

Perturbation of Hausdorff moment sequences, and an application to the theory of C^* -algebras of real rank zero

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Abstract

We investigate the class of unital C^* -algebras that admit a unital embedding into *every* unital C^* -algebra of real rank zero, that has no finite-dimensional quotients. We refer to a C^* -algebra in this class as an *initial object*. We show that there are many initial objects, including for example some unital, simple, infinite-dimensional AF-algebras, the Jiang-Su algebra, and the GICAR-algebra.

That the GICAR-algebra is an initial object follows from an analysis of Hausdorff moment sequences. It is shown that a dense set of Hausdorff moment sequences belong to a given dense subgroup of the real numbers, and hence that the Hausdorff moment problem can be solved (in a non-trivial way) when the moments are required to belong to an arbitrary simple dimension group (i.e., unperforated simple ordered group with the Riesz decomposition property).

1 Introduction

The following three questions concerning an arbitrary unital C^* -algebra A , that is “large” in the sense that it has no finite-dimensional representation, are open.

Question 1.1 Does A contain a simple, unital, infinite-dimensional sub- C^* -algebra?

Question 1.2 (The Global Glimm Halving Problem) Does A contain a full¹ sub-C*-algebra isomorphic to $C_0((0, 1], M_2)$?

Question 1.3 Is there a unital embedding of the Jiang-Su algebra \mathcal{Z} into A ?

The Jiang-Su algebra (see [8]) is a simple, unital, infinite-dimensional C*-algebra, which is KK -equivalent to the complex numbers (and hence at least from a K -theoretical point of view could be an initial object as suggested in Question 1.3). An affirmative answer to Question 1.3 clearly would yield an affirmative answer to Question 1.1. A version of a lemma of Glimm (see [10, Proposition 4.10]) confirms Question 1.2 in the special case that A is simple (and not isomorphic to \mathbb{C}); so Question 1.2 is weaker than Question 1.1.

Question 1.2 was raised in [11, Section 4] because a positive answer will imply that every weakly purely infinite C*-algebra is automatically purely infinite.

The Jiang-Su algebra plays a role in the classification program for amenable C*-algebras (a role that may well become more important in the future). An affirmative answer to Question 1.3 will, besides also answering the two other questions, shed more light on the Jiang-Su algebra. It would for example follow that the Jiang-Su algebra is the (necessarily unique) unital, simple, separable infinite-dimensional C*-algebra with the property stipulated in Question 1.3 and with the property (established in [8]) that every unital endomorphism can be approximated in the pointwise-norm topology by inner automorphisms.

We provide in this paper an affirmative answer to the three questions above in the special case in which the target C*-algebra A is required to be of real rank zero (in addition to being unital and with no finite-dimensional representations).

Zhang proved in [15] that in any unital simple non-elementary C*-algebra of real rank zero and for any natural number n one can find pairwise orthogonal projections p_0, p_1, \dots, p_n that sum up to 1 and satisfy $p_0 \precsim p_1 \sim p_2 \sim \dots \sim p_n$. In other words, one can divide the unit into $n+1$ pieces where n of the pieces are of the same size, and the last piece is smaller. This result was improved in [12] where it was shown that for every natural number n one can unitaly embed $M_n \oplus M_{n+1}$ into any unital C*-algebra of real rank

¹A subset of a C*-algebra is called *full* if it is not contained in any proper closed two-sided ideal of the C*-algebra.

zero, that has no non-zero representation of dimension $< n$. Thus, in the terminology of the abstract, $M_n \oplus M_{n+1}$ is an initial object for every n . We shall here extend this result and show that also the infinite tensor product $P = \bigotimes_{n=1}^{\infty} M_2 \oplus M_3$ is an initial object.

We shall give an algorithm which to an arbitrary unital AF-algebra, that has no finite-dimensional representations, assigns a unital *simple* infinite-dimensional AF-algebra that embeds unitaly into A . This leads to the existence of a unital infinite-dimensional simple AF-algebra that unitaly embeds into P , and hence is an initial object. The Jiang-Su algebra was known by Jiang and Su to embed unitaly into any unital simple non-elementary AF-algebra, and so is also an initial object.

In Section 4 we shall show that the Gauge Invariant CAR-algebra is an initial object. Along the way to this result we shall prove a perturbation result that may be of independent interest: the set of Hausdorff moment sequences, with the property that all terms belong to an arbitrary fixed dense subset of the real numbers, is a dense subset of the Choquet simplex of all Hausdorff moment sequences.

We shall show in Section 5 that a simple, unital, infinite-dimensional C^* -algebra of real rank zero must have infinite-dimensional trace simplex if it is an initial object. This leads to the open question if one can characterise initial objects among (simple) unital infinite-dimensional C^* -algebras of real rank zero (or among simple AF-algebras).

We hope that our results will find applications in the future study of real rank zero C^* -algebras; and we hope to have cast some light on the three fundamental questions raised above.

2 Initial objects in unital real rank zero C^* -algebras

Definition 2.1 A unital C^* -algebra A will be said in this paper to be an *initial object* if it embeds unitaly into any unital C^* -algebra of real rank zero which has no non-zero finite-dimensional representations. (Note that we do not require A to belong to the class of algebras with these properties.) (Also we do not require the embedding to be unique in any way.)

It is clear that the algebra of complex numbers \mathbb{C} is an initial object, even in the category of all unital C^* -algebras—and that it is the unique initial object

in this larger category. It will be shown in Proposition 2.3 below that the infinite C^* -algebra tensor product $P = \bigotimes_{n=1}^{\infty} M_2 \oplus M_3$ is also an initial object in the sense of the present paper. Note that this C^* -algebra in fact belongs to the category we are considering, i.e., unital C^* -algebras of real rank zero with no non-zero finite-dimensional quotients. It follows that a C^* -algebra is an initial object in our sense if and only if it is (isomorphic to) a unital sub- C^* -algebra of P . We shall use this fact to exhibit a perhaps surprisingly large number of initial objects, including many simple AF-algebras, the Jiang-Su algebra, and the GICAR-algebra (the gauge invariant subalgebra of the CAR-algebra).

Let us begin by recalling the following standard fact.

Lemma 2.2 *Let A be a unital C^* -algebra and let F be a unital finite-dimensional sub- C^* -algebra of A . Let g_1, \dots, g_n denote the minimal (non-zero) central projections in F and let e_1, \dots, e_n be minimal (non-zero) projections in Fg_1, \dots, Fg_n , respectively.*

The map consisting of multiplying by the sum $e_1 + \dots + e_n$ is an isomorphism from the relative commutant $A \cap F'$ of F in A onto the sub- C^ -algebra $e_1 A e_1 \oplus e_2 A e_2 \oplus \dots \oplus e_n A e_n$ of A . Moreover, if B is another unital C^* -algebra and $\rho_j: B \rightarrow e_j A e_j$ are unital $*$ -homomorphisms, then there is a unique unital $*$ -homomorphism $\rho: B \rightarrow A \cap F'$ such that $\rho(b)e_j = e_j \rho(b) = \rho_j(b)$ for all $b \in B$ and all $j = 1, \dots, n$.*

Proposition 2.3 *The C^* -algebra $P = \bigotimes_{n=1}^{\infty} M_2 \oplus M_3$ is an initial object (in the sense of Definition 2.1).*

Proof. Let A be a unital C^* -algebra of real rank zero with no non-zero finite-dimensional representations. We must find a unital embedding of P into A .

Set $\bigotimes_{j=1}^n M_2 \oplus M_3 = P_n$, so that $P_{n+1} = P_n \otimes (M_2 \oplus M_3)$. Let us construct for each n a unital embedding $\varphi_n: P_n \rightarrow A$ in such a way that $\varphi_{n+1}(x \otimes 1) = \varphi_n(x)$ for each $x \in P_n$. This will yield a unital embedding of P into A as desired. In order to be able construct these maps inductively we must require in addition that they be full.²

²By a full $*$ -homomorphism we mean a $*$ -homomorphism that maps each non-zero element to a full element in the codomain algebra. (A full element is one not belonging to any proper closed two-sided ideal.)

For each full projection e in A there is a full unital embedding $\psi: M_2 \oplus M_3 \rightarrow eAe$. Indeed, eAe cannot have any non-zero finite-dimensional representation since any such representation would extend to a finite-dimensional representation of A (on a larger Hilbert space). Hence by [12, Proposition 5.3], there is a unital $*$ -homomorphism from $M_5 \oplus M_7$ into eAe . Composing this with a full unital embedding $M_2 \oplus M_3 \rightarrow M_5 \oplus M_7$ yields the desired full embedding ψ .

The preceding argument shows that there is a full unital embedding $\varphi_1: P_1 = M_2 \oplus M_3 \rightarrow A$. Suppose that $n \geq 1$ and that maps $\varphi_1, \varphi_2, \dots, \varphi_n$ have been found with the desired properties.

Choose minimal projections f_1, f_2, \dots, f_{2^n} in P_n , one in each minimal non-zero direct summand, and set $\varphi_n(f_j) = e_j$. Each e_j is then a full projection in A . Choose a full unital embedding $\rho_j: M_2 \oplus M_3 \rightarrow e_j A e_j$ for each j , and note that by Lemma 2.2 there exists a unital $*$ -homomorphism $\rho: M_2 \oplus M_3 \rightarrow A \cap \varphi_n(P_n)'$ such that $\rho(b)e_j = e_j\rho(b) = \rho_j(b)$ for all $b \in M_2 \oplus M_3$ and all j . There is now a unique $*$ -homomorphism $\varphi_{n+1}: P_{n+1} = P_n \otimes (M_2 \oplus M_3) \rightarrow A$ with the property that $\varphi_{n+1}(a \otimes b) = \varphi_n(a)\rho(b)$ for $a \in P_n$ and $b \in M_2 \oplus M_3$. To show that φ_{n+1} is full it suffices to check that $\varphi_{n+1}(f_j \otimes b)$ is full in A for all j and for all non-zero b in $M_2 \oplus M_3$; this follows from the identity $\varphi_{n+1}(f_n \otimes b) = \varphi_n(f_n)\rho(b) = e_j\rho(b) = \rho_j(b)$ and the fact that ρ_j is full. \square

Corollary 2.4 *Let A be a unital C^* -algebra of real rank zero. The following three conditions are equivalent.*

- (i) *A has no non-zero finite-dimensional representations.*
- (ii) *There is a unital embedding of $\bigotimes_{n=1}^{\infty} M_2 \oplus M_3$ into A .*
- (iii) *There is a unital embedding of each initial object³ into A .*

Proof. (i) \Rightarrow (iii) is true by Definition 2.1. (iii) \Rightarrow (ii) follows from Proposition 2.3. (ii) \Rightarrow (i) holds because any finite-dimensional representation of A would restrict to a finite-dimensional representation of $\bigotimes_{n=1}^{\infty} M_2 \oplus M_3$, and no such exists. \square

As remarked above, a C^* -algebra is an initial object if and only if it embeds unitaly into the C^* -algebra $P = \bigotimes_{n=1}^{\infty} M_2 \oplus M_3$. The ordered K_0 -group of

³The list of initial objects includes some simple unital infinite-dimensional AF-algebras and the Jiang-Su algebra \mathcal{Z} as shown in Section 2.

P can be described as follows. Consider the Cantor set $X = \prod_{n=1}^{\infty} \{0, 1\}$. Consider the maps $\nu_0, \nu_1: X \rightarrow \mathbb{N}_0 \cup \{\infty\}$ that for each $x \in X$ count the number of 0s and 1s, respectively, among the coordinates of x , and note that $\nu_0(x) + \nu_1(x) = \infty$ for every $x \in X$. For each supernatural number n denote by $\mathcal{Q}(n)$ the set of rational numbers p/q with q dividing n , and consider the subgroup $G \subseteq C(X, \mathbb{R})$ consisting of those functions g for which $g(x) \in \mathcal{Q}(2^{\nu_0(x)} 3^{\nu_1(x)})$ for every $x \in X$. Equip G with the pointwise order, i.e., $g \geq 0$ if $g(x) \geq 0$ for all $x \in X$. Then $(K_0(P), K_0(P)^+, [1])$ is isomorphic to $(G, G^+, 1)$. Note in particular that G is a dense subgroup of $C(X, \mathbb{R})$.

3 Simple initial objects

We shall show in this section that the class of initial objects, in the sense of the previous section, includes several simple unital (infinite-dimensional) AF-algebras.

Lemma 3.1 *The following two conditions are equivalent for any dimension group G .*

- (i) *For each order unit x in G there exists an order unit y in G such that $2y \leq x$.*
- (ii) *For each finite set of order units x_1, \dots, x_k in G and for each set of natural numbers n_1, \dots, n_k there is an order unit y in G such that $n_j y \leq x_j$ for $j = 1, 2, \dots, k$.*

Proof. The implication (i) \Rightarrow (ii) follows from the well-known fact (which is also easy to prove—using the Effros-Handelman-Shen theorem) that if x_1, x_2, \dots, x_k are order units in a dimension group G , then there is an order unit y_0 in G such that $y_0 \leq x_j$ for all j . The implication (ii) \Rightarrow (i) is immediate. \square

A dimension group will be said to have the property (D) if it satisfies the two equivalent conditions of Lemma 3.1.

Lemma 3.2 *Let A be a unital AF-algebra. The ordered group $K_0(A)$ has the property (D) if and only if A has no non-zero finite-dimensional representations.*

Proof. Suppose that A has no non-zero finite-dimensional representation, and let x be an order unit in $K_0(A)$. Then $x = [e]$ for some full projection e in $M_n(A)$ for some n . Since any finite-dimensional representation of $eM_n(A)e$ would induce a finite-dimensional representation of A (on a different Hilbert space), $eM_n(A)e$ has no non-zero finite-dimensional representation. By [12, Proposition 5.3] there is a unital $*$ -homomorphism from $M_2 \oplus M_3$ into $eM_n(A)e$. (Cf. proof of Proposition 2.3 above.) Let $f = (f_1, f_2)$ be a projection in $M_2 \oplus M_3$, with f_1 and f_2 one-dimensional, and denote by $\tilde{f} \in eM_n(A)e$ the image of f under the unital $*$ -homomorphism $M_2 \oplus M_3 \rightarrow eM_n(A)e$. Then \tilde{f} is full in $eM_n(A)e$ (because f is full in $M_2 \oplus M_3$), and $2[\tilde{f}] \leq [e]$, as desired.

Suppose conversely that $K_0(A)$ has the property (D). Condition 2.1 (ii) with $k = 1$ implies immediately that every non-zero representation of A is infinite-dimensional. \square

We present below a more direct alternative proof (purely in terms of ordered group theory) of the first implication of the lemma above. Consider a decomposition of $K_0(A)$ as the ordered group inductive limit of a sequence of ordered groups $G_1 \rightarrow G_2 \rightarrow \cdots$ with each G_i isomorphic to a finite ordered group direct sum of copies of \mathbb{Z} , and let x be an order unit in $K_0(A)$. Modifying the inductive limit decomposition of $K_0(A)$, we may suppose that x is the image of an order unit x_1 in G_1 , and that the image x_n of x_1 in G_n is an order unit for G_n for each $n \geq 2$. Let us show that for some n the condition 2.1 (i) holds for x_n in G_n —or else, if not, then G has a non-zero quotient ordered group isomorphic to \mathbb{Z} . If not, then for every n there exists at least one coordinate of x_n in G_n equal to one, and the inductive limit of the sequence consisting, at the n th stage, of the largest quotient of the ordered group G_n in which every coordinate of x_n is equal to one is a non-zero quotient of G every prime quotient of which is \mathbb{Z} . As soon as Condition 2.1 (i) holds for x_n in G_n , then it holds for x in G . In other words, if G has no non-zero quotient isomorphic to \mathbb{Z} , then it has the property (D).

Proposition 3.3 *Let (G, G^+) be a dimension group with the property (D). Denote by G^{++} the set of all order units in G , and suppose that $G^{++} \neq \emptyset$. Then $(G, G^{++} \cup \{0\})$ is a simple dimension group.*

Proof. Observe first that $G^{++} + G^+ = G^{++}$. With this fact (and with the assumption that G^{++} is non-empty) it is straightforward to check that $(G, G^{++} \cup \{0\})$ is an ordered abelian group. We proceed to show that it is a

dimension group. This ordered group is unperforated as (G, G^+) is, and so we need only show that it has the Riesz decomposition property. Equip G with the two orderings \leq and \lesssim given by $x \leq y$ if $y - x \in G^+$ and $x \lesssim y$ if $y - x \in G^{++} \cup \{0\}$. Suppose that $x \lesssim y_1 + y_2$ where $x, y_1, y_2 \in G^{++} \cup \{0\}$. We must find $x_1, x_2 \in G^{++} \cup \{0\}$ such that $x = x_1 + x_2$ and $x_j \lesssim y_j$, $j = 1, 2$. It is trivial to find x_1 and x_2 in the cases that one of x, y_1, y_2 , and $y_1 + y_2 - x$ is zero. Suppose that the four elements above are non-zero, in which case by hypothesis they all are order units. By hypothesis (and by Lemma 3.1) there is $z \in G^{++}$ such that

$$2z \leq x, \quad z \leq y_1, \quad z \leq y_2, \quad 2z \leq y_1 + y_2 - x.$$

Then $x - 2z \leq (y_1 - 2z) + (y_2 - 2z)$. Since (G, G^+) has the Riesz decomposition property there are $v_1, v_2 \in G^+$ such that

$$x - 2z = v_1 + v_2, \quad v_1 \leq y_1 - 2z, \quad v_2 \leq y_2 - 2z.$$

Set $v_1 + z = x_1$ and $v_2 + z = x_2$. Then x_1, x_2 belong to G^{++} , $x = x_1 + x_2$, $x_1 \lesssim y_1$, and $x_2 \lesssim y_2$; the latter two inequalities hold because

$$y_j - x_j = y_j - v_j - z = (y_j - v_j - 2z) + z \in G^+ + G^{++} = G^{++}.$$

□

Proposition 3.4 *Let A be a unital AF-algebra A with no non-zero finite-dimensional representation. There exists a unital sub- C^* -algebra B of A which is a simple, infinite-dimensional AF-algebra, and for which the inclusion mapping $B \rightarrow A$ gives rise to*

- (i) *an isomorphism of simplices $T(A) \rightarrow T(B)$, and*
- (ii) *an isomorphism of groups $K_0(B) \rightarrow K_0(A)$ which maps $K_0(B)^+$ onto $K_0(A)^{++} \cup \{0\}$, and so in particular,*

$$(K_0(B), K_0(B)^+, [1]) \cong (K_0(A), K_0(A)^{++} \cup \{0\}, [1]).$$

If A is an initial object, then so also is B .

Proof. We derive from Lemma 3.2 that $K_0(A)$ has property (D), and we then conclude from Proposition 3.3 that $K_0(A)$ equipped with the positive cone $G^+ := K_0(A)^{++} \cup \{0\}$ is a simple dimension group. Let B_1 be

the simple, unital, infinite-dimensional AF-algebra with dimension group $(K_0(A), G^+, [1_A])$, and use the homomorphism theorem for AF-algebras ([13, Proposition 1.3.4 (iii)]), to find a unital (necessarily injective) *-homomorphism $\varphi: B_1 \rightarrow A$ which induces the (canonical) homomorphism $K_0(B_1) \rightarrow K_0(A)$ that maps $K_0(B_1)^+$ onto G^+ and $[1_{B_1}]$ onto $[1_A]$. Set $\varphi(B_1) = B$. Then B is a unital sub-C*-algebra of A , B is isomorphic to B_1 , and (ii) holds.

The property (i) follows from (ii) and the fact, that we shall prove, that the state spaces of $(K_0(A), K_0(A)^+, [1_A])$ and $(K_0(A), G^+, [1_A])$ coincide. The former space is contained in the latter because G^+ is contained in $K_0(A)^+$. To show the reverse inclusion take a state f on $(K_0(A), G^+, [1_A])$ and take $g \in K_0(A)^+$. We must show that $f(g) \geq 0$. Use Lemmas 3.1 and 3.2 to find for each natural number n an element v_n in $K_0(A)^{++}$ such that $nv_n \leq [1_A]$. Then $nf(v_n) \leq 1$, so $f(v_n) \leq 1/n$; and $g + v_n$ belongs to $K_0(A)^{++}$, so $f(g+v_n) \geq 0$. These two inequalities, that hold for all n , imply that $f(g) \geq 0$. \square

Corollary 3.5

- (i) *There is a simple unital infinite-dimensional AF-algebra which is an initial object.*
- (ii) *The Jiang-Su algebra \mathcal{Z} is an initial object.*

Proof. The assertion (i) follows immediately from Propositions 2.3 and 3.4.

The assertion (ii) follows from (i) and the fact, proved in [8], that the Jiang-Su algebra \mathcal{Z} embeds in (actually is tensorially absorbed by) any unital simple infinite-dimensional AF-algebra. \square

The corollary above provides an affirmative answer to Question 1.3 (and hence also to Questions 1.1 and 1.2) of the introduction in the case that the target C*-algebra A is assumed to be of real rank zero.

The question of initial objects may perhaps be pertinent in the classification program, where properties such as approximate divisibility and being able to absorb the Jiang-Su algebra \mathcal{Z} are of interest. We remind the reader that a C*-algebra A is approximately divisible if for each natural number n there is a sequence $\varphi_k: M_n \oplus M_{n+1} \rightarrow \mathcal{M}(A)$ of unital *-homomorphisms (where $\mathcal{M}(A)$ denotes the multiplier algebra of A) such that $[\varphi_k(x), a] \rightarrow 0$ for all $a \in A$ and all $x \in M_n \oplus M_{n+1}$. (It turns out that if A is unital, then we

need only find such a sequence of $*$ -homomorphisms for $n = 2$.) It is easily seen that a separable C^* -algebra A is approximately divisible if, and only if, there is a unital $*$ -homomorphism

$$\prod_{n \in \mathbb{N}} (M_n \oplus M_{n+1}) / \sum_{n \in \mathbb{N}} (M_n \oplus M_{n+1}) \rightarrow \mathcal{M}(A)_\omega \cap A', \quad (3.1)$$

and it follows from [13, Theorem 7.2.2] and [8] that A is \mathcal{Z} -absorbing if and only if there is a unital embedding of \mathcal{Z} into $\mathcal{M}(A)_\omega \cap A'$; here, ω is any free ultrafilter on \mathbb{N} , and $\mathcal{M}(A)$ is identified with a sub- C^* -algebra of the ultrapower $\mathcal{M}(A)_\omega$ (the C^* -algebra of bounded sequences in $\mathcal{M}(A)$, modulo the ideal of bounded sequences convergent to 0 along ω).

Toms and Winter recently observed ([14]) that any separable approximately divisible C^* -algebra is \mathcal{Z} -absorbing, because one can embed \mathcal{Z} unitaly into the C^* -algebra on the left-hand side of (3.1). (The latter fact follows from our Corollary 3.5, but it can also be proved directly, as was done in [14].) In the general case, when A need not be approximately divisible, it is of interest to decide when A is \mathcal{Z} -absorbing, or, equivalently, when one can find a unital embedding of \mathcal{Z} into $\mathcal{M}(A)_\omega \cap A'$. Here it would be extremely useful if one knew that \mathcal{Z} was an initial object in the category of all unital C^* -algebras with no non-zero finite-dimensional representations.

The proof of Corollary 3.5 yields an explicit—at the level of the invariant—simple unital AF-algebra which is an initial object. Indeed, consider the initial object $P = \bigotimes_{n=1}^{\infty} M_2 \oplus M_3$, the K_0 -group of which is the dense subset G of $C(X, \mathbb{R})$ described above (after Corollary 2.4), with the relative order, where $C(X, \mathbb{R})$ is equipped with the standard pointwise ordering. The simple dimension group $(G, G^{++} \cup \{0\})$ of Proposition 3.3 is obtained by again viewing G as a subgroup of $C(X, \mathbb{R})$ but this time endowing $C(X, \mathbb{R})$ with the strict pointwise ordering (in which an element $f \in C(X, \mathbb{R})$ is positive if $f = 0$ or if $f(x) > 0$ for all $x \in X$). Any other simple dimension group which maps onto this may also be used.

It would of course be nice to have an even more explicit (or natural) example of a simple unital infinite-dimensional AF-algebra which is an initial object in the sense of this paper.

The trace simplex of the simple unital AF-algebra referred to above is the simplex of probability measures on the Cantor set. We shall show in Section 5 that the trace simplex of an initial object, that has sufficiently many projections, must be infinite-dimensional. Let us now note that a large

class of infinite-dimensional Choquet simplices arise as the trace simplex of an initial object.

Proposition 3.6 *Let X be a metrizable compact Hausdorff space which admits an embedding of the Cantor set.⁴ There exists a simple unital AF-algebra A which is an initial object, such that $T(A)$ is affinely homeomorphic to the simplex $\mathcal{M}_1(X)$ of (Borel) probability measures on X .*

Proof. By hypothesis X has a closed subset X_0 which is (homeomorphic to) the Cantor set. The dimension group of the known initial object $\bigotimes_{n=1}^{\infty} M_2 \oplus M_3$ is isomorphic in a natural way to a dense subgroup G of $C(X_0, \mathbb{R})$ (equipped with the standard pointwise ordering), with canonical order unit corresponding to the constant function 1_{X_0} , cf. the remark after Corollary 2.4. We shall construct below a countable dense subgroup H of $C(X, \mathbb{R})$ such that the constant function 1_X belongs to H , and such that the restriction $f|_{X_0}$ belongs to G for every $f \in H$. Equip H with the strict pointwise ordering on $C(X, \mathbb{R})$ and with the order unit 1_X . Then we have an ordered group homomorphism $\varphi: H \rightarrow G$ given by $\varphi(f) = f|_{X_0}$, which maps 1_X into 1_{X_0} . It follows that we may take A to be the unital, simple AF-algebra with invariant $(H, H^+, 1_X)$, as by the homomorphism theorem for AF-algebras (cf. above) φ induces a unital embedding of A into $\bigotimes_{n=1}^{\infty} M_2 \oplus M_3$, whence A is an initial object, and the trace simplex of A is homeomorphic to the state space of $(H, H^+, 1_X)$, which is $\mathcal{M}_1(X)$.

Let us now pass to the construction of H . Each $g \in G$ extends to $\tilde{g} \in C(X, \mathbb{R})$ (we do not make any assumption concerning the mapping $g \mapsto \tilde{g}$). Choose a countable dense subgroup H_0 of $C_0(X \setminus X_0, \mathbb{R}) \subseteq C(X, \mathbb{R})$, and consider the countable subgroup of $C(X, \mathbb{R})$ generated by H_0 and the countable set $\{\tilde{g} : g \in G\}$. Denote this group, with the relative (strict pointwise) order, by H ; let us check that this choice of H fulfils the requirements. First, $f|_{X_0} \in G$ for every $f \in H$. To see that H is dense in $C(X, \mathbb{R})$, let there be given $f \in C(X, \mathbb{R})$ and $\varepsilon > 0$. Choose $g \in G$ such that $\|f|_{X_0} - g\|_{\infty} < \varepsilon/2$. Extend $f|_{X_0} - g$ to a function $f_0 \in C(X, \mathbb{R})$ with $\|f_0\|_{\infty} = \|f|_{X_0} - g\|_{\infty} < \varepsilon/2$. Note that $f - \tilde{g} - f_0$ belongs to $C_0(X \setminus X_0, \mathbb{R})$. Choose $h_0 \in H_0$ such that $\|f - \tilde{g} - f_0 - h_0\|_{\infty} < \varepsilon/2$, and consider the function $h = \tilde{g} + h_0 \in H$. We have $\|f - h\|_{\infty} \leq \|f - \tilde{g} - f_0 - h_0\|_{\infty} + \|f_0\|_{\infty} < \varepsilon$, as desired. \square

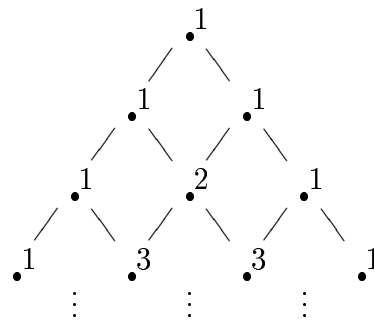
⁴An equivalent formulation of this (rather weak) property is that X has a non-empty closed subset with no isolated points.

4 Hausdorff moments, the GICAR-algebra, and Pascal's triangle

In this section we shall establish the following result.

Theorem 4.1 *The GICAR-algebra is an initial object (in the sense of Definition 2.1).*

We review some of the background material. Consider the Bratteli diagram given by Pascal's triangle,



and denote by

$$\mathbb{C} = B_0 \rightarrow B_1 \rightarrow B_2 \rightarrow \cdots \rightarrow \varinjlim B_n (= B)$$

the inductive system of finite-dimensional C^* -algebras associated with that Bratteli diagram. The C^* -algebra B is the GICAR-algebra. (It can also, more naturally, be realized as the fixed point algebra of the CAR-algebra under a certain action of the circle referred to as the gauge invariant action, cf. [3, Section 5.5] and [4].)

For each $n \geq 0$ and $0 \leq k \leq n$, choose a minimal projection $e(n, k)$ in the k th minimal direct summand of B_n . Note that $e(0, 0) = 1_B$ and that $e(n, k)$ is Murray-von Neumann equivalent to $e(n+1, k) + e(n+1, k+1)$ in B_{n+1} . A trace τ on B_n is determined by its values on the projections $e(n, k)$, $0 \leq k \leq n$.

The group $K_0(B)$ is generated, as an ordered abelian group, by the elements $[e(n, k)]$, with $n \geq 0$ and $0 \leq k \leq n$; that is, these elements span $K_0(B)$ as an abelian group, and the semigroup spanned by the elements $[e(n, k)]$ is equal to $K_0(B)^+$. Our generators satisfy the relations

$$[e(n, k)] = [e(n+1, k)] + [e(n+1, k+1)], \quad n \geq 0, \quad 0 \leq k \leq n. \quad (4.1)$$

Moreover, $(K_0(B), K_0(B)^+)$ is the universal ordered abelian group generated, as an ordered abelian group, by elements $g(n, k)$, $n \geq 0$ and $0 \leq k \leq n$, with the relations $g(n, k) = g(n+1, k) + g(n+1, k+1)$.

For brevity we shall set $(K_0(B), K_0(B)^+, [1_B]_0) = (H, H^+, v)$.

For each abelian (additively written) group G and for each sequence $t: \mathbb{N}_0 \rightarrow G$ associate the discrete derivative $t': \mathbb{N}_0 \rightarrow G$ given by $t'(k) = t(k) - t(k+1)$. Denote the n th derivative of t by $t^{(n)}$, and apply the convention $t^{(0)} = t$.

We remind the reader of the following classical result. The equivalence of (i) and (iv) is the solution to the Hausdorff Moment problem (see e.g. [1, Proposition 6.11]). The equivalence of (i), (ii), and (iii) follows from Proposition 4.3 below (with $(G, G^+, u) = (\mathbb{R}, \mathbb{R}^+, 1)$).

Proposition 4.2 (Hausdorff Moments) *The following four conditions are equivalent for any sequence $t: \mathbb{N}_0 \rightarrow \mathbb{R}$.*

- (i) $t^{(k)}(n) \geq 0$ for all $n, k \geq 0$.
- (ii) *There is a system, $\{t(n, k)\}_{0 \leq k \leq n}$, of positive real numbers (necessarily unique) such that*

$$t(n+1, k) + t(n+1, k+1) = t(n, k), \quad t(n, n) = t(n),$$

for $n \geq 0$ and $0 \leq k \leq n$.

- (iii) *There is a (unique) tracial state τ on the GICAR-algebra such that $t(n) = \tau(e(n, n))$ for all $n \geq 0$.*
- (iv) *There is a Borel probability measure μ on the interval $[0, 1]$ such that*

$$t(n) = \int_0^1 \lambda^n d\mu(\lambda),$$

for all $n \geq 0$.

It follows from Proposition 4.3 below and from (iv) that the coefficients $t(n, k)$ from (ii) are given by

$$t(n, k) = t^{(n-k)}(k) = \int_0^1 \lambda^k (1-\lambda)^{n-k} d\mu(\lambda). \quad (4.2)$$

A sequence $t = (t(0), t(1), \dots)$ satisfying the condition in Proposition 4.2 (iv) (or, equivalently, the three other conditions of Proposition 4.2) is called a *Hausdorff moment sequence*. Note that $t(0) = 1$ in every Hausdorff moment sequence. Let us denote the set of all moment sequences by \mathcal{M} . Note that \mathcal{M} is a compact convex set and in fact a Choquet simplex. For each $n \in \mathbb{N}_0$ let us set

$$\mathcal{M}_n = \{(t(0), t(1), t(2), \dots, t(n)) : (t(0), t(1), t(2), \dots) \in \mathcal{M}\} \subseteq \mathbb{R}^{n+1},$$

and denote by π_n the canonical surjective affine mapping $\mathcal{M}_{n+1} \rightarrow \mathcal{M}_n$.

Let us say that a moment sequence $t = (t(0), t(1), t(2), \dots)$ is *trivial* if the corresponding measure in Proposition 4.2 (iv) is supported in $\{0, 1\}$, and say that t is *non-trivial* otherwise. A sequence t is trivial if and only if it is a convex combination of the two trivial sequences $(1, 1, 1, \dots)$ and $(1, 0, 0, \dots)$. It follows from this and (iv) above that t is non-trivial if and only if $t(2) < t(1)$. One can use Equation (4.2) to see that t is non-trivial if and only if $t(n, k) \neq 0$ for all n and k .

We seek unital embeddings from the GICAR algebra B into unital AF-algebras (and into unital C^* -algebras of real rank zero). At the level of the invariant we are thus seeking positive unit preserving group homomorphisms from the dimension group with distinguished unit (H, H^+, v) associated to the GICAR algebra into the ordered K_0 -group with distinguished unit of the target algebra; call this invariant (G, G^+, u) . The proposition below rephrases this problem as that of the existence of a function $g: \mathbb{N}_0 \rightarrow G$ with certain properties.

Proposition 4.3 *Let (H, H^+, v) be as above, and let (G, G^+, u) be an ordered abelian group with a distinguished order unit u . Let $g: \mathbb{N}_0 \rightarrow G$ be given, and assume that $g(0) = v$. The following conditions are equivalent.*

- (i) $g^{(k)}(n) \in G^+$ for all $n, k \geq 0$.
- (ii) *There is a system, $\{g(n, k)\}_{0 \leq k \leq n}$, of elements in G^+ (necessarily unique) such that*

$$g(n+1, k) + g(n+1, k+1) = g(n, k), \quad g(n, n) = g(n),$$

for all $n \geq 0$ and $0 \leq k \leq n$.

- (iii) *There is a (unique) homomorphism of ordered groups $\varphi: H \rightarrow G$ with $\varphi(v) = u$ such that $\varphi([e(n, n)]) = g(n)$ for all $n \geq 0$.*

If the three conditions above are satisfied, then

$$\varphi([e(n, k)]) = g(n, k) = g^{(n-k)}(k)$$

for all $n \geq 0$ and $0 \leq k \leq n$; and the homomorphism φ is faithful if and only if $g(n, k)$ is non-zero for all $n \geq 0$ and $0 \leq k \leq n$.

Proof. (i) \Rightarrow (ii). Set $g(n, k) = g^{(n-k)}(k) \in G^+$. Then $g(n, n) = g^{(0)}(n) = g(n)$, and

$$\begin{aligned} g(n, k) - g(n+1, k+1) &= g^{(n-k)}(k) - g^{(n-k)}(k+1) = g^{(n-k+1)}(k) \\ &= g(n+1, k). \end{aligned}$$

(ii) \Rightarrow (iii). We noted after Theorem 4.1 that $H = K_0(B)$ is generated, as an ordered abelian group, by the elements $[e(n, k)]$, $n \geq 0$ and $0 \leq k \leq n$, and that H is the universal ordered abelian group generated by these elements subject to the relations (4.1). Accordingly, by (ii), there exists a (unique) positive group homomorphism $\varphi: H \rightarrow G$ with $\varphi([e(n, k)]) = g(n, k)$. Also, $\varphi(v) = \varphi([e(0, 0)]) = g(0, 0) = g(0) = u$.

To complete the proof we must show that φ is uniquely determined by its value on the elements $[e(n, n)]$, $n \geq 0$. But this follows from the fact that the elements $[e(n, k)]$, with $n \geq 0$ and $0 \leq k \leq n$, belong to the subgroup generated by the elements $[e(n, n)]$, for $n \geq 0$, by the relations (4.1).

(iii) \Rightarrow (i). This implication follows from the identity $\varphi([e(n+k, n)]) = g^{(k)}(n)$, that we shall proceed to prove by induction on k . The case $k = 0$ is explicitly contained in (iii). Assume that the identity has been shown to hold for some $k \geq 0$. Then, by (4.1),

$$\begin{aligned} g^{(k+1)}(n) &= g^{(k)}(n) - g^{(k)}(n+1) = \varphi([e(n+k, n)] - [e(n+k+1, n+1)]) \\ &= \varphi([e(n+k+1, n)]). \end{aligned}$$

To prove the two last claims of the proposition, assume that g satisfies the three equivalent conditions, and consider the homomorphism of ordered groups $\varphi: H \rightarrow G$ asserted to exist in (iii). It follows from the proofs of (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) that $\varphi([e(n, k)]) = g(n, k) = g^{(n-k)}(k)$. Any non-zero positive element h of H is a finite (non-empty) sum of elements of the form $[e(n, k)]$. Thus $\varphi(h)$ is a finite (non-empty) sum of elements of the form $g(n, k)$. This shows that $\varphi(h)$ is non-zero for all non-zero positive elements h in H if and only if $g(n, k)$ is non-zero for all n and k . \square

Let us now return to the convex set \mathcal{M} of Hausdorff moment sequences in \mathbb{R}^+ and to the truncated finite-dimensional convex sets \mathcal{M}_n .

Lemma 4.4 $\dim(\mathcal{M}_n) = n$.

Proof. The convex set \mathcal{M}_n is a subset of $\{1\} \times \mathbb{R}^n$, and has therefore dimension at most n . On the other hand, the points $(1, \lambda, \lambda^2, \dots, \lambda^n)$ belong to \mathcal{M}_n for each $\lambda \in (0, 1)$, and these points span an n -dimensional convex set. \square

Let \mathcal{M}_n° denote the relative interior⁵ of \mathcal{M}_n . By standard theory for finite-dimensional convex sets, see e.g. [5, Theorem 3.4], $\dim(\mathcal{M}_n^\circ) = \dim(\mathcal{M}_n) = n$. Note that

$$\mathcal{M}_1 = \{(1, \lambda) : \lambda \in [0, 1]\}, \quad \mathcal{M}_1^\circ = \{(1, \lambda) : \lambda \in (0, 1)\}.$$

For $n \geq 2$ we can use Lemma 4.4 to conclude that $\mathcal{M}_n^\circ = \{1\} \times U_n$ for some open convex subset U_n of \mathbb{R}^n .

Lemma 4.5 $\pi_n(\mathcal{M}_{n+1}^\circ) = \mathcal{M}_n^\circ$.

Proof. This follows from the standard fact from the theory for finite-dimensional convex sets (see e.g. [5, §3 and Exercise 3.3]) that the relative interior of the image of π_n is the image under π_n of the relative interior of \mathcal{M}_{n+1} (combined with the fact that π_n is surjective). \square

Theorem 4.6 *Let G be a dense subset of the reals that contains 1. Then there is a non-trivial moment sequence (t_0, t_1, t_2, \dots) such that t_n belongs to G for every $n \in \mathbb{N}_0$. Furthermore, the moment sequences with all terms belonging to G constitute a dense⁶ subset of \mathcal{M} . If G also is a group, and has infinite rank over \mathbb{Q} , then there exists a moment sequence in G the terms of which are independent over \mathbb{Q} .*

Proof. Let (s_0, s_1, s_2, \dots) be a moment sequence, let m be a natural number, and let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m$ be strictly positive real numbers. Since (s_0, s_1, \dots, s_m) belongs to \mathcal{M}_m , since \mathcal{M}_m° is dense in \mathcal{M}_m (cf. [5, Theorem 3.4]) and is equal

⁵The relative interior of a finite-dimensional convex set is its interior relatively to the affine set it generates.

⁶In the standard pointwise (or product) topology.

to $\{1\} \times U_m$ for some open subset U_m of \mathbb{R}^m , since $1 \in G$, and since G is dense in \mathbb{R} , we can find (t_0, t_1, \dots, t_m) in \mathcal{M}_m° such that t_j belongs to G for $j = 0, 1, \dots, m$ and $|t_j - s_j| < \varepsilon_j$ for $j = 1, \dots, m$.

Let us choose inductively $t_n, n > m$, such that $t_n \in G$ and $(t_0, t_1, \dots, t_n) \in \mathcal{M}_n^\circ$. Suppose that $n \geq m$ and that t_0, t_1, \dots, t_n have been found. The set

$$\{s \in \mathbb{R} : (t_0, t_1, \dots, t_n, s) \in \mathcal{M}_{n+1}^\circ\}$$

is non-empty (by Lemma 4.5) and open (because $\mathcal{M}_{n+1}^\circ = \{1\} \times U_{n+1}$ for some open subset U_{n+1} of \mathbb{R}^{n+1}). Hence there exists $t_{n+1} \in G$ such that $(t_0, t_1, \dots, t_{n+1}) \in \mathcal{M}_{n+1}^\circ$.

The resulting sequence (t_0, t_1, t_2, \dots) in G is a moment sequence by construction and is close to the given moment sequence (s_0, s_1, s_2, \dots) .

The inequality $t_2 < t_1$ holds because (t_0, t_1, t_2) belongs to the open set $\mathcal{M}_2^\circ = \{1\} \times U_2$. (Indeed, note that $t_1 \leq t_2$ whenever (t_0, t_1, t_2) belongs to \mathcal{M}_2 and, hence, that the element $(1, t_1, t_1)$ of \mathcal{M}_2 belongs to the boundary.)

Concerning the desired independence of the terms of the moment sequence when G is a group, of infinite rank, it will suffice to choose each t_n in the set

$$G \setminus \text{span}_{\mathbb{Q}}\{t_0, t_1, \dots, t_{n-1}\}.$$

This is possible because this set is dense in \mathbb{R} by the assumption on G . \square

Corollary 4.7 *Let G be a dense subgroup of \mathbb{R} with $1 \in G$. There is a faithful homomorphism of ordered groups from the dimension group H associated with the Pascal triangle to G (with the order inherited from \mathbb{R}) that maps the canonical order unit of H to 1. Furthermore, the set of such maps into G is dense in the set of such maps just into \mathbb{R} , in the topology of pointwise convergence on H . If G is of infinite rank there exists such a map which is injective.*

Proof. Propositions 4.2 and 4.3 give a one-to-one correspondence between moment sequences $t: \mathbb{N}_0 \rightarrow G$ and homomorphisms $\varphi: H \rightarrow G$ of ordered abelian groups that map the canonical order unit $v \in H$ into $1 \in G$, such that $\varphi([e(n, k)]) = t(n, k)$ for all $n \geq 0$ and $0 \leq k \leq n$. If t is non-trivial, then $t(n, k)$ is non-zero for all n, k , whence $\varphi(g) > 0$ for every non-zero positive element g in H (because each such element g is a sum of elements of the form $[e(n, k)]$).

A pointwise converging net of moment sequences corresponds to a pointwise converging net of homomorphisms $H \rightarrow G$.

The first two claims now follow from Theorem 4.6.

A homomorphism $\varphi: H \rightarrow G$ is injective if the restriction of φ to the sub-group spanned by $\{[e(n, k)] : k = 0, 1, \dots, n\}$ is injective for every n . The latter holds, for a specific n , if and only if $t(n, 0), t(n, 1), \dots, t(n, n)$ are independent over \mathbb{Q} , or, equivalently, if and only if $t(0), t(1), \dots, t(n)$ are independent over \mathbb{Q} . (Use the relation in Proposition 4.2 (ii) to see the second equivalence.) This shows that a moment sequence $t: \mathbb{N}_0 \rightarrow G$, where $t(0), t(1), \dots$ are independent over \mathbb{Q} , gives rise to an injective homomorphism $\varphi: H \rightarrow G$. The existence of such a moment sequence t , under the assumption that G has infinite rank, follows from Theorem 4.6. \square

Lemma 4.8 *With X the Cantor set, let $f_1, \dots, f_n: X \rightarrow \mathbb{R}$ be continuous functions, and let $U \subseteq \mathbb{R}^{n+1}$ be an open subset such that*

$$\{s \in \mathbb{R} : (f_1(x), f_2(x), \dots, f_n(x), s) \in U\}$$

is non-empty for every $x \in X$. It follows that there exists a continuous function $f_{n+1}: X \rightarrow \mathbb{R}$ such that

$$(f_1(x), f_2(x), \dots, f_n(x), f_{n+1}(x)) \in U$$

for all $x \in X$.

Proof. For each $s \in \mathbb{R}$ consider the set V_s of those $x \in X$ for which $(f_1(x), f_2(x), \dots, f_n(x), s)$ belongs to U . Then $(V_s)_{s \in \mathbb{R}}$ is an open cover of X , and so by compactness, X has a finite subcover $V_{s_1}, V_{s_2}, \dots, V_{s_k}$. Because X is totally disconnected there are clopen subsets $W_j \subseteq V_{s_j}$ such that W_1, W_2, \dots, W_k partition X . The function $f_{n+1} = \sum_{j=1}^k s_j 1_{W_j}$ is as desired. \square

Proposition 4.9 *With X the Cantor set, let G be a norm-dense subset of $C(X, [0, 1])$ that contains the constant function 1. There exists a sequence (g_0, g_1, g_2, \dots) in G such that $(g_0(x), g_1(x), g_2(x), \dots)$ is a non-trivial moment sequence for every $x \in X$.*

Proof. Choose g_0, g_1, \dots in G inductively such that $(g_0(x), g_1(x), \dots, g_n(x))$ belongs to \mathcal{M}_n° for every $x \in X$. Begin by choosing g_0 to be the constant function 1 (as it must be). Suppose that $n \geq 0$ and that g_0, g_1, \dots, g_n as

above have been found. As observed earlier, $\mathcal{M}_{n+1}^\circ = \{1\} \times U_{n+1}$ for some open subset U_{n+1} of \mathbb{R}^{n+1} . The set

$$\begin{aligned} & \{s \in \mathbb{R} : (g_1(x), \dots, g_n(x), s) \in U_{n+1}\} \\ &= \{s \in \mathbb{R} : (g_0(x), g_1(x), \dots, g_n(x), s) \in \mathcal{M}_{n+1}^\circ\} \end{aligned}$$

is non-empty for each $x \in X$ (by Lemma 4.5), and so we can use Lemma 4.8 to find a continuous function $f: X \rightarrow \mathbb{R}$ such that $(g_1(x), \dots, g_n(x), f(x))$ belongs to U_{n+1} for all $x \in X$. By compactness of X , continuity of the functions g_1, \dots, g_n, f , and because U_{n+1} is open, there exists $\delta > 0$ such that $(g_1(x), \dots, g_n(x), h(x))$ belongs to U_{n+1} for all $x \in X$ whenever $\|f - h\|_\infty < \delta$. As G is dense in $C(X, \mathbb{R})$ we can find $g_{n+1} \in G$ with $\|f - g_{n+1}\|_\infty < \delta$, and this function has the desired properties.

As in the proof of Proposition 4.2, since $(g_0(x), g_1(x), g_2(x))$ belongs to \mathcal{M}_2° , we get $g_2(x) < g_1(x)$, which in turns implies that the moment sequence $(g_0(x), g_1(x), g_2(x), \dots)$ is non-trivial for every $x \in X$. \square

Proposition 4.10 *With X the Cantor set, let G be a norm-dense subgroup of $C(X, \mathbb{R})$ that contains the constant function 1. There exists a faithful homomorphism of ordered groups from the dimension group H associated with the Pascal triangle to G (with the strict pointwise order) that takes the distinguished order unit v of H into the constant function 1.*

Proof. Choose a sequence g_0, g_1, g_2, \dots in G as specified in Proposition 4.9, and consider the (unique) system $\{g(n, k)\}_{0 \leq k \leq n}$ in G such that

$$g(n+1, k) + g(n+1, k+1) = g(n, k), \quad g(n, n) = g_n$$

for $n \geq 0$ and $0 \leq k \leq n$. Use Proposition 4.2 and the non-triviality of the moment sequence $(g_0(x), g_1(x), g_2(x), \dots)$ to conclude that $g(n, k)(x) > 0$ for all $x \in X$. Hence, by Proposition 4.3, there exists a homomorphism of ordered groups $\varphi: H \rightarrow G$ such that $\varphi([e(n, k)]) = g(n, k)$ for all $n \geq 0$ and $0 \leq k \leq n$.

Each function $g(n, k)$ is strictly positive, and hence non-zero, so it follows from Proposition 4.3 that φ is faithful. \square

Proof of Theorem 4.1. By Corollary 2.4 it suffices to find a unital embedding of the GICAR-algebra B into the AF-algebra $P = \bigotimes_{n=1}^\infty M_2 \oplus M_3$. The ordered K_0 -group of P is (isomorphic to) a dense subgroup G of $C(X, \mathbb{R})$ which contains the constant function 1 (as shown immediately after Corollary 2.4). The existence of a unital embedding of the GICAR-algebra into the AF-algebra P now follows from Proposition 4.10. \square

5 Properties of initial objects

We shall show in this last section that initial objects in the sense of this paper, although abundant, form at the same time a rather special class of C^* -algebras.

An element g in an abelian group G will be said to be *infinitely divisible* if the set of natural numbers n for which the equation $nh = g$ has a solution $h \in G$ is unbounded.

Proposition 5.1 *If B is an initial object, then $K_0(B)^+$ contains no non-zero infinitely divisible elements.*

Proof. There exists a unital C^* -algebra A of real rank zero and with no non-zero finite-dimensional representations, such that no non-zero element in $K_0(A)$ is infinitely divisible, and such that any non-zero projection has a non-zero class in $K_0(A)$. (For example, any irrational rotation C^* -algebra.) If B is an initial object, then B embeds into A , and by choice of A the corresponding ordered group homomorphism $K_0(B) \rightarrow K_0(A)$ takes any non-zero positive element of $K_0(B)$ into a non-zero positive element of $K_0(A)$. Since the image of an infinitely divisible element is again infinitely divisible, no non-zero element of $K_0(B)^+$ can be infinitely divisible. \square

Lemma 5.2 *Let (G, G^+, u) be an ordered abelian group with order unit. Let p_1, p_2, \dots, p_n be distinct primes and suppose that f_1, \dots, f_n are states on (G, G^+, u) such that $f_j(G) = \mathbb{Z}[1/p_j]$ for $j = 1, \dots, n$. Then f_1, \dots, f_n are affinely independent.*

Proof. The assertion is proved by induction on n . It suffices to show that for every natural number n , for every set of distinct primes p_1, \dots, p_n, q , and for every set of states f_1, \dots, f_n, f on (G, G^+, u) , with $f_j(G) = \mathbb{Z}[1/p_j]$ and $f(G) = \mathbb{Z}[1/q]$ and with f_1, \dots, f_n affinely independent, f is not an affine combination of f_1, \dots, f_n .

Suppose, to reach a contradiction, that $f = \alpha_1 f_1 + \dots + \alpha_n f_n$, with $\alpha_1, \dots, \alpha_n$ real numbers with sum 1. If $n = 1$, then $f = f_1$, which clearly is impossible. Consider the case $n \geq 2$. Since f_1, \dots, f_n are assumed to be affinely independent, there are $g_1, \dots, g_{n-1} \in G$ such that the vectors

$$x_j = (f_j(g_1), f_j(g_2), \dots, f_j(g_{n-1})) \in \mathbb{Q}^{n-1}, \quad j = 1, 2, \dots, n,$$

are affinely independent. The coefficients α_j above therefore constitute the unique solution to the equations

$$\begin{aligned} f_1(g_j)\alpha_1 + f_2(g_j)\alpha_2 + \cdots + f_n(g_j)\alpha_n &= f(g_j), & j = 1, 2, \dots, n-1, \\ \alpha_1 + \cdots + \alpha_n &= 1. \end{aligned}$$

As these n equations in the n unknowns $\alpha_1, \alpha_2, \dots, \alpha_n$ are linearly independent, and all the coefficients are rational, also $\alpha_1, \alpha_2, \dots, \alpha_n$ must be rational.

Denote by $\mathcal{Q}'(q)$ the ring of all rational numbers with denominator (in reduced form) not divisible by q . Observe that $f_j(g) \in \mathcal{Q}'(q)$ for all $j = 1, \dots, n$ and for all $g \in G$. There is a natural number k such that $q^k \alpha_j \in \mathcal{Q}'(q)$ for all $j = 1, \dots, n$. Then

$$q^k f(g) = q^k \alpha_1 f_1(g) + \cdots + q^k \alpha_n f_n(g) \in \mathcal{Q}'(q),$$

for all $g \in G$. But this is impossible as, by hypothesis, $f(g) = 1/q^{k+1}$ for some $g \in G$. \square

Proposition 5.3 *Let B be an initial object (in the sense of Definition 2.1), and suppose that no quotient of B has a minimal non-zero projection. Then the trace simplex $T(B)$ of B is necessarily infinite-dimensional.*

It follows in particular that any simple unital C^* -algebra of real rank zero, other than \mathbb{C} , which is an initial object has infinite-dimensional trace simplex. (Note for this that no matrix algebra M_n with $n \geq 2$ is an initial object.)

Proof. Any initial object embeds by definition into a large class of C^* -algebras that includes exact C^* -algebras (such as for example any UHF-algebra), and is therefore itself exact, being a sub- C^* -algebra of an exact C^* -algebra (see [9, Proposition 7.1]). It follows (from [2] and [6], or from [7]) that the canonical affine map from the trace simplex $T(B)$ to the state space of $(K_0(B), K_0(B)^+, [1])$ is surjective. It is therefore sufficient to show that the latter space is infinite-dimensional. For each prime p there is a unital embedding of B into the UHF-algebra of type p^∞ , and hence a homomorphism of ordered groups $f_p: K_0(B) \rightarrow \mathbb{Z}[1/p]$ with $f_p([1]) = 1$. Let us show that the homomorphisms f_p , when considered as states (i.e., homomorphisms of ordered groups with order unit from $(K_0(A), [1])$ to $(\mathbb{R}, 1)$), are affinely independent.

For each prime number p , the image of f_p is a subgroup of $\mathbb{Z}[1/p]$ which contains 1, but the only such subgroups are $\mathbb{Z}[1/p]$ itself and the subgroups

$p^{-k}\mathbb{Z}$ for some $k \geq 0$. The latter cannot be the image of f_p because the image of B in our UHF-algebra, being isomorphic to a quotient of B , is assumed to have no minimal non-zero projections. (Indeed, if $\{p_n\}$ is a strictly decreasing sequence of projections in the sub-algebra of the UHF-algebra, and if τ is the tracial state on the UHF-algebra, then $\{\tau(p_n - p_{n+1})\}$ is a sequence of strictly positive real numbers which converges to 0.)

Hence $f_p(K_0(B)) = \mathbb{Z}[1/p]$ for each prime p . It now follows from Lemma 5.2 that the states $\{f_p : p \text{ prime}\}$ are affinely independent. This shows that the state space of $(K_0(B), K_0(B)^+, [1])$ is infinite-dimensional, as desired. \square

We end our paper by raising the following question:

Problem 5.4 Characterise initial objects (in the sense of Definition 2.1) among (simple) unital AF-algebras.

We could of course extend the problem above to include all (simple) real rank zero C^* -algebras, but we expect a nice(r) answer when we restrict our attention to AF-algebras. Propositions 5.1 and 5.3 give necessary, but not sufficient, conditions for being an initial object. (A simple AF-algebra that satisfies the conditions of Propositions 5.1 and 5.3 can contain a unital simple sub-AF-algebra that does not satisfy the condition in Proposition 5.3, and hence is not an initial object.)

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