $*$ -homomorphisms  $\varphi_j: \mathcal{A} \rightarrow \mathcal{B}_j$ , and tracial states  $\sigma_j$  on  $\mathcal{B}_j$  such that  $\tau_j = \sigma_j \circ \varphi_j$ , for  $j = 1, 2$ . Consider the  $*$ -homomorphism  $\varphi = \varphi_1 \oplus \varphi_2: \mathcal{A} \rightarrow \mathcal{B}_1 \oplus \mathcal{B}_2$ . Fix  $0 < c < 1$  and let  $\sigma$  be the tracial state on  $\mathcal{B}_1 \oplus \mathcal{B}_2$  given by  $\sigma(b_1, b_2) = c\sigma_1(b_1) + (1-c)\sigma_2(b_2)$ , for  $b_1 \in \mathcal{B}_1$  and  $b_2 \in \mathcal{B}_2$ . Then  $c\tau_1 + (1-c)\tau_2 = \sigma \circ \varphi$ , which belongs to  $T_{\text{fin}}(\mathcal{A})$ .

Suppose, conversely, that  $\tau_1, \tau_2$  belong to  $T(\mathcal{A})$ , and that  $c\tau_1 + (1-c)\tau_2$  belongs to  $T_{\text{fin}}(\mathcal{A})$ , for some  $0 < c < 1$ . Then  $\mathcal{A}/I_{c\tau_1 + (1-c)\tau_2}$  is finite dimensional. But  $I_{c\tau_1 + (1-c)\tau_2} = I_{\tau_1} \cap I_{\tau_2}$ , so  $\mathcal{A}/I_{\tau_1}$  and  $\mathcal{A}/I_{\tau_2}$  are both finite dimensional, whence  $\tau_1, \tau_2$  belong to  $T_{\text{fin}}(\mathcal{A})$ .

The last claim follows from the fact that the closure of a face of any compact convex set is, again, a face.

(ii). It is well-known that  $\tau$  is an extreme point in  $T(\mathcal{A})$  if and only if  $\pi_\tau(\mathcal{A})''$  is a factor. If  $\pi_\tau(\mathcal{A})$  is finite dimensional, then this happens if and only if  $\pi_\tau(\mathcal{A})$  is a full matrix algebra, whence  $\tau$  is as desired.

Let  $\tau$  be an arbitrary finite dimensional trace on  $\mathcal{A}$ , and write it as  $\tau = \tau_0 \circ \varphi$ , for some unital  $*$ -homomorphism  $\varphi: \mathcal{A} \rightarrow \mathcal{B}$  onto some finite dimensional  $C^*$ -algebra  $\mathcal{B}$ , and some tracial state  $\tau_0$  on  $\mathcal{B}$ . Write  $\mathcal{B} = \bigoplus_{j=1}^r \mathcal{B}_j$ , where each  $\mathcal{B}_j$  is a full matrix algebra equipped with tracial state  $\text{tr}_{\mathcal{B}_j}$ , and let  $\pi_j: \mathcal{B} \rightarrow \mathcal{B}_j$  be the canonical projection. Then  $\tau = \sum_{j=1}^r c_j \tau_j$ , where  $\tau_j = \text{tr}_{\mathcal{B}_j} \circ \pi_j \circ \varphi$  and  $c_j = \tau(e_j)$ , where  $e_j \in \mathcal{B}$  is the unit of  $\mathcal{B}_j$ .  $\square$

We have the following inclusions of tracial states on any unital  $C^*$ -algebra  $\mathcal{A}$ :

$$T_{\text{fin}}(\mathcal{A}) \subseteq \overline{T_{\text{fin}}(\mathcal{A})} \subseteq T_{\text{qd}}(\mathcal{A}) \subseteq T_{\text{am}}(\mathcal{A}) \subseteq T_{\text{hyp}}(\mathcal{A}) \subseteq T(\mathcal{A}),$$

where  $T_{\text{qd}}(\mathcal{A})$ ,  $T_{\text{am}}(\mathcal{A})$  and  $T_{\text{hyp}}(\mathcal{A})$  are the sets of *quasi-diagonal*, *amenable*, respectively, *hyperlinear* tracial states on  $\mathcal{A}$ . Recall, e.g., from [7], see also the introduction of [27], that a tracial state  $\tau$  on  $\mathcal{A}$  is *hyperlinear* if it factors through an ultrapower  $\mathcal{R}^\omega$  of the hyperfinite  $\text{II}_1$ -factor  $\mathcal{R}$ . Equivalently,  $\tau$  is hyperlinear if  $\pi_\tau(\mathcal{A})''$  embeds in a trace-preserving way into  $\mathcal{R}^\omega$ . If the embedding  $\pi_\tau(\mathcal{A})'' \rightarrow \mathcal{R}^\omega$  moreover can be chosen to admit a u.c.p. lift  $\pi_\tau(\mathcal{A})'' \rightarrow \ell^\infty(\mathcal{R})$ , then  $\tau$  is *amenable* (or *liftable*, in the terminology of Kirchberg, [20]). A tracial state  $\tau$  on an *exact*  $C^*$ -algebra  $\mathcal{A}$  is amenable if and only if  $\pi_\tau(\mathcal{A})''$  is hyperfinite, see [7, Corollary 4.3.6]. A trace  $\tau$  is *quasi-diagonal* if it factors through the  $C^*$ -algebra  $\prod_{n=1}^\infty M_{k_n}(\mathbb{C}) / \bigoplus_{k=1}^\infty M_{k_n}(\mathbb{C})$  with a u.c.p. lift  $\mathcal{A} \rightarrow \prod_{n=1}^\infty M_{k_n}(\mathbb{C})$ , for some sequence of integers  $k_n \geq 1$ .

Each of the three sets  $T_{\text{qd}}(\mathcal{A})$ ,  $T_{\text{am}}(\mathcal{A})$ , and  $T_{\text{hyp}}(\mathcal{A})$  is compact and convex. Kirchberg proved in [20] that, moreover,  $T_{\text{am}}(\mathcal{A})$  is a face of  $T(\mathcal{A})$ . The Connes Embedding Problem is equivalent to  $T_{\text{hyp}}(\mathcal{A})$  being equal to  $T(\mathcal{A})$ , for all  $C^*$ -algebras  $\mathcal{A}$ . It is not known if  $T_{\text{qd}}(\mathcal{A}) = T_{\text{am}}(\mathcal{A})$  in general, but in recent remarkable works, e.g., [29], [14] and [27], it has been resolved that amenable traces are quasi-diagonal in many important cases of interest. The inclusions  $\overline{T_{\text{fin}}(\mathcal{A})} \subseteq T_{\text{qd}}(\mathcal{A})$  and  $T_{\text{am}}(\mathcal{A}) \subseteq T_{\text{hyp}}(\mathcal{A})$  are in general strict, even for residually finite dimensional  $C^*$ -algebras  $\mathcal{A}$ , cf. Propositions 2.3 and Example 2.8 (as well as the remarks above Example 2.8). In particular,  $T_{\text{fin}}(\mathcal{A})$  need not be weak\*-dense in  $T(\mathcal{A})$ , for residually finite dimensional  $C^*$ -algebras  $\mathcal{A}$ .

Recall that a  $C^*$ -algebra  $\mathcal{A}$  is *residually finite dimensional* if it admits a separating family of finite dimensional representations  $\varphi_i: \mathcal{A} \rightarrow M_{k(i)}(\mathbb{C})$ ,  $i \in I$ . The index set  $I$  can be taken to be countable if  $\mathcal{A}$  is separable. Equivalently,  $\mathcal{A}$  is residually finite dimensional if and only if the set of finite dimensional traces on  $\mathcal{A}$  is *separating*, in the sense that  $\bigcap_{\tau \in T_{\text{fin}}(\mathcal{A})} I_\tau = \{0\}$ . We can sharpen this statement, as follows:

**Proposition 2.2.** *A unital  $C^*$ -algebra  $\mathcal{A}$  is residually finite dimensional if and only if  $\overline{T_{\text{fin}}(\mathcal{A})}$  is separating. If  $\mathcal{A}$  is separable, then  $\overline{T_{\text{fin}}(\mathcal{A})}$  is separating if and only if it contains a faithful trace.*

*Proof.* The first part of the statement follows from the remark above, and the fact that  $\bigcap_{\tau \in \mathcal{T}} I_\tau = \bigcap_{\tau \in \overline{\mathcal{T}}} I_\tau$ , for every subset  $\mathcal{T}$  of  $T(\mathcal{A})$ .

Assume now that  $\mathcal{A}$  is separable and that  $\overline{T_{\text{fin}}(\mathcal{A})}$  is separating. Observe first that for each positive element  $a \in \mathcal{A}$ , there is  $\tau \in \overline{T_{\text{fin}}(\mathcal{A})}$  such that  $\|a + I_\tau\| \geq \|a\|/2$ . To see this we may assume that  $\|a\| = 1$ . Let  $g: [0, 1] \rightarrow [0, 1]$  be a continuous function which is zero on  $[0, 1/2]$  and with  $g(1) = 1$ . Then  $g(a)$  is positive and non-zero, so there is  $\tau \in \overline{T_{\text{fin}}(\mathcal{A})}$  such that  $\tau(g(a)) > 0$ . It follows that  $g(a + I_\tau) = g(a) + I_\tau \neq 0$ . This entails that  $\|a + I_\tau\| > 1/2$ .

Let  $\{a_n\}_{n \geq 1}$  be a dense set of positive contractions in  $\mathcal{A}$ . For each  $n \geq 1$ , find  $\tau_n \in \overline{T_{\text{fin}}(\mathcal{A})}$  such that  $\|a_n + I_{\tau_n}\| \geq \|a_n\|/2$ . Set  $\tau = \sum_{n=1}^{\infty} 2^{-n} \tau_n$ . Since  $\overline{T_{\text{fin}}(\mathcal{A})}$  is convex and weak\*-closed (and hence norm-closed), it follows that  $\tau$  belongs to  $\overline{T_{\text{fin}}(\mathcal{A})}$ . Moreover,  $I_\tau = \bigcap_{n=1}^{\infty} I_{\tau_n}$ , whence  $\|a_n + I_\tau\| \geq \|a_n + I_{\tau_n}\| \geq \|a_n\|/2$ , for all  $n \geq 1$ . This implies that  $\|a + I_\tau\| \geq \|a\|/2$ , for all positive contractions  $a \in \mathcal{A}$ , from which we see that  $\tau$  is faithful.  $\square$

Kirchberg proved in [20] that  $\overline{T_{\text{fin}}(C^*(G))} = T_{\text{am}}(C^*(G))$ , for all discrete groups  $G$  with Kazhdan's property (T); he also proved therein that a property (T) group is residually finite if and only if it possesses the *factorization property*. Hence, by Proposition 2.2, a countably infinite property (T) group  $G$  has residually finite dimensional group  $C^*$ -algebra  $C^*(G)$  if and only if  $C^*(G)$  has a faithful amenable trace. As it turns out, there are no known examples of infinite property (T) groups where  $C^*(G)$  is residually finite dimensional, or even quasi-diagonal, or where  $C^*(G)$  has a faithful tracial state (amenable or not). Bekka proved in [3] that  $C^*(G)$  has no faithful tracial state, and hence is not residually finite dimensional, when  $G = \text{SL}(n, \mathbb{Z})$ , for  $n \geq 3$ .

**Proposition 2.3** (N. Brown, [7, Corollary 4.3.8]). *There exists an exact unital residually finite dimensional  $C^*$ -algebra  $\mathcal{A}$  for which  $T_{\text{am}}(\mathcal{A}) \neq T_{\text{hyp}}(\mathcal{A})$ .*

In more detail, Brown showed in [6] that there is a separable unital exact residually finite dimensional  $C^*$ -algebra  $\mathcal{A}$  that surjects onto  $C_\lambda^*(\mathbb{F}_2)$ . The (unique) tracial state  $\tau$  on  $\mathcal{A}$  that factors through  $C_\lambda^*(\mathbb{F}_2)$  is not amenable, because  $\pi_\tau(\mathcal{A})''$  (which is equal to the group von Neumann algebra  $\mathcal{L}(\mathbb{F}_2)$ ) is not hyperfinite, and  $\mathcal{A}$  is exact. It is hyperfinite because  $\mathcal{L}(\mathbb{F}_2)$  admits an embedding into  $\mathcal{R}^\omega$ , as observed by Connes in [9].

**Remark 2.4.** We note that the convex set of finite dimensional tracial states on a unital  $C^*$ -algebra is almost never closed. More precisely, if  $\mathcal{A}$  is a unital residually finite dimensional  $C^*$ -algebra, then  $T_{\text{fin}}(\mathcal{A})$  is closed if and only if  $\mathcal{A}$  is finite dimensional. Indeed, if  $\mathcal{A}$  is infinite dimensional, then it admits a sequence  $\{\pi_n\}_{n \geq 1}$  of pairwise inequivalent finite dimensional irreducible representations. For  $n \geq 1$ , set  $\tau_n = \text{tr}_{k(n)} \circ \pi_n$ , where  $k(n)$  is the dimension of the representation  $\pi_n$ . Then  $\{\tau_n\}_{n \geq 1}$  is a sequence of distinct extreme points of  $T_{\text{fin}}(\mathcal{A})$ , and  $\tau := \sum_{n=1}^{\infty} 2^{-n} \tau_n$  belongs to the closure of  $T_{\text{fin}}(\mathcal{A})$ , but not to  $T_{\text{fin}}(\mathcal{A})$  itself.

A separable  $C^*$ -algebra is residually finite dimensional if and only if it embeds into a  $C^*$ -algebra of the form  $\mathcal{M} = \prod_{n=1}^{\infty} M_{k(n)}(\mathbb{C})$ , for some sequence  $\{k(n)\}_{n \geq 1}$  of positive integers. The (non-separable)  $C^*$ -algebra  $\mathcal{M}$  is itself residually finite dimensional. The following result is contained in Ozawa, [24, Theorem 8], but also follows from [30], as explained below:

**Proposition 2.5.** *The set  $T_{\text{fin}}(\mathcal{M})$  is weak\*-dense in  $T(\mathcal{M})$ , when  $\mathcal{M} = \prod_{n=1}^{\infty} M_{k(n)}(\mathbb{C})$ .*

*Proof.* Since  $\mathcal{M}$  is an AW\*-algebra (in fact, a finite von Neumann algebra), we can use [30] to see that any tracial state on  $\mathcal{M}$  factors through the center  $\mathcal{Z}(\mathcal{M})$  of  $\mathcal{M}$ , via the (unique) center-valued trace. The center  $\mathcal{Z}(\mathcal{M})$  can be identified with  $C(\beta\mathbb{N})$ , the continuous functions on the Stone-Ćech compactification of  $\mathbb{N}$ . Hence, tracial states on  $\mathcal{M}$  are in one-to-one correspondence with probability measures on  $\beta\mathbb{N}$ . Furthermore, finite dimensional traces correspond to convex combinations of Dirac measures at points of  $\mathbb{N}$ . Since  $\mathbb{N}$  is dense in  $\beta\mathbb{N}$ , and since the convex hull of Dirac measures (at points of  $\beta\mathbb{N}$ ) is dense in the set of all probability measures on  $\beta\mathbb{N}$ , we reach the desired conclusion.  $\square$

Consider a unital embedding  $\varphi$  of a unital residually finite dimensional  $C^*$ -algebra  $\mathcal{A}$  into  $\mathcal{M} := \prod_{n=1}^{\infty} M_{k(n)}(\mathbb{C})$ , for some sequence of positive integers  $k(n) \geq 1$ . Let  $\pi_n$  denote the  $n$ th coordinate map  $\mathcal{M} \rightarrow M_{k(n)}(\mathbb{C})$ . We say that the inclusion  $\varphi$  is *saturated* if

$$\left( \bigoplus_{n=1}^N \pi_n \right) (\mathcal{A}) = \bigoplus_{n=1}^N M_{k(n)}(\mathbb{C}), \quad (2.2)$$

for all  $N \geq 1$ , and  $\{\tau_n : n \geq 1\}$  is weak\*-dense in  $\partial_e T(\mathcal{A}) \cap T_{\text{fin}}(\mathcal{A})$ , where  $\tau_n = \text{tr}_{k(n)} \circ \pi_n \circ \varphi$ .

Each separable residually finite dimensional unital  $C^*$ -algebra admits a saturated embedding into some  $\prod_{n=1}^{\infty} M_{k(n)}(\mathbb{C})$ . Indeed, if  $\mathcal{A}$  is separable, then  $T(\mathcal{A})$  is separable, too, and hence so is  $\partial_e T(\mathcal{A}) \cap T_{\text{fin}}(\mathcal{A})$ . Pick a countable dense subset  $\{\tau_n : n \geq 1\}$  of this set. Then  $\tau_n = \text{tr}_{k(n)} \circ \varphi_n$ , for some surjective \*-homomorphism  $\varphi_n : \mathcal{A} \rightarrow M_{k(n)}(\mathbb{C})$ , cf. Proposition 2.1, and  $\varphi := \bigoplus_{n \geq 1} \varphi_n$  is a saturated embedding of  $\mathcal{A}$ . Injectivity of  $\varphi$  follows from  $\ker(\varphi) = \bigcap_{n=1}^{\infty} I_{\tau_n} = \bigcap_{\tau \in T_{\text{fin}}(\mathcal{A})} I_{\tau} = \{0\}$ , where the second equality follows from density of  $\{\tau_n : n \geq 1\}$  in  $\partial_e T(\mathcal{A}) \cap T_{\text{fin}}(\mathcal{A})$  and Proposition 2.1.

**Proposition 2.6.** *Let  $\mathcal{A}$  be a unital  $C^*$ -algebra.*

- (i) *If  $\tau \in T(\mathcal{A})$  factors through  $\prod_{n=1}^{\infty} M_{k(n)}(\mathbb{C})$ , for some sequence of integers  $k(n) \geq 1$ , then  $\tau \in \overline{T_{\text{fin}}(\mathcal{A})}$ .*
- (ii) *If  $\mathcal{A}$  is a separable residually finite dimensional  $C^*$ -algebra with saturated embedding  $\varphi : \mathcal{A} \rightarrow \prod_{n=1}^{\infty} M_{k(n)}(\mathbb{C}) := \mathcal{M}$ , then  $\overline{T_{\text{fin}}(\mathcal{A})}$  consists precisely of those traces  $\tau$  on  $\mathcal{A}$  that extend to a trace on  $\mathcal{M}$  (in the sense that  $\tau = \tau' \circ \varphi$ , for some  $\tau' \in T(\mathcal{M})$ ).*

*Proof.* Part (i) follows from Proposition 2.5 and the fact that if  $\tau'$  belongs to  $T_{\text{fin}}(\mathcal{M})$ , then  $\tau' \circ \varphi$  belongs to  $T_{\text{fin}}(\mathcal{A})$ .

(ii). Denote, as above, the  $n$ th coordinate map  $\mathcal{M} \rightarrow M_{k(n)}(\mathbb{C})$  by  $\pi_n$ . Each of the tracial states  $\text{tr}_{k(n)} \circ \pi_n \circ \varphi$  extends to the tracial state  $\text{tr}_{k(n)} \circ \pi_n$  on  $\mathcal{M}$ . By assumption,  $\{\text{tr}_{k(n)} \circ \pi_n \circ \varphi : n \geq 1\}$  is dense in  $\partial_e T(\mathcal{A}) \cap T_{\text{fin}}(\mathcal{A})$ . The set of tracial states on  $\mathcal{A}$  that extend to a tracial state on  $\mathcal{M}$  is closed and convex. (Indeed, it is equal to the image of the continuous affine homomorphism  $T(\mathcal{M}) \rightarrow T(\mathcal{A})$  induced by the embedding  $\varphi : \mathcal{A} \rightarrow \mathcal{M}$ .) We may therefore conclude from Proposition 2.1 (ii) that each tracial state in the closure of  $T_{\text{fin}}(\mathcal{A})$  extends to a tracial state on  $\mathcal{M}$ . The other inclusion follows from (i).  $\square$

Proposition 2.6 raises the question when a tracial state on a unital sub- $C^*$ -algebra  $\mathcal{B}$  of a unital  $C^*$ -algebra  $\mathcal{A}$  can be extended to  $\mathcal{A}$ . This is well-understood when  $\mathcal{B} \subseteq \mathcal{A}$  are von Neumann algebras: each tracial state on  $\mathcal{B}$  extends to a tracial state on  $\mathcal{A}$  if and only if finite central projections in  $\mathcal{A}$  separate finite central projections in  $\mathcal{B}$ , i.e., if  $p, p'$  are distinct finite central projections in  $\mathcal{B}$ , then there exists a finite central projection  $q$  in  $\mathcal{A}$  such that  $pq$  is zero and  $p'q$  is non-zero, or vice versa. The corresponding question for  $C^*$ -algebras is more subtle: Take  $\mathcal{B}$  to be a unital  $C^*$ -algebra with a faithful extremal tracial state  $\tau$ . Then  $\mathcal{B} \subseteq \pi_\tau(\mathcal{B})''$ , and  $\pi_\tau(\mathcal{B})''$  is a type II<sub>1</sub>-factor. Hence  $\tau$  is the only tracial state on  $\mathcal{B}$  that extends to a tracial state on  $\pi_\tau(\mathcal{B})''$ , while  $\mathcal{B}$  need not have unique trace (even for simple  $C^*$ -algebras  $\mathcal{B}$ ). We pursue this issue in Example 2.8 below.

As it was remarked in [11, Section 2.1] (see also relevant definitions therein), a matricially weakly semiprojective  $C^*$ -algebra is residually finite dimensional if and only if it is quasi-diagonal if and only if it is MF (defined in [5]). Every weakly semiprojective  $C^*$ -algebra is also matricially weakly semiprojective.

**Proposition 2.7.** *Let  $\mathcal{A}$  be a unital (matricially) weakly semiprojective  $C^*$ -algebra. Then  $\overline{T_{\text{fin}}(\mathcal{A})} = T_{\text{qd}}(\mathcal{A})$ .*

*Proof.* If  $\tau \in T_{\text{qd}}(\mathcal{A})$ , then  $\tau$  factors through  $\mathcal{M} := \prod_{n=1}^{\infty} M_{k(n)}(\mathbb{C}) / \bigoplus_{n=1}^{\infty} M_{k(n)}(\mathbb{C})$  via a  $*$ -homomorphism  $\varphi$  and a tracial state  $\tau_0$  on  $\mathcal{M}$ , for some sequence of positive integers  $k(n)$ . Since  $\mathcal{A}$  is matricially weakly semiprojective,  $\varphi$  lifts to a  $*$ -homomorphism  $\psi$ :

$$\begin{array}{ccccc} & & \prod_{n=1}^{\infty} M_{k(n)}(\mathbb{C}) & & \\ & \nearrow \psi & \downarrow \pi & & \\ \mathcal{A} & \xrightarrow{\varphi} & \prod_{n=1}^{\infty} M_{k(n)}(\mathbb{C}) / \bigoplus_{n=1}^{\infty} M_{k(n)}(\mathbb{C}) & \xrightarrow{\tau_0} & \mathbb{C}. \end{array}$$

Hence  $\tau$  factors through  $\prod_{n=1}^{\infty} M_{k(n)}(\mathbb{C})$ , so it belongs to  $\overline{T_{\text{fin}}(\mathcal{A})}$  by Proposition 2.6 (i).  $\square$

There exists unital residually finite dimensional  $C^*$ -algebras  $\mathcal{A}$  for which  $\overline{T_{\text{fin}}(\mathcal{A})} \neq T_{\text{qd}}(\mathcal{A})$ , see [17, Example 3.11] by Hadwin and Shulman. As an application of Proposition 2.6, we exhibit here a larger class of  $C^*$ -algebras with these properties.

**Example 2.8.** Take a unital MF-algebra  $\mathcal{B}$  witnessed by an embedding  $\varphi$  as in the diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{n \geq 1} M_{k(n)}(\mathbb{C}) & \longrightarrow & \mathcal{A} & \xrightarrow{\rho = \varphi^{-1} \circ \pi} & \mathcal{B} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \varphi \\ 0 & \longrightarrow & \bigoplus_{n \geq 1} M_{k(n)}(\mathbb{C}) & \longrightarrow & \prod_{n \geq 1} M_{k(n)}(\mathbb{C}) & \xrightarrow{\pi} & \frac{\prod_{n \geq 1} M_{k(n)}(\mathbb{C})}{\bigoplus_{n \geq 1} M_{k(n)}(\mathbb{C})} \longrightarrow 0. \end{array}$$

The pull-back  $C^*$ -algebra  $\mathcal{A} := \pi^{-1}(\varphi(\mathcal{B}))$  is then unital and residually finite dimensional. If  $\mathcal{B}$  has no finite dimensional representations, then the inclusion of  $\mathcal{A}$  into  $\prod_{n=1}^{\infty} M_{k(n)}(\mathbb{C})$  is saturated. Each tracial state on  $\mathcal{B}$  gives rise to a tracial state on  $\mathcal{A}$  by composition with  $\rho$ . The resulting tracial state on  $\mathcal{A}$  will not always extend to a tracial state on  $\prod_{n=1}^{\infty} M_{k(n)}(\mathbb{C})$ , as shown below.

Take now  $\mathcal{B}$  to be a unital MF-algebra with no finite dimensional representations, and which admits a faithful quasi-diagonal tracial state  $\tau$ . Then there is a sequence  $\{k(n)\}_{n \geq 1}$  of positive integers and u.c.p. maps  $\mu_n: \mathcal{B} \rightarrow M_{k(n)}(\mathbb{C})$ ,  $n \geq 1$ , such that for all  $a, b \in \mathcal{B}$ ,

$$\lim_{n \rightarrow \infty} \|\mu_n(ab) - \mu_n(a)\mu_n(b)\| = 0, \quad \lim_{n \rightarrow \infty} \|\mu_n(a)\| = \|a\|, \quad \lim_{n \rightarrow \infty} \text{tr}_{k(n)}(\mu_n(a)) = \tau(a).$$

Hence  $\varphi := \pi \circ \bigoplus_{n \geq 1} \mu_n$  is an injective  $*$ -homomorphism as in the diagram above. Let also  $\mathcal{A}$  be as above. If  $\tau'$  is a tracial state on  $\mathcal{B}$ , then  $\tau' \circ \rho$  extends to a trace on  $\prod_{n \geq 1} M_{k(n)}(\mathbb{C})$  if and only if  $\tau = \tau'$ . Indeed, if  $\sigma$  is a tracial state on  $\prod_{n \geq 1} M_{k(n)}(\mathbb{C})$  that extends  $\tau' \circ \rho$ , then, e.g., by Proposition 2.5 and its proof,  $\sigma(\{x_n\}) = \omega(\{\text{tr}_{k(n)}(x_n)\})$ , for all  $\{x_n\}_{n \geq 1} \in \prod_{n \geq 1} M_{k(n)}(\mathbb{C})$ , for some state  $\omega$  on  $\ell^\infty(\mathbb{N})$ , which vanishes on  $c_0(\mathbb{N})$ , since  $\tau' \circ \rho$ , and therefore also  $\sigma$ , vanish on  $\bigoplus_{n \geq 1} M_{k(n)}(\mathbb{C})$ . It follows that

$$\tau'(b) = (\tau' \circ \rho)(\{\mu_n(b)\}) = \sigma(\{\mu_n(b)\}) = \omega(\{\text{tr}_{k(n)}(\mu_n(b))\}) = \tau(b),$$

for all  $b \in \mathcal{B}$ .

We conclude that if  $\tau' \in T(\mathcal{B})$  and  $\tau' \neq \tau$ , then  $\tau' \circ \rho$  does not extend to  $\prod_{n \geq 1} M_{k(n)}(\mathbb{C})$ , whence  $\tau' \circ \rho$  does not belong to  $\overline{T_{\text{fin}}(\mathcal{A})}$ , cf. Proposition 2.6, while  $\tau' \circ \rho$  does belong to  $T_{\text{qd}}(\mathcal{A})$  whenever  $\mathcal{B}$ , moreover, is quasi-diagonal and nuclear, and  $\tau'$  is faithful.

We now turn our interest to the particular example of the unital universal free product  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$  of two copies of the full matrix algebra  $M_n(\mathbb{C})$ . It was shown in [12] that  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$  is residually finite dimensional, while Blackadar proved that it is semiprojective, see [4, Corollary 2.28 and Proposition 2.31].

**Lemma 2.9.** *Let  $n \geq 1$  be an integer and let  $\pi: \mathcal{A} \rightarrow \mathcal{B}$  be a surjective  $*$ -homomorphism between unital  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ . Suppose, furthermore, that the following conditions hold:*

- (a) *the unitary group of  $\mathcal{B}$  is connected;*
- (b) *whenever  $p, q \in \mathcal{B}$  are projections such that the  $n$ -fold direct sum of  $p$  is equivalent to the  $n$ -fold direct sum of  $q$ , then  $p \sim q$ ;*
- (c) *there is a unital embedding of  $M_n(\mathbb{C})$  into  $\mathcal{A}$ .*

*Then any unital  $*$ -homomorphism  $M_n(\mathbb{C}) \rightarrow \mathcal{B}$  lifts to a unital  $*$ -homomorphism  $M_n(\mathbb{C}) \rightarrow \mathcal{A}$ , and any unital  $*$ -homomorphism  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C}) \rightarrow \mathcal{B}$  lifts to a unital  $*$ -homomorphism  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C}) \rightarrow \mathcal{A}$ .*

*Proof.* Fix a unital  $*$ -homomorphism  $\beta: M_n(\mathbb{C}) \rightarrow \mathcal{B}$ , and pick any unital  $*$ -homomorphism  $\alpha': M_n(\mathbb{C}) \rightarrow \mathcal{A}$ , cf. (c). Set  $\beta' = \pi \circ \alpha'$ . It follows from assumption (b) that  $\beta(e_{11}) \sim \beta'(e_{11})$ , where  $e_{ij}$ ,  $1 \leq i, j \leq n$ , are the standard matrix units for  $M_n(\mathbb{C})$ . It is a well-known fact, see, e.g., [26, Lemma 7.3.2(ii)], that  $\beta$  and  $\beta'$  are unitarily equivalent, i.e., there is a unitary  $u \in \mathcal{B}$  such that  $u\beta'(a)u^* = \beta(a)$ , for all  $a \in \mathcal{A}$ . By (a),  $u$  lifts to a unitary  $v \in \mathcal{A}$ . It follows that  $\alpha: \mathcal{A} \rightarrow M_n(\mathbb{C})$  given by  $\alpha(a) = v\alpha'(a)v^*$ ,  $a \in \mathcal{A}$ , is a lift of  $\beta$ .

By the universal property of free products, there is a bijective correspondence between unital  $*$ -homomorphisms from  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$  into a given unital  $C^*$ -algebra and pairs of unital  $*$ -homomorphisms from  $M_n(\mathbb{C})$  into the same unital  $C^*$ -algebra. The second statement about  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$  follows therefore from the first one.  $\square$

**Theorem 2.10.** *The closure of  $T_{\text{fin}}(M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C}))$  is equal to  $T_{\text{hyp}}(M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C}))$ .*

*Proof.* Let  $\tau \in T_{\text{hyp}}(M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C}))$ . By the definition of hyperlinear traces, there is a unital embedding  $\varphi: M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C}) \rightarrow \mathcal{R}^\omega$  such that  $\tau = \tau_{\mathcal{R}^\omega} \circ \varphi$ . Let  $\mathcal{Q}$  denote the universal UHF algebra, and view it as a dense subalgebra of the hyperfinite  $\text{II}_1$ -factor  $\mathcal{R}$ , with respect to  $\|\cdot\|_2$ . Composing the inclusion  $\prod_{k=1}^\infty \mathcal{Q} \rightarrow \prod_{k=1}^\infty \mathcal{R}$  with the natural surjection from  $\prod_{k=1}^\infty \mathcal{R}$  onto  $\mathcal{R}^\omega$  yields the  $*$ -homomorphism  $\pi$  in the following diagram:

$$\begin{array}{ccc} & & \prod_{k=1}^\infty \mathcal{Q} \\ & \nearrow \psi & \downarrow \pi \\ M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C}) & \xrightarrow{\varphi} & \mathcal{R}^\omega. \end{array}$$

Since  $\mathcal{Q}$  is  $\|\cdot\|_2$ -dense in  $\mathcal{R}$ , we see that  $\pi$  is surjective. The  $*$ -homomorphism  $\varphi$  lifts to a  $*$ -homomorphism  $\psi = \bigoplus_{k=1}^\infty \psi_k$ , by Lemma 2.9. Moreover,  $(\tau_{\mathcal{R}^\omega} \circ \pi)(\{b_k\}_{k \geq 1}) = \lim_{k \rightarrow \omega} \tau_{\mathcal{Q}}(b_k)$ , for all  $\{b_k\}_{k \geq 1} \in \prod_{k=1}^\infty \mathcal{Q}$ . It follows that

$$\tau = \tau_{\mathcal{R}^\omega} \circ \varphi = \tau_{\mathcal{R}^\omega} \circ \pi \circ \psi = \lim_{k \rightarrow \omega} \tau_{\mathcal{Q}} \circ \psi_k.$$

This shows that  $\tau$  is the limit of a net of tracial states that factor through the universal UHF-algebra  $\mathcal{Q}$ . As every tracial state that factors through  $\mathcal{Q}$  is quasi-diagonal, and the set of quasi-diagonal traces is closed, we conclude that  $\tau$  is quasi-diagonal. By Proposition 2.7, this completes the proof.  $\square$

The theorem above implies that the set of finite dimensional tracial states on  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$  is weak $*$ -dense if the Connes Embedding Problem has an affirmative answer.

### 3 Factorizable channels and the Connes Embedding Problem: a new viewpoint

We give in this section a new characterization of factorizable channels in terms of certain properties of their Choi matrix, that bears resemblance with the matrices of quantum correlations that appear in Tsirelson's conjecture. From this viewpoint, we then establish a new link to the Connes Embedding Problem.

Keeping consistent notation with previous papers on this topic, let  $T: M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  be a linear map. One associates to it its Choi matrix

$$C_T = \sum_{i,j=1}^n e_{ij} \otimes T(e_{ij}) \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}),$$

where  $e_{ij}$ ,  $1 \leq i, j \leq n$ , are, as before, the standard matrix units for  $M_n(\mathbb{C})$ . Choi's celebrated theorem, [8], states that  $T$  is completely positive if and only if  $C_T$  is a positive matrix. Furthermore, we can recover  $T$  from the matrix  $C_T$  by the formula

$$T(e_{ij}) = \sum_{k,\ell=1}^n C_T(i, j; k, \ell) e_{k\ell}, \quad 1 \leq i, j \leq n, \quad (3.1)$$

where  $C_T(i, j; k, \ell)$  are the matrix coefficients of  $C_T$ , cf. (3.2) below, which we briefly justify: Equip the vector space  $M_n(\mathbb{C})$  with the inner product  $\langle \cdot, \cdot \rangle_{\text{Tr}_n}$  coming from the standard



trace  $\text{Tr}_n$  on  $M_n(\mathbb{C})$ . (We reserve the notation  $\text{tr}_n$  for the normalized trace on  $M_n(\mathbb{C})$ .) The set of standard matrix units  $\{e_{ij}\}$  is then an orthonormal basis for  $M_n(\mathbb{C})$ , and

$$C_T(i, j; k, \ell) = \langle T(e_{ij}), e_{k\ell} \rangle_{\text{Tr}_n} = \langle C_T, e_{ij} \otimes e_{k\ell} \rangle_{\text{Tr}_n \otimes \text{Tr}_n}. \quad (3.2)$$

A unital completely positive trace-preserving map  $T: M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  is *factorizable*, cf. [2], if there exist a finite von Neumann algebra  $M$  with normal faithful tracial state  $\tau_M$  and unital  $*$ -homomorphisms  $\alpha, \beta: M_n(\mathbb{C}) \rightarrow M$  such that  $T = \beta^* \circ \alpha$ , where  $\beta^*: M \rightarrow M_n(\mathbb{C})$  is the adjoint of  $\beta$ . The map  $\beta^*$  is formally defined by the identity  $\langle \beta(x), y \rangle_{\tau_M} = \langle x, \beta^*(y) \rangle_{\text{tr}_n}$ , for  $x \in M_n(\mathbb{C})$  and  $y \in M$ , and it is obtained by composing the (unique) trace-preserving conditional expectation  $E: M \rightarrow \beta(M_n(\mathbb{C}))$  with  $\beta^{-1}$ , see [15]. In this case,  $T$  is said to *exactly factor through*  $(M, \tau_M)$ .

As explained in [15], if  $T$  factors through  $(M, \tau_M)$ , then we may write  $M = M_n(\mathbb{C}) \otimes N$ , for some ancillary finite von Neumann algebra  $(N, \tau_N)$ , and we may take  $\beta$  to be given by  $\beta(x) = x \otimes 1_N$ , for  $x \in M_n(\mathbb{C})$ . In this case,  $\alpha(x) = u(x \otimes 1_N)u^*$ , for some unitary  $u \in M_n(\mathbb{C}) \otimes N$ , and  $T(x) = (\text{id}_n \otimes \tau_N)(u(x \otimes 1_N)u^*)$ , for  $x \in M_n(\mathbb{C})$ . This gives a more transparent definition of  $T$  being factorizable. The finite von Neumann algebra  $N$  above is called the *ancilla*. The reader should be warned that the ancilla is far from being unique, and determining the “minimal” ancilla for a given factorizable map  $T$  seems to be a difficult task.

We shall now rephrase the notion of factorizability of a linear map  $T: M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  in terms of a certain property of its associated Choi matrix.

**Proposition 3.1.** *Let  $n \geq 2$  be an integer, and let  $T: M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  be a linear map with Choi matrix  $C_T = [C_T(i, j; k, \ell)]_{(i,k),(j,\ell)}$  as above. Then the following are equivalent:*

- (i)  $T$  is factorizable.
- (ii) There is a von Neumann algebra  $M$  with normal faithful tracial state  $\tau_M$ , a unital  $*$ -homomorphism  $\alpha: M_n(\mathbb{C}) \rightarrow M$ , and a set of matrix units  $\{f_{ij}\}_{i,j=1}^n$  in  $M$ , so that

$$T(x) = \sum_{i,j=1}^n n \langle \alpha(x), f_{ij} \rangle_{\tau_M} e_{ij}, \quad x \in M_n(\mathbb{C}).$$

- (iii) There is a von Neumann algebra  $M$  with normal faithful tracial state  $\tau_M$  and sets of matrix units  $\{f_{ij}\}_{i,j=1}^n$  and  $\{g_{ij}\}_{i,j=1}^n$  in  $M$ , so that

$$n^{-1}C_T(i, j; k, \ell) = \tau_M(f_{k\ell}^* g_{ij}), \quad 1 \leq i, j, k, \ell \leq n.$$

- (iv) There is a tracial state  $\tau$  on the unital free product  $C^*$ -algebra  $M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C})$ , so that

$$n^{-1}C_T(i, j; k, \ell) = \tau(\iota_2(e_{k\ell})^* \iota_1(e_{ij})), \quad 1 \leq i, j, k, \ell \leq n, \quad (3.3)$$

where  $\iota_1$  and  $\iota_2$  are the two canonical inclusions of  $M_n(\mathbb{C})$  into  $M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C})$ .

*Proof.* (i)  $\Leftrightarrow$  (ii). Suppose that  $T$  is factors through a finite von Neumann algebra  $M$  equipped with normal faithful tracial state  $\tau_M$  via unital  $*$ -homomorphisms  $\alpha, \beta: M_n(\mathbb{C}) \rightarrow M$ . Let  $E: M \rightarrow \beta(M_n(\mathbb{C}))$  be the trace-preserving conditional expectation. Set  $f_{ij} = \beta(e_{ij})$ , for

$1 \leq i, j \leq n$ . Then  $\{\sqrt{n} f_{ij}\}$  is an orthonormal basis for  $\beta(M_n(\mathbb{C}))$  with respect to the inner product  $\langle \cdot, \cdot \rangle_{\tau_M}$  on  $M$ , induced by  $\tau_M$ . It follows that

$$E(x) = \sum_{i,j=1}^n n \langle x, f_{ij} \rangle_{\tau_M} f_{ij}, \quad x \in M.$$

This proves (ii), since  $T = \beta^{-1} \circ E \circ \alpha$ .

For the converse direction, let  $\beta: M_n(\mathbb{C}) \rightarrow M$  be given by  $\beta(e_{ij}) = f_{ij}$ ,  $1 \leq i, j \leq n$ . By reversing the argument above, one can verify that  $T = \beta^{-1} \circ E \circ \alpha = \beta^* \circ \alpha$ .

(ii)  $\Leftrightarrow$  (iii). Let  $M$ ,  $\tau_M$  and  $\alpha$  be as in (ii). For  $1 \leq i, j \leq n$ , set  $g_{ij} = \alpha(e_{ij})$ . Then

$$\begin{aligned} C_T(i, j; k, \ell) &= \langle T(e_{ij}), e_{k\ell} \rangle_{\text{Tr}_n} = n \sum_{s,t=1}^n \langle \alpha(e_{ij}), f_{st} \rangle_{\tau_M} \cdot \langle e_{st}, e_{k\ell} \rangle_{\text{Tr}_n} \\ &= n \langle g_{ij}, f_{k\ell} \rangle_{\tau_M} = n \tau_M(f_{k\ell}^* g_{ij}), \quad 1 \leq i, j, k, \ell \leq n. \end{aligned}$$

Conversely, if (iii) holds, then define  $\alpha: M_n(\mathbb{C}) \rightarrow M$  by  $\alpha(e_{ij}) = g_{ij}$ ,  $1 \leq i, j \leq n$ . Then

$$T(e_{ij}) = \sum_{k,\ell=1}^n C_T(i, j; k, \ell) e_{k,\ell} = n \sum_{k,\ell=1}^n \langle g_{ij}, f_{k\ell} \rangle_{\tau_M} e_{k\ell} = n \sum_{k,\ell=1}^n \langle \alpha(e_{ij}), f_{k\ell} \rangle_{\tau_M} e_{k\ell}.$$

(iii)  $\Leftrightarrow$  (iv). Assuming that (iii) holds, let  $\pi: M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C}) \rightarrow M$  be the \*-homomorphism satisfying  $\pi(\iota_1(e_{ij})) = g_{ij}$  and  $\pi(\iota_2(e_{ij})) = f_{ij}$ ,  $1 \leq i, j \leq n$ . Set  $\tau = \tau_M \circ \pi$ . Then  $\tau$  is a tracial state on  $M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C})$ , satisfying  $\tau(\iota_2(e_{k\ell})^* \iota_1(e_{ij})) = \tau_M(f_{k\ell}^* g_{ij})$ ,  $1 \leq i, j, k, \ell \leq n$ .

Conversely, if (iv) holds with respect to some tracial state  $\tau$  on  $M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C})$ , let  $M$  be the finite von Neumann algebra  $\pi_{\tau}(M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C}))''$ , equipped with the extension  $\tau_M$  of  $\tau$  to  $M$ . Then (iii) holds with  $g_{ij} = \pi_{\tau}(\iota_1(e_{ij}))$  and  $f_{ij} = \pi_{\tau}(\iota_2(e_{ij}))$ , for  $1 \leq i, j \leq n$ .  $\square$

Note that by (iii) one can identify the set  $\mathcal{FM}(n)$  of factorizable maps on  $M_n(\mathbb{C})$ , via their Choi matrix, with the set consisting of complex correlation matrices  $[\tau(f_{k\ell}^* g_{ij})]_{i,j,k,\ell}$ , where  $\{f_{k\ell}\}_{k,\ell}$  and  $\{g_{ij}\}_{i,j}$  are systems  $n \times n$  matrix units in some von Neumann algebra  $(M, \tau)$ . This latter set bears resemblance to the set of matrices of quantum correlations arising, for example, in Tsirelson's conjecture.

Using Proposition 3.1, for  $n \geq 2$  we can define a map  $\Phi: T(M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C})) \rightarrow \mathcal{FM}(n)$  by letting  $T = \Phi(\tau)$  be the factorizable channel determined, via its Choi matrix, by (3.3), for each tracial state  $\tau$  on  $M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C})$ . More precisely, for all  $x \in M_n(\mathbb{C})$ ,

$$\Phi(\tau)(x) = \sum_{i,j=1}^n n \tau(\iota_2(e_{ij})^* \iota_1(x)) e_{ij}. \quad (3.4)$$

Following the notation of [23], denote by  $\mathcal{FM}_{\text{fin}}(n)$  the set of maps in  $\mathcal{FM}(n)$  that admit a factorization through a finite dimensional  $C^*$ -algebra.

**Proposition 3.2.** *The map  $\Phi: T(M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C})) \rightarrow \mathcal{FM}(n)$  defined above is continuous, affine and surjective. Moreover,*

- (i)  $\mathcal{FM}_{\text{fin}}(n) = \Phi(T_{\text{fin}}(M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C})))$ ;
- (ii)  $\overline{\mathcal{FM}_{\text{fin}}(n)} = \Phi(\overline{T_{\text{fin}}(M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C}))}) = \Phi(T_{\text{hyp}}(M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C})))$ .

*Proof.* Surjectivity of  $\Phi$  follows from Proposition 3.1. To prove it is continuous and affine, it suffices to show that the map  $\tau \mapsto \Phi(\tau)(x)$  is continuous and affine, for all  $x \in M_n(\mathbb{C})$ . This follows easily from (3.4).

(i). If  $\tau$  belongs to  $T_{\text{fin}}(M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C}))$ , then  $M := \pi_\tau(M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C}))''$  is finite dimensional. It follows from the proofs of the implications (iv)  $\Rightarrow$  (iii)  $\Rightarrow$  (ii)  $\Rightarrow$  (i) in Proposition 3.1 that  $T = \Phi(\tau)$  admits a factorization through  $(M, \tau)$ , so  $T$  belongs to  $\mathcal{FM}_{\text{fin}}(n)$ .

Likewise, if  $T$  belongs to  $\mathcal{FM}_{\text{fin}}(n)$ , then we can take the finite von Neumann algebra  $M$  with normal faithful tracial state  $\tau_M$  in (iii) of Proposition 3.1 to be finite dimensional. Let  $\pi: M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C}) \rightarrow M$  be as in the proof of (iii)  $\Rightarrow$  (iv) in Proposition 3.1, and let  $\tau = \tau_M \circ \pi$ . Then  $\tau$  is a tracial state on  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$  with kernel  $I_{\tau'} = \ker(\pi)$ . Hence  $(M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C}))/I_{\tau'}$  is finite dimensional, so  $\tau \in T_{\text{fin}}(M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C}))$ . It follows from the proof of (iii)  $\Rightarrow$  (iv) in Proposition 3.1 that  $T = \Phi(\tau)$ .

Finally, (ii) follows from (i), continuity of  $\Phi$ , compactness of  $\overline{T_{\text{fin}}(M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C}))}$ , and Theorem 2.10.  $\square$

**Remark 3.3.** Proposition 3.2 provides a direct proof, avoiding ultraproduct arguments, of the well-known fact that the set  $\mathcal{FM}(n)$  is a compact convex subset of the normed vector space of all linear maps on  $M_n(\mathbb{C})$ .

Note that the map  $\Phi$  is not injective. More precisely, if we let  $V_n$  denote the  $n^4$ -dimensional operator subspace of  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$  spanned by  $\{\iota_1(x)\iota_2(y) : x, y \in M_n(\mathbb{C})\}$ , then, for  $\tau, \tau' \in T(M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C}))$ ,

$$\Phi(\tau) = \Phi(\tau') \text{ if and only if } \tau|_{V_n} = \tau'|_{V_n}. \quad (3.5)$$

The next corollary extends and sheds new light on [16, Theorem 3.7], which states that (i) and (ii) below are equivalent.

**Corollary 3.4.** *The following statements are equivalent:*

- (i) *The Connes Embedding Problem has an affirmative answer;*
- (ii)  *$\mathcal{FM}_{\text{fin}}(n)$  is dense in  $\mathcal{FM}(n)$ , for all  $n \geq 3$ ;*
- (iii)  *$T_{\text{hyp}}(M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})) = T(M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C}))$ , for all  $n \geq 2$ ;*
- (iv) *For each  $n \geq 2$ , and each  $\tau$  in  $T(M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C}))$ , there is  $\tau'$  in  $T_{\text{hyp}}(M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C}))$  such that  $\tau|_{V_n} = \tau'|_{V_n}$ .*

*Proof.* It is clear that an affirmative answer to the Connes Embedding Problem is equivalent to all traces on all  $C^*$ -algebras being hyperlinear, thus proving (i)  $\Rightarrow$  (iii), while the implication (iii)  $\Rightarrow$  (iv) is trivial. It follows from Proposition 3.2 (ii) and (3.5) that (iv)  $\Rightarrow$  (ii). Finally, (ii)  $\Rightarrow$  (i) is contained in [16, Theorem 3.7].  $\square$

**Remark 3.5.** Suppose that  $\tau$  is a tracial state on  $M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C})$ , and that  $T = \Phi(\tau)$  is the corresponding factorizable map on  $M_n(\mathbb{C})$ . Then, by the proof of Proposition 3.1, we see that  $T$  admits a factorization through the finite von Neumann algebra  $M = \pi_\tau(M_n(\mathbb{C}) *_\mathbb{C} M_n(\mathbb{C}))''$ , equipped with the trace  $\tau$ . In particular, we see that  $M$  admits an embedding into  $\mathcal{R}^\omega$  if and only if  $\tau$  is hyperlinear. It was shown in [16] that  $T$  belongs to  $\mathcal{FM}_{\text{fin}}(n)$  if and only if it admits a factorization through a finite von Neumann algebra that embeds into  $\mathcal{R}^\omega$ .

**Remark 3.6.** J. Peterson mentioned to us that one can prove the implication (iii)  $\Rightarrow$  (i) of Corollary 3.4 directly as follows: Assume that (iii) holds and that  $(M, \tau)$  is a separable  $\text{II}_1$ -factor. Upon replacing  $M$  by  $M \otimes \mathcal{R}$ , we may assume that  $M$  is singly generated, and hence generated by two self-adjoint elements  $a$  and  $b$ , that can be taken to be contractions. Take sequences  $\{a_n\}_{n \geq 1}$  and  $\{b_n\}_{n \geq 1}$  of self-adjoint contractions converging with respect to  $\|\cdot\|_2$  to  $a$  and  $b$ , respectively, so that  $C^*(1_M, a_n)$  and  $C^*(1_M, b_n)$  admit unital embeddings (necessarily trace-preserving) into  $M_n(\mathbb{C})$ . (Such unital embeddings exist precisely when  $a_n$  and  $b_n$  are of the form  $\sum_{j=1}^n \lambda_j e_j$ , for some real numbers  $\lambda_j$  and some pairwise orthogonal and pairwise equivalent projections  $e_1, \dots, e_n$  summing up to 1.) Then  $C^*(1_M, a_n, b_n)$  admits a unital embedding into  $M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C})$ , that is trace-preserving with respect to some tracial state  $\tau$  on  $M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C})$ , which, by assumption, is hyperlinear. This shows that  $C^*(1_M, a_n, b_n)$  admits a unital trace-preserving embedding into  $\mathcal{R}^\omega$ . Consequently,  $M$  embeds into the double ultrapower  $(\mathcal{R}^\omega)^\omega$ , and therefore into  $\mathcal{R}^\omega$ , by a diagonal argument.

We end this paper with a result concerning the structure of the simplex  $T(M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C}))$ , and a related result describing which unital  $C^*$ -algebras are quotients of  $M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C})$ , or, equivalently, which unital  $C^*$ -algebras can be generated by two copies of  $M_n(\mathbb{C})$ . Recall that a unital  $C^*$ -algebra is generated by  $n \geq 1$  elements if and only if it is generated by  $2n$  self-adjoint elements; and if a unital  $C^*$ -algebra is generated by  $n$  self-adjoint elements, then it is also generated by  $n$  unitary elements.

**Proposition 3.7.** *Let  $\mathcal{A}$  be a unital  $C^*$ -algebra, and let  $n \geq 2$  be an integer.*

- (i) *If there exists a unital surjective  $*$ -homomorphism  $M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \otimes \mathcal{A}$ , then  $\mathcal{A}$  is generated by at most  $n^2$  elements.*
- (ii) *If  $\mathcal{A}$  is generated by  $n-1$  unitaries, then there exists a unital surjective  $*$ -homomorphism  $M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \otimes \mathcal{A}$ .*

*Proof.* (i). The unital  $*$ -homomorphism  $\varphi: M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \otimes \mathcal{A}$  is determined by two unital  $*$ -homomorphisms  $\alpha, \beta: M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \otimes \mathcal{A}$ , and we may take  $\alpha(x) = x \otimes 1_{\mathcal{A}}$ , for  $x \in M_n(\mathbb{C})$ . Now,  $M_n(\mathbb{C})$  is singly generated, say by an element  $g \in M_n(\mathbb{C})$ , and

$$\beta(g) = \sum_{i,j=1}^n e_{ij} \otimes g_{ij},$$

for some elements  $g_{ij} \in \mathcal{A}$ ,  $1 \leq i, j \leq n$ . Since  $\varphi$  is surjective, it follows that  $\mathcal{A}$  must be generated by the set  $\{g_{ij} : 1 \leq i, j \leq n\}$ .

(ii). Suppose that  $\mathcal{A}$  is generated by unitaries  $u_2, \dots, u_n$  in  $\mathcal{A}$ . Set  $f_{11} = e_{11} \otimes 1_{\mathcal{A}}$  and  $f_{1j} = e_{1j} \otimes u_j$ , for  $2 \leq j \leq n$ . Note that  $f_{1j} f_{1j}^* = e_{11} \otimes 1_{\mathcal{A}}$ , and that  $f_{1j}^* f_{1j} = e_{jj} \otimes 1_{\mathcal{A}}$ , for  $1 \leq j \leq n$ . Further, set  $f_{ij} = f_{1i}^* f_{1j}$ ,  $1 \leq i, j \leq n$ . Observe that  $\{f_{ij}\}$  is a set of  $n \times n$  matrix units in  $M_n(\mathbb{C}) \otimes \mathcal{A}$ . Hence there exists a unital  $*$ -homomorphism  $\beta: M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \otimes \mathcal{A}$  satisfying  $\beta(e_{ij}) = f_{ij}$ ,  $1 \leq i, j \leq n$ . Let  $\gamma: M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \otimes \mathcal{A}$  be determined by  $\alpha$  and  $\beta$  (i.e.,  $\gamma(\iota_1(x)) = \alpha(x)$  and  $\gamma(\iota_2(x)) = \beta(x)$ , for  $x \in M_n(\mathbb{C})$ ). It is then easy to see that  $1 \otimes u_j$  belongs to the image of  $\gamma$ , for  $2 \leq j \leq n$ , and that  $M_n(\mathbb{C}) \otimes 1_{\mathcal{A}}$  is contained in the image of  $\gamma$ . This shows that  $\gamma$  is surjective.  $\square$

It follows in particular that  $M_n(\mathbb{C}) \otimes \mathcal{A}$  is a quotient of  $M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C})$ , for every singly generated unital  $C^*$ -algebra  $\mathcal{A}$ , when  $n \geq 3$ . It was shown in [28] that every unital separable

$\mathcal{Z}$ -stable  $C^*$ -algebra is singly generated. It is easy to see that every finite dimensional  $C^*$ -algebra is singly generated, so  $M_n(\mathbb{C}) \otimes \mathcal{A}$  is generated by two copies of  $M_n(\mathbb{C})$ , whenever  $\mathcal{A}$  is finite dimensional and  $n \geq 3$ .

**Remark 3.8.** We know, e.g., from Remark 2.4 that  $T_{\text{fin}}(M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C}))$  is not closed, for all  $n \geq 2$ . For  $n \geq 11$ , this also follows from Proposition 3.2 and the main result from [23], which states that  $\mathcal{FM}_{\text{fin}}(n)$  is non-closed.

One can exhibit many traces in  $T(M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C}))$ , and also in  $\overline{T_{\text{fin}}(M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C}))}$ , which are of type  $\text{II}_1$ . Indeed, take any unital separable tracial  $C^*$ -algebra  $(\mathcal{A}, \tau)$ . Then, by Proposition 3.7, there is a trace  $\tau'$  on  $M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C})$  that factors through the trace  $\tau \otimes \text{tr}_n \otimes \tau_{\mathcal{Z}}$  on  $\mathcal{A} \otimes M_n(\mathbb{C}) \otimes \mathcal{Z}$ . The trace  $\tau'$  is always of type  $\text{II}_1$ , it is a factor trace if  $\tau$  is, and  $\tau'$  belongs to the closure of  $T_{\text{fin}}(M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C}))$ , which equals  $T_{\text{hyp}}(M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C}))$ , if  $\pi_{\tau}(\mathcal{A})''$  embeds into  $\mathcal{R}^{\omega}$ .

As an interesting application of Proposition 3.7, we show next that the trace simplex of  $M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C})$ , for  $n \geq 3$ , is as large as possible. For the proof of this result we make use of the following elementary fact: Any surjective unital  $*$ -homomorphism  $\varphi: \mathcal{A} \rightarrow \mathcal{B}$  between unital  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  induces an affine continuous injective map  $T(\varphi): T(\mathcal{B}) \rightarrow T(\mathcal{A})$ , by  $T(\tau) = \tau \circ \varphi$ , for  $\tau \in T(\mathcal{B})$ . Moreover,  $T(\varphi)$  maps extreme points of  $T(\mathcal{B})$  into extreme points of  $T(\mathcal{A})$ , and hence faces of  $T(\mathcal{B})$  onto faces of  $T(\mathcal{A})$ . Indeed, if  $\tau$  is an extreme point of  $T(\mathcal{B})$ , then  $\pi_{\tau}(\mathcal{B})''$  is a factor. As  $\mathcal{B} = \varphi(\mathcal{A})$ , we infer that  $\pi_{\tau \circ \varphi}(\mathcal{A}) = \pi_{\tau}(\mathcal{B})$ , so  $\pi_{\tau \circ \varphi}(\mathcal{A})'' = \pi_{\tau}(\mathcal{B})''$  is a factor, which implies that  $T(\tau) = \tau \circ \varphi$  is an extreme point.

**Theorem 3.9.** *Let  $n \geq 3$  be an integer. Then each metrizable Choquet simplex is affinely homeomorphic to a (closed) face of  $T(M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C}))$ .*

*Proof.* Let  $S$  be a metrizable Choquet simplex. Then there is a simple infinite dimensional unital AF-algebra  $\mathcal{A}$  such that  $T(\mathcal{A})$  is affinely homeomorphic to  $S$ , see, e.g., [10] or [21]. Every simple infinite dimensional unital AF-algebra is  $\mathcal{Z}$ -absorbing, see [18, Theorem 5], and hence singly generated. It follows from Proposition 3.7 above that there is a unital surjective  $*$ -homomorphism  $\varphi: M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \otimes \mathcal{A}$ , which, in turn, induces an injective affine continuous map  $T(\varphi): T(\mathcal{A}) \rightarrow T(M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C}))$ . As remarked above, the image of  $T(\varphi)$  is a face of  $T(M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C}))$  which is affinely homeomorphic to  $S$ .  $\square$

It was shown in [22, Theorems 2.3, 2.5 and 2.11] that a metrizable Choquet simplex  $S$  is the Poulsen simplex if and only if the following two conditions hold:

- (i) Each metrizable Choquet simplex is affinely homeomorphic to a face of  $S$ .
- (ii) (Homogeneity) For every pair  $F, F'$  of faces of  $S$  with  $\dim(F) = \dim(F') < \infty$ , there is an affine homomorphism of  $S$  that maps  $F$  onto  $F'$ .

We would like to point out that property (i) by itself does not characterize the Poulsen simplex, and hence one cannot conclude from Proposition 3.9 that  $T(M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C}))$  is the Poulsen simplex. Indeed, if  $S$  is the Poulsen simplex embedded in a locally convex topological vector space  $V$ , then the suspension  $S' := \{(ts, 1-t) : s \in S, 0 \leq t \leq 1\} \subseteq V \times \mathbb{R}$  of  $S$  is a Choquet simplex that contains  $S$  as a face, but it is not itself the Poulsen simplex, as the extreme points are not dense.

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