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PROJECTIONS IN FREE PRODUCT C*-ALGEBRAS

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Abstract

Consider the reduced free product of C*-algebras, $(A,\varphi) = (A_1,\varphi_1) * (A_2,\varphi_2)$, with respect to states φ_1 and φ_2 that are faithful. If φ_1 and φ_2 are traces, if the so–called Avitzour conditions are satisfied, (i.e. A_1 and A_2 are not "too small" in a specific sense) and if A_1 and A_2 are nuclear, then it is shown that the positive cone, $K_0(A)^+$, of the K_0–group of A consists of those elements $g \in K_0(A)$ for which $g = 0$ or $K_0(\varphi)(g) > 0$. Thus, the ordered group $K_0(A)$ is weakly unperforated.

If, on the other hand, φ_1 or φ_2 is not a trace and if a certain condition weaker than the Avitzour conditions holds, then A is properly infinite.

Introduction and statement of the main results

The reduced free product of C*-algebras [Vo1], [Av], (see also the book [VoDN]), is the appropriate construction in Voiculescu's theory of freeness [Vo1], [VoDN]. Given unital C*-algebras A_1 and A_2 with states φ_1 and, respectively, φ_2, whose GNS representations are faithful, we denote the corresponding reduced free product by

$$(A,\varphi) = (A_1,\varphi_1) * (A_2,\varphi_2)\,. \tag{1}$$

Recall that φ is a trace if and only if both φ_1 and φ_2 are traces. Moreover, [D2], φ is faithful if and only if both φ_1 and φ_2 are faithful.

The K-theory of A can be calculated, at least when A_1 and A_2 are nuclear, by the following theorem of Emmanuel Germain.

Theorem 1 ([G1,2]). *Let A be a reduced free product C*-algebra as in (1), and suppose that A_1 and A_2 are nuclear. Then there is an exact sequence of K–groups,*

$$\begin{array}{ccccccc}
\mathbf{Z} \cong K_0(\mathbf{C}) & \xrightarrow{(K_0(i_1),-K_0(i_2))} & K_0(A_1)\oplus K_0(A_2) & \xrightarrow{K_0(j_1)+K_0(j_2)} & K_0(A) & & \\
\uparrow & & & & \downarrow & & \\
K_1(A) & \xleftarrow{K_1(j_1)+K_1(j_2)} & K_1(A_1)\oplus K_1(A_2) & \xleftarrow{(K_1(i_1),-K_1(i_2))} & K_1(\mathbf{C}) & = 0\,, &
\end{array}$$

*where $i_k\colon \mathbf{C} \to A_k$ is the unital *-homomorphism and where $j_k\colon A_k \to A$ is the unital embedding arising from the construction of the reduced free product* (1).

It follows in particular that

$$K_0(A) = K_0(j_1)(K_0(A_1)) + K_0(j_2)(K_0(A_2)) \tag{2}$$

whenever A_1 and A_2 are nuclear.

Let us consider the case where both φ_1 and φ_2 are traces. The pair $(A_1, \varphi_1), (A_2, \varphi_2)$ is then said to satisfy the Avitzour condition (cf. [Av]) if there exist unitaries

$$a \in A_1, \qquad b, c \in A_2 \tag{3}$$

which satisfy

$$\varphi_1(a) = 0\,, \qquad \varphi_2(b) = \varphi_2(c) = 0 = \varphi_2(b^*c)\,. \tag{4}$$

If this is the case, then A is simple and φ is the unique tracial state on A ([Av]). It was proved in [DHR] that under the same conditions, A is of stable rank one.

Let $P(A)$ denote the set of projections in A, set $P_n(A) = P(M_n(A))$, and set $P_\infty(A) = \bigcup_{n=1}^\infty P_n(A)$. Murray–von Neumann equivalence of projections yields an equivalence relation on $P_\infty(A)$, denoted by $\sim$. (If $p \in M_n(A)$ and $q \in M_m(A)$, then $p \sim q$ if there exists $v \in M_{n,m}(A)$ such that $vv^* = p$ and $v^*v = q$.) Moreover, if $p, q \in P_\infty(A)$, then $p \precsim q$ if p is equivalent to a subprojection of q.

By the definition of the K_0-group of a (unital) C*-algebra, we have a map $[\,\cdot\,]_0\colon P_\infty(A) \to K_0(A)$, and $[p]_0 = [q]_0$ if $p \sim q$. The image of this map is called *the positive cone* of $K_0(A)$, and is denoted by $K_0(A)^+$. The *scale* of $K_0(A)$ is defined to be the subset $\{\,[p]_0 \mid p \in P(A)\}$ of $K_0(A)^+$, and it is denoted by $\Sigma(A)$. In the present case, where A is unital and has a faithful trace, $(K_0(A), K_0(A)^+)$ is an ordered abelian group (with $g \le h$ if $h - g \in K_0(A)^+$). Notice that the trace φ on A induces a state $K_0(\varphi)\colon K_0(A) \to \mathbf{R}$ given by $K_0(\varphi)([p]_0 - [q]_0) = \varphi(p) - \varphi(q)$. Notice also that $[p]_0 \le [q]_0$ if $p \precsim q$.

The ordered group $K_0(A)$ is said to be *weakly unperforated* if whenever $ng \ge 0$ for some positive integer n and some $g \in K_0(A)$ it follows that $g \ge 0$.

The order structure of the K_0-group of the Choi algebra and its generalizations, $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$, $n \ge 2$, $m \ge 3$, was considered in [ABH]. They proved that for each $t \in \left(\frac{1}{n}\mathbf{Z} + \frac{1}{m}\mathbf{Z}\right) \cap (0,1)$ there is a projection in

$C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$ of trace t. They could not decide whether all elements $g \in K_0(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m))$ of positive trace can be realized by projections, (i.e. are positive) or if $K_0(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m))$ is weakly unperforated. Theorem 2 below answers both of these questions in the affirmative.

Jesper Villadsen [V] has recently proven that not all simple C*-algebras have weakly unperforated K_0-groups. This shows that it will be hard (if not impossible) to obtain general theorems about the positive cone of the K_0-group of arbitrary simple C*-algebras. In the light of Villadsen's example and of our theorem below, we conclude that the class of reduced free product C*-algebras (that we are considering) behaves rather nicely.

Theorem 2. *Let A be a reduced free product C*-algebra as in* (1), *where φ_1 and φ_2 are faithful traces. Suppose that there are unitaries as in* (3) *which satisfy* (4), *and suppose that* (2) *holds* (*which is the case if A_1 and A_2 are nuclear*). *It follows that*

$$K_0(A)^+ = \{g \in K_0(A) \mid K_0(\varphi)(g) > 0\} \cup \{0\}$$

and

$$\Sigma(A) = \{g \in K_0(A) \mid 0 < K_0(\varphi)(g) < 1\} \cup \{0, 1\}\,.$$

It follows easily from this theorem that $K_0(A)$ is weakly unperforated. Moreover,

$$K_0(\varphi)(K_0(A)) = K_0(\varphi_1)(K_0(A_1)) + K_0(\varphi_2)(K_0(A_2)) \subseteq \mathbf{R}\,.$$

If we set $\Gamma = K_0(\varphi)(K_0(A)) \subseteq \mathbf{R}$ and let G be the subgroup $\{g \in K_0(A) \mid K_0(\varphi)(g) = 0\}$ of $K_0(A)$, then we have a short exact sequence

$$0 \to G \to K_0(A) \to \Gamma \to 0\,,$$

and the positive cone of $K_0(A)$ consists precisely of those elements g for which either $g = 0$ or g is mapped into $\Gamma \cap (0, \infty)$. The group G is the infinitesimal subgroup of $K_0(A)$.

It is a consequence of A being of stable rank one, that the map $[\,\cdot\,]_0 : P_\infty(A) \to K_0(A)$ induces an injection (and hence a bijection)

$$P_\infty(A)/\!\sim\; \longrightarrow K_0(A)^+\,.$$

Under the conditions of Theorem 2 this bijection is an order isomorphism when $P_\infty(A)/\sim$ is equipped with the order relation $\precsim$ and when $K_0(A)^+$ is equipped with the algebraic order relation (as above). Thus, by using Theorem 2 to find $K_0(A)^+$ and $\Sigma(A)$, we can also classify Murray–von Neumann equivalence classes of projections in A and find their ordering.

Remark 3. The assumption in Germain's Theorem above that A_1 and A_2 are nuclear can be loosened somewhat to the assumption that A_k are

K-nuclear in the GNS representation of φ_k $(k = 1, 2)$. For the definition of this concept and the properties mentioned below, see [G3, §4]. A C*-algebra A is K-nuclear in a given representation π if either A is nuclear or if $A = C_r^*(G)$ for a K-amenable group G and π is the reduced regular representation. Moreover, if $(A, \varphi) = (A_1, \varphi_1) * (A_2, \varphi_2)$ where each A_j is K-nuclear in the GNS representation of φ_j, then A is K-nuclear in the GNS representation of φ. Thus, Theorem 2 also applies to the case of the reduced free product of finitely many C*-algebras.

Consider now the case where either φ_1 or φ_2 is not a trace. Then, as remarked above, φ is not a trace. If there are unitaries as in (3) satisfying (4) and if a and b are in the centralizer of φ_1, respectively, φ_2, then by Avitzour's theorem ([Av]), A is simple and does not admit a tracial state.

Recall that a projection p in a C*-algebra A is said to be infinite is it is equivalent to a proper subprojection of itself. If there exist two mutually orthogonal subprojections p_1 and p_2 of p such that $p \sim p_1 \sim p_2$, then p is said to be *properly infinite*. A unital C*-algebra A is called infinite, respectively, properly infinite, if its unit is infinite, respectively, properly infinite. Note that no infinite C*-algebra has a faithful, tracial state and that no properly infinite C*-algebra has a tracial state. A simple unital C*-algebra is said to be *purely infinite* if all its non-zero hereditary subalgebras contain an infinite projection (see [C3]). It is an open problem whether every simple unital C*-algebra, that does not admit a trace, must be purely infinite. The theorem below should be viewed in the light of that question.

Theorem 4. *Let A be a reduced free product C*-algebra as in* (1). *Suppose that φ is faithful and is not a trace and suppose there are unitaries as in* (3) *satisfying* (4). *Then A is properly infinite. Hence*

$$\Sigma(A) = K_0(A)^+ = K_0(A)\,. \tag{5}$$

(Note that we assume less than Avitzour required, because we do not ask unitaries to be in the centralizers of states.)

When proving these theorems, we define and make use of the intermediate concept of *eigenfree* C*-algebras. Although the only C*-algebras we prove are eigenfree are the reduced free product C*-algebras having unitaries as required in Theorems 2 and 4, we believe that more general reduced free product C*-algebras are eigenfree. We have written our proofs in terms of eigenfreeness in order to allow for easy use in possible generalizations.

1 Preliminaries

Basic to our investigation of projections in free product C*-algebras is the following result, which can be found in [ABH], and can also be derived from Voiculescu's multiplicative free convolution [Vo2]. See [D3, 2,7] for a related result and see [D4] for a detailed proof.

PROPOSITION 1.1. *Let A be a C*-algebra and φ a faithful state on A having faithful GNS representation. Suppose $p, q \in A$ are projections that are free with respect to φ. If $\varphi(p) < \varphi(q)$ then $||p(1-q)|| < 1$ and there is $v \in A$ such that $v^*v = p$ and $vv^* \leq q$.*

Another useful and well-known fact is the following. (The proof is a particularly easy example of free etymology.) As usual, we employ the notation, for subsets $S_1, S_2, \ldots, S_m$ of A,

$$\Lambda^o(S_1, S_2, \ldots, S_m)$$
$$\stackrel{\text{def}}{=} \{a_1 a_2 a_3 \cdots a_n \mid n \in \mathbf{N},\, a_j \in S_{\iota_j},\, \iota_1 \neq \iota_2, \iota_2 \neq \iota_3, \ldots, \iota_{n-1} \neq \iota_n\}.$$

LEMMA 1.2. *Let (A, φ) be a C*-noncommutative probability space and let B be a subset of A. Suppose $u \in A$ is a unitary, that $\varphi(u) = 0$ and that B and $\{u, u^*\}$ are free. Then B and u^*Bu are free.*

Proof. We may suppose B is a subalgebra of A containing the unit of A. By the freeness assumption, for $b \in B$, $\varphi(u^*bu) = 0$ if and only if $\varphi(b) = 0$. To show that B and u^*Bu are free it thus suffices to show that

$$\varphi\big(b_1(u^*b_2u)b_3(u^*b_4u)\cdots b_{n-2}(u^*b_{n-1}u)b_n\big) = 0 \tag{6}$$

whenever n is odd, $n \geq 3$, $b_j \in B$, $\varphi(b_j) = 0$ $(2 \leq j \leq n-1)$, and for $j = 1$ and $j = n$ either $b_j = 1$ or $\varphi(b_j) = 0$. However, the word $b_1u^*b_2ub_3\cdots u^*b_{n-1}ub_n$ is easily seen to equal an alternating product in $\{b \in B \mid \varphi(b) = 0\}$ and $\{u, u^*\}$. By freeness of B and $\{u, u^*\}$, (6) holds. □

LEMMA 1.3. *Let I be an index set and let (A_ι, φ_ι) be a C*-noncommutative probability space $(\iota \in I)$, where each φ_ι is faithful. Let (B, ψ) be a C*-noncommutative probability space with ψ faithful. Let*

$$(A, \varphi) = \mathop{*}_{\iota \in I}(A_\iota, \varphi_\iota)$$

be the reduced free product C-algebra. Given unital *-homomorphisms, $\pi_\iota : A_\iota \to B$, such that $\psi \circ \pi_\iota = \varphi_\iota$ and $(\pi_\iota(A_\iota))_{\iota \in I}$ is free in (B, ψ), there is a *-homomorphism, $\pi : A \to B$ such that $\pi|_{A_\iota} = \pi_\iota$ and $\psi \circ \pi = \varphi$.*

Proof. Let B_0 be the C*-subalgebra of B generated by $\bigcup_{\iota \in I} \pi_\iota(A_\iota)$ and let $\psi_0 = \psi|_{B_0}$. If the GNS representation associated to ψ_0 is faithful on

B_0, then by Voiculescu's construction [Vo1], (see also [VoDN]), (B_0, ψ_0) is canonically isomorphic to

$$\mathop{*}_{\iota \in I} (A_\iota, \varphi_\iota)$$

via an isomorphism having the desired properties. The faithfulness of ψ, however, shows that ψ_0 is faithful, hence has faithful GNS representation. □

The following example shows that the hypothesis that ψ be faithful is essential, and cannot be replaced with the weaker hypothesis that the GNS representation of ψ be faithful.

EXAMPLE 1.4. Using the notation of [D3], let

$$(B_1, \psi_1) \stackrel{\text{def}}{=} \Big(\underset{3/4}{\overset{p}{\mathbf{C}}} \oplus \underset{1/4}{\mathbf{C}} \Big) * \Big(\underset{2/3}{\overset{q}{\mathbf{C}}} \oplus \underset{1/3}{\mathbf{C}} \Big) .$$

Let $B = M_2(B_1)$ and let ψ be the state on $M_2(B_1)$ given by $\psi\big(\begin{smallmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{smallmatrix}\big) = \psi_1(b_{11})$. Although ψ is not faithful, clearly the GNS representation of ψ is faithful on B. Let

$$(A_1, \varphi_1) = \Big(\underset{3/4}{\overset{r}{\mathbf{C}}} \oplus \underset{1/4}{\mathbf{C}} \Big) ,$$

$$(A_2, \varphi_2) = \Big(\underset{2/3}{\overset{s}{\mathbf{C}}} \oplus \underset{1/3}{\mathbf{C}} \Big) ,$$

and

$$(A, \varphi) \stackrel{\text{def}}{=} (A_1, \varphi_1) * (A_2, \varphi_2) .$$

Then s is Murray–von Neumann equivalent in A to a proper subprojection of r. Let $\rho_j : A_j \to B$ be the unital *-homomorphisms such that $\rho_1(r) = \big(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\big)$ and $\rho_2(s) = \big(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\big)$. Then $\psi \circ \rho_j = \varphi_j$. But there cannot be a *-homomorphism $\rho \colon A \to B$ such that $\rho(r) = \rho_1(r)$ and $\rho(s) = \rho_2(s)$, because $\rho_2(s)$ is not equivalent to a subprojection of $\rho_1(r)$.

2 Matrices of Free Random Variables

Theorem 2.1. *Let $n \in \mathbf{N}$ and suppose (A, φ) is a *-noncommutative probability space having random variables $x_{ij} \in A$ $(1 \le i \le j \le n)$ and a unital subalgebra $B \subseteq A$ such that*

(i) *x_{ii} is a semicircular element with $\varphi(x_{ii}^2) = 1$ $(1 \le i \le n)$;*
(ii) *x_{ij} is a circular element with $\varphi(x_{ij}^* x_{ij}) = 1$ $(1 \le i < j \le n)$;*
(iii) *the family of sets of random variables,*

$$\big(B, (\{x_{ii}\})_{1 \le i \le n}, (\{x_{ij}^*, x_{ij}\})_{1 \le i < j \le n} \big) \tag{7}$$

is free.

Let $(e_{ij})_{1\le i,j\le n}$ be a system of matrix units for $M_n(\mathbf{C})$. Consider the noncommutative probability space $(A\otimes M_n(\mathbf{C}),\varphi\otimes\mathrm{tr}_n)$, where tr_n is the tracial state on $M_n(\mathbf{C})$. Consider also the random variable

$$x=\frac{1}{\sqrt{n}}\bigg(\sum_{1\le i\le n}x_{ii}\otimes e_{ii}+\sum_{1\le i<j\le n}(x_{ij}\otimes e_{ij}+x_{ij}^*\otimes e_{ji})\bigg)\,.$$

Then x is semicircular and $\{x\}$ and $B\otimes M_n(\mathbf{C})$ are free.

Proof. Voiculescu, using his matrix model [Vo4], proved in [Vo3] that x is semicircular. Moreover, from [D1] it follows that $\{x\}$ and $1\otimes M_n(\mathbf{C})$ are free.

Let 1_n denote the unit of $M_n(\mathbf{C})$ and let id_n denote the identity map on $M_n(\mathbf{C})$. For $p\in\mathbf{N}$ and $a\in A\otimes M_n(\mathbf{C})$, let

$$\begin{aligned}[x^p]_{\varphi\otimes\mathrm{tr}}&=x^p-(\varphi\otimes\mathrm{tr}_n)(x^p)\cdot(1\otimes 1_n)\,,\\ [a]_{\varphi\otimes\mathrm{id}}&=a-(\varphi\otimes\mathrm{id}_n)(a)\,.\end{aligned}$$

LEMMA 2.2. *Let $m\in\mathbf{N}$ and $p_1,p_2,\ldots,p_m\in\mathbf{N}$. Let $d_0,d_1,\ldots,d_m\in 1\otimes M_n(\mathbf{C})$ and assume that $(\mathrm{id}\otimes\mathrm{tr}_n)(d_j)=0$ whenever $1\le j\le m-1$. Let*

$$y=d_0[x^{p_1}]_{\varphi\otimes\mathrm{tr}}d_1[x^{p_2}]_{\varphi\otimes\mathrm{tr}}\cdots d_{m-1}[x^{p_m}]_{\varphi\otimes\mathrm{tr}}d_m\,. \tag{8}$$

Then $(\varphi\otimes\mathrm{id}_n)(y)=0$.

Note that $(\varphi\otimes\mathrm{tr}_n)(y)=0$ by freeness of $\{x\}$ and $1\otimes M_n(\mathbf{C})$. The lemma gives more, namely that every matrix entry of y has zero expectation.

Proof. To show that $(\varphi\otimes\mathrm{id}_n)(y)=0$ it will suffice to show that

$$(\varphi\otimes\mathrm{tr}_n)\big((1\otimes e_{1i})y(1\otimes e_{j1})\big)=0\,. \tag{9}$$

whenever $1\le i,j\le n$. But absorbing $1\otimes e_{1i}$ into d_0 and $1\otimes e_{j1}$ into d_m, we see that $(1\otimes e_{1i})y(1\otimes e_{j1})$ is a word having the same form as y. As mentioned above, the freeness of $\{x\}$ and $1\otimes M_n(\mathbf{C})$ then implies (9). □

Continuing with the proof of Theorem 2.1, take arbitrary $m\in\mathbf{N}$, $p_1,p_2,\ldots,p_m\in\mathbf{N}$ and $a_0,a_1,\ldots,a_m\in B\otimes M_n(\mathbf{C})$ such that $(\varphi\otimes\mathrm{tr}_n)(a_j)=0$ for every $1\le j\le m-1$. To prove the theorem it will suffice to show that $(\varphi\otimes\mathrm{tr}_n)(z)=0$, where

$$z=a_0[x^{p_1}]_{\varphi\otimes\mathrm{tr}}a_1[x^{p_2}]_{\varphi\otimes\mathrm{tr}}\cdots a_{m-1}[x^{p_m}]_{\varphi\otimes\mathrm{tr}}a_m\,.$$

Writing $a_j=[a_j]_{\varphi\otimes\mathrm{id}}+(\varphi\otimes\mathrm{id}_n)(a_j)$ and distributing, we write z as a sum of 2^{m+1} terms,

$$\begin{aligned}
z &= [a_0]_{\varphi\otimes\mathrm{id}}[x^{p_1}]_{\varphi\otimes\mathrm{tr}}[a_1]_{\varphi\otimes\mathrm{id}}[x^{p_2}]_{\varphi\otimes\mathrm{tr}}\cdots[a_{m-1}]_{\varphi\otimes\mathrm{id}}[x^{p_m}]_{\varphi\otimes\mathrm{tr}}[a_m]_{\varphi\otimes\mathrm{id}} \\
&\quad + [a_0]_{\varphi\otimes\mathrm{id}}[x^{p_1}]_{\varphi\otimes\mathrm{tr}}[a_1]_{\varphi\otimes\mathrm{id}}[x^{p_2}]_{\varphi\otimes\mathrm{tr}}\cdots[a_{m-1}]_{\varphi\otimes\mathrm{id}}[x^{p_m}]_{\varphi\otimes\mathrm{tr}}(\varphi\otimes\mathrm{id}_n)(a_m) \\
&\quad + \quad \cdots \quad + \\
&\quad + (\varphi\otimes\mathrm{id}_n)(a_0)[x^{p_1}]_{\varphi\otimes\mathrm{tr}}(\varphi\otimes\mathrm{id}_n)(a_1)[x^{p_2}]_{\varphi\otimes\mathrm{tr}} \\
&\qquad\qquad \cdots(\varphi\otimes\mathrm{id}_n)(a_{m-1})[x^{p_m}]_{\varphi\otimes\mathrm{tr}}[a_m]_{\varphi\otimes\mathrm{id}} \\
&\quad + (\varphi\otimes\mathrm{id}_n)(a_0)[x^{p_1}]_{\varphi\otimes\mathrm{tr}}(\varphi\otimes\mathrm{id}_n)(a_1)[x^{p_2}]_{\varphi\otimes\mathrm{tr}}\cdots \\
&\qquad\qquad \cdots(\varphi\otimes\mathrm{id}_n)(a_{m-1})[x^{p_m}]_{\varphi\otimes\mathrm{tr}}(\varphi\otimes\mathrm{id}_n)(a_m)\,.
\end{aligned}$$

Since each $(\varphi\otimes\mathrm{id}_n)(a_j) \in 1\otimes M_n(\mathbf{C})$ and since if $1 \le j \le m-1$ we have $(\varphi\otimes\mathrm{tr}_n)\circ(\varphi\otimes\mathrm{id}_n)(a_j) = 0$, it follows from the freeness of $\{x\}$ and $1\otimes M_n(\mathbf{C})$ that $\varphi\otimes\mathrm{tr}_n$ of the last term is zero. Each of the remaining $2^{m+1}-1$ terms is of the form

$$t = b_0\left[a_{j(1)}\right]_{\varphi\otimes\mathrm{id}} b_1\left[a_{j(2)}\right]_{\varphi\otimes\mathrm{id}}\cdots b_{k-1}\left[a_{j(k)}\right]_{\varphi\otimes\mathrm{id}} b_k$$

where $1\le k\le m$, $0\le j(1)<j(2)<\cdots<j(k)\le m$ and

$$b_0 = \begin{cases} 1\otimes 1_n & \text{if } j(1)=0 \\ (\varphi\otimes\mathrm{id}_n)(a_0)[x^{p_1}]_{\varphi\otimes\mathrm{tr}}\cdots(\varphi\otimes\mathrm{id}_n)(a_{j(1)-1})[x^{p_{j(1)}}]_{\varphi\otimes\mathrm{tr}} & \text{if } j(1)>0\,, \end{cases}$$

if $1\le l\le k-1$ then

$$\begin{aligned} b_l = [x^{p_{j(l)+1}}]_{\varphi\otimes\mathrm{tr}}(\varphi\otimes\mathrm{id}_n)(a_{j(l)+1})[x^{p_{j(l)+2}}]_{\varphi\otimes\mathrm{tr}} \\ \cdots(\varphi\otimes\mathrm{id}_n)(a_{j(l+1)-1})[x^{p_{j(l+1)}}]_{\varphi\otimes\mathrm{tr}} \end{aligned}$$

and

$$b_k = \begin{cases} [x^{p_{j(k)+1}}]_{\varphi\otimes\mathrm{tr}}(\varphi\otimes\mathrm{id}_n)(a_{j(k)+1})[x^{p_{j(k)+2}}]_{\varphi\otimes\mathrm{tr}} & \\ \qquad\cdots(\varphi\otimes\mathrm{id}_n)(a_{m-1})[x^{p_m}]_{\varphi\otimes\mathrm{tr}}(\varphi\otimes\mathrm{id}_n)(a_m) & \text{if } j(k)<m \\ 1\otimes 1_n & \text{if } j(k)=m\,. \end{cases}$$

From Lemma 2.2, we see that $(\varphi\otimes\mathrm{id}_n)(b_l) = 0$ whenever $0\le l\le k$, except when $l=0$ and $j(0)=0$ or when $l=k$ and $j(k)=m$ (i.e. when $b_0 = 1\otimes 1_n$ or $b_k = 1\otimes 1_n$). Thus, excepting the cases just mentioned, every matrix entry of each b_l belongs to the $*$-subalgebra of A generated by $\{x_{ij} \mid 1\le i\le j\le n\}$ and evaluates to 0 under φ. In addition, every matrix entry of each $[a_j]_{\varphi\otimes\mathrm{id}}$ belongs to B and evaluates to zero under φ. Hence it follows from the freeness of (7) that every matrix entry of t evaluates to zero under φ. Therefore, each $(\varphi\otimes\mathrm{tr}_n)(t)=0$ and hence $(\varphi\otimes\mathrm{tr}_n)(z)=0$. □

3 Eigenfreeness

DEFINITION 3.1. Let A be a unital C*-algebra with state φ. We say that (A, φ) is *eigenfree* if there is a unital *-endomorphism, ρ, of A and a Haar unitary with respect to φ, $u \in A$, such that $\rho(A)$ and $\{u\}$ are *-free in (A, φ) and $\varphi \circ \rho = \varphi$.

REMARK. Actually, one could weaken the definition somewhat by requiring only $\varphi(u) = 0$ (and not assuming that u is a Haar unitary). For then $u\rho(u)$ is a Haar unitary, and Definition 3.1 is satisfied with the unitary $u\rho(u)$ and the endomorphism ρ^2.

PROPOSITION 3.2. *Let $A_1 \neq \mathbf{C}$ and $A_2 \neq \mathbf{C}$ be unital C*-algebras having faithful states φ_1 and, respectively, φ_2. Let*

$$(A, \varphi) = (A_1, \varphi_1) * (A_2, \varphi_2)$$

be the C-algebra reduced free product. If there are unitaries $a \in A_1$ and $b, c \in A_2$ such that $\varphi_1(a) = 0$ and $\varphi_2(b) = \varphi_2(c) = 0 = \varphi_2(b^*c)$ then (A, φ) is eigenfree by an endomorphism ρ such that $K_0(\rho)\colon K_0(A) \to K_0(A)$ restricts to the identity map on the image of $K_0(A_j) \to K_0(A)$ $(j = 1, 2)$.*

Proof. Let ρ_k $(k = 1, 2)$, be the *-homomorphisms

$$\rho_k\colon A_k \to A$$

defined by $\rho_1(x) = a^*bxb^*a$ and $\rho_2(y) = ba^*yab^*$. Then each ρ_k is injective. Moreover, since $\varphi_1(x) = 0$ implies $\varphi(a^*bxb^*a) = 0$ and $\varphi_2(y) = 0$ implies $\varphi(ba^*yab^*) = 0$, we have $\varphi \circ \rho_k = \varphi_k$ $(k = 1, 2)$. Let us show that $\rho_1(A_1)$ and $\rho_2(A_2)$ are free. It will suffice to show that $\varphi(z) = 0$ whenever $z \in \Lambda^\circ(\rho_1(A_1^\circ), \rho_2(A_2^\circ))$. But for such a word, z, no cancellation occurs and we see that z is equal to an element of $\Lambda^\circ(A_1^\circ, A_2^\circ)$, so by freeness, $\varphi(z) = 0$. By [D2], the state φ is faithful. It then follows from Lemma 1.3 that there is a *-endomorphism, ρ, of A such that $\rho \circ j_k = \rho_k$ $(k = 1, 2)$ and $\varphi \circ \rho = \varphi$.

Let $u = a^*cac^*$. Clearly u is a Haar unitary in (A, φ). We now show that $\{u\}$ and $\rho(A)$ are *-free in (A, φ), which will complete the proof that (A, φ) is eigenfree. By the freeness of $\rho_1(A_1)$ and $\rho_2(A_2)$, $\operatorname{span}\Lambda^\circ(\rho_1(A_1^\circ), \rho(A_2^\circ))$ is dense in $\rho(A)^\circ$. Therefore, it will suffice to show that $\varphi(z) = 0$ whenever

$$z \in \Lambda^\circ\big(a^*b(A_1^\circ)b^*a,\ ba^*(A_2^\circ)ab^*,\ \{u^n \mid n \in \mathbf{N}\} \cup \{(u^*)^n \mid n \in \mathbf{N}\}\big).$$

Expand each u as a^*cac^* and each u^* as ca^*c^*a. Now the only cancellations which may occur are

$$\begin{aligned}(a^*bxb^*a)(a^*cac^*) &= a^*bx(b^*c)ac^* \\ (ca^*c^*a)(a^*bxb^*a) &= ca^*(c^*b)xb^*a \\ (a^*cac^*)(ba^*yab^*) &= a^*ca(c^*b)a^*yab^* \\ (ba^*yab^*)(ca^*c^*a) &= ba^*ya(b^*c)a^*c^*a\,,\end{aligned}$$

for $x \in A_1^\circ$ and $y \in A_2^\circ$. Making these cancellations, z is seen to be equal to an element of $\Lambda^\circ(A_1^\circ, A_2^\circ)$, so by freeness $\varphi(z) = 0$.

Since ρ on the copies of A_j in A is conjugation by a unitary, we easily see that $K_0(\rho)$ is the identity map on the image of $K_0(A_j) \to K_0(A)$ ($j = 1, 2$). □

PROPOSITION 3.3. *Let A be a C^*-algebra with state φ and let $n \in \mathbf{N}$. If (A, φ) is eigenfree by an endomorphism ρ and some unitary then $(A \otimes M_n(\mathbf{C}), \varphi \otimes \mathrm{tr}_n)$ is eigenfree by the endomorphism $\rho^{n^2} \otimes \mathrm{id}_n$, where tr_n is the tracial state on $M_n(\mathbf{C})$ and id_n is the identity map on $M_n(\mathbf{C})$.*

Proof. Let ρ and u be as in Definition 3.1. Because $\varphi \circ \rho = \varphi$ and using [VoDN, 2.5.5(iii)], the n^2 unitaries,

$$u, \rho(u), \rho^2(u), \ldots, \rho^{n^2-1}(u)\,,$$

are Haar unitaries and the family

$$\big(\rho^{n^2}(A), \{u\}, \{\rho(u)\}, \ldots, \{\rho^{n^2-1}(u)\}\big)$$

is *-free. Using the continuous functional calculus, we find semicircular elements $x_{ii} \in A$ such that $\varphi(x_{ii}^2) = 1$ $(1 \le i \le n)$, and circular elements $x_{ij} \in A$ such that $\varphi(x_{ij}^* x_{ij}) = 1$ $(1 \le i < j \le n)$, such that

$$\big(\rho^{n^2}(A), (\{x_{ii}\})_{1 \le i \le n}, (\{x_{ij}\})_{1 \le i < j \le n}\big)$$

is *-free. Let $(e_{ij})_{1 \le i,j \le n}$ be a system of matrix units for $M_n(\mathbf{C})$ and let

$$x = \frac{1}{\sqrt{n}}\bigg(\sum_{1 \le i \le n} x_{ii} \otimes e_{ii} + \sum_{1 \le i < j \le n} (x_{ij} \otimes e_{ij} + x_{ij}^* \otimes e_{ji})\bigg)\,.$$

Then by Theorem 2.1, x is semicircular and $\rho^{n^2}(A) \otimes M_n(\mathbf{C})$ and $\{x\}$ are free. Using the continuous functional calculus we obtain a Haar unitary, $v \in C^*(\{x\})$, which then satisfies that $\rho^{n^2}(A) \otimes M_n(\mathbf{C})$ and $\{v\}$ are *-free.

Thus the endomorphism $\rho^{n^2} \otimes \mathrm{id}_n$ of $A \otimes M_n(\mathbf{C})$ and the Haar unitary v give that $(A \otimes M_n(\mathbf{C}), \varphi \otimes \mathrm{tr}_n)$ is eigenfree. □

4 The Tracial Case

In this section we prove Theorem 2, which follows from the proposition below, in conjunction with Proposition 3.2.

PROPOSITION 4.1. *Let A be a unital C^*-algebra with faithful, tracial state τ and suppose that (A,τ) is eigenfree by an endomorphism ρ. Suppose that G is a subgroup of $K_0(A)$ on which $K_0(\rho)$ is the identity map. Then*

$$G \cap K_0(A)^+ = \{x \in G \mid K_0(\tau)(x) > 0\} \cup \{0\} \tag{10}$$

and

$$G \cap \Sigma(A) = \{x \in G \mid 0 < K_0(\tau)(x) < 1\} \cup \{0,1\}\,. \tag{11}$$

Proof. Since τ is a faithful trace on A, the inclusion $\subseteq$ is clear in both (10) and (11). To show $\supseteq$ in (10), let $x \in G$ have $K_0(\tau)(x) > 0$. Then there is $n \in \mathbf{N}$ and there are projections $p, q \in A \otimes M_n(\mathbf{C})$ such that $x = [p] - [q]$. We now use Proposition 3.3; let ρ_n denote the endomorphism $\rho^{n^2} \otimes \mathrm{id}_n$ and let $u_n \in A \otimes M_n(\mathbf{C})$ denote the Haar unitary such that $\rho_n(A \otimes M_n(\mathbf{C}))$ and $\{u_n\}$ are *-free. Then $x = [\rho_n(p)] - [\rho_n(q)]$. Thus $(\tau \otimes \mathrm{tr}_n)(\rho_n(p)) > (\tau \otimes \mathrm{tr}_n)(\rho_n(q))$. Moreover, by Lemma 1.2, $\rho_n(p)$ and $u_n^*\rho_n(q)u_n$ are free and clearly the traces of $\rho_n(q)$ and $u_n^*\rho_n(q)u_n$ are the same. So by Proposition 1.1, $u_n^*\rho_n(q)u_n$, and thus also $\rho_n(q)$, is equivalent in $A \otimes M_n(\mathbf{C})$ to a subprojection, say r, of $\rho_n(p)$. Then $x = [\rho_n(p) - r] \in K_0(A)^+$. This proves (10).

The truth of (11) now follows similarly. Indeed, if $x \in G$ satisfies $0 < K_0(\tau)(x) < 1$ then by (10) there is $n \in \mathbf{N}$ and a projection $p \in A \otimes M_n(\mathbf{C})$ such that $x = [p] = [\rho_n(p)]$. Let $e = 1 \otimes e_{11} \in A \otimes M_n(\mathbf{C})$, where e_{11} is a rank-one projection in $M_n(\mathbf{C})$. Then since $K_0(\tau)(x) < 1$ we have $(\tau \otimes \mathrm{tr}_n)(p) < (\tau \otimes \mathrm{tr}_n)(e)$. Clearly $e = \rho_n(e)$. By the same argument as was applied above in the proof of (10), we see that $\rho_n(p)$ is equivalent to a subprojection, say s, of e. Thus $x = [s] \in \Sigma(A)$. □

In fact, the above proposition together with Proposition 3.2 proves the following result, which is more general than Theorem 2.

PROPOSITION 4.2. *Let A be a reduced free product C^*-algebra as in (1), where φ_1 and φ_2 are faithful traces. Suppose that there are unitaries as in (3) which satisfy (4). Consider the subgroup*

$$G = K_0(j_1)(K_0(A_1)) + K_0(j_2)(K_0(A_2)) \subseteq K_0(A)\,.$$

Then

$$G \cap K_0(A)^+ = \{x \in G \mid K_0(\varphi)(x) > 0\} \cup \{0\}$$

and

$$G \cap \Sigma(A) = \{x \in G \mid 0 < K_0(\varphi)(x) < 1\} \cup \{0,1\}.$$

5 The Non-tracial Case

In this section we will prove Theorem 4.

We will make use of the comparison theory for positive elements in a C*-algebra that was introduced by J. Cuntz [C1,2] (see also [R]), and which we describe in the definition and proposition below.

DEFINITION 5.1. Let A be a unital C*-algebra with positive cone denoted A^+ and let $a, b \in A^+$. Write

$$a \precsim b$$

if there is a sequence, $(x_n)_{n=1}^\infty$, in A such that $\lim x_n^* b x_n = a$.

If φ is a state on A, define the function, $D_\varphi \colon A^+ \to [0,1]$ by

$$D_\varphi(a) = \lim_{\epsilon \searrow 0} \varphi(f_\epsilon(a)),$$

where

$$f_\epsilon(t) = \begin{cases} 0 & \text{if } 0 \le t \le \epsilon \\ (t-\epsilon)/\epsilon & \text{if } \epsilon \le t \le 2\epsilon \\ 1 & \text{if } t \ge 2\epsilon. \end{cases}$$

(If φ were a trace then D_φ would be a dimension function.)

PROPOSITION 5.2 ([C1,2], [R]). *Let A be a unital C*-algebra. Then*

- (i) *the relation $\precsim$ on A is transitive and reflexive;*
- (ii) *$\precsim$, when restricted to the projections of A, gives the usual Murray-von Neumann ordering.*

Let $a, b \in A^+$ and $x \in A$. Then

- (iii) *if $f_\epsilon(a) \precsim b$ for every $\epsilon > 0$, then $a \precsim b$;*
- (iv) *if $a \le b$ then $a \precsim b$;*
- (v) *if $f \colon \mathbf{R}_+ \to \mathbf{R}_+$ is a continuous function with $f(0) = 0$ then $f(a) \precsim a$;*
- (vi) *$xx^* \precsim x^*x$;*
- (vii) *$D_\varphi(p) = \varphi(p)$ if $p \in A$ is a projection;*
- (viii) *$\varphi(a) = \int_0^{\|a\|} D_\varphi(f_t(a))\,dt$.*

LEMMA 5.3. *Let A be a unital C*-algebra and let φ be a faithful state on A. Suppose that p is a projection in A, $a \in A^+$ and p and a are free with respect to φ.*

- (i) *If $D_\varphi(a) < \varphi(p)$ then $a \precsim p$.*

(ii) *If* $\varphi(p) < D_\varphi(a)$ *then* $p \lesssim a$.

Proof. Let A'' denote the von Neumann algebra generated by the image of A under the GNS representation of φ, and denote also by φ the normal extension of φ to A''.

For (i), it will suffice to show that $f_\epsilon(a) \lesssim p$ for every $\epsilon > 0$. Set

$$q = \chi_{[\epsilon,\infty)}(a) \in A''.$$

Then $f_\epsilon(a) \le q \le f_{\epsilon/2}(a)$, $f_\epsilon(a)q = f_\epsilon(a)$, and p and q are free with respect to φ. Therefore

$$\varphi(q) \le \varphi(f_{\epsilon/2}(a)) \le D_\varphi(a) < \varphi(p),$$

and hence, by Proposition 1.1, $||q(1-p)|| < 1$. Setting $\lambda = 1 - ||q(1-p)||^2$, it follows that

$$q(1-p)q \le ||q(1-p)||^2 q = (1-\lambda)q,$$

so $qpq \ge \lambda q$. Hence

$$\lambda f_\epsilon(a) = \lambda f_\epsilon(a)^{1/2} q f_\epsilon(a)^{1/2} \le f_\epsilon(a)^{1/2} qpq f_\epsilon(a)^{1/2} = f_\epsilon(a)^{1/2} p f_\epsilon(a)^{1/2},$$

which implies that

$$f_\epsilon(a) \lesssim \lambda f_\epsilon(a) \le f_\epsilon(a)^{1/2} p f_\epsilon(a)^{1/2} \lesssim p f_\epsilon(a) p \le p.$$

For (ii), choose $\epsilon > 0$ such that $\varphi(f_\epsilon(a)) > \varphi(p)$. Set

$$q = \chi_{[\epsilon, ||a||]}(a) \in A''.$$

Then, as above, p and q are free with respect to φ and $f_\epsilon(a) \le q \le f_{\epsilon/2}(a)$, so $\varphi(p) < \varphi(q)$, whence $||p(1-q)|| < 1$. Setting $\lambda = 1 - ||p(1-q)||^2$ we get $\lambda p \le pqp \le p f_{\epsilon/2}(a) p$ which gives that

$$p \lesssim \lambda p \le p f_{\epsilon/2}(a) p \lesssim f_{\epsilon/2}^{1/2}(a) p f_{\epsilon/2}^{1/2}(a) \le f_{\epsilon/2}(a) \lesssim a. \qquad \square$$

LEMMA 5.4. *Let A be a unital C*-algebra and let φ be a state on A that is not a trace. Then there are $a \in A^+$ and a unitary $u \in A$ such that*

$$D_\varphi(a) < D_\varphi(u^*au).$$

Proof. Suppose for contradiction that $D_\varphi(a) = D_\varphi(u^*au)$ for every $a \in A^+$ and unitary $u \in A$. Then by Proposition 5.2(vii),

$$\varphi(a) = \int_0^{||a||} D_\varphi(f_t(a))\,dt = \int_0^{||a||} D_\varphi(u^* f_t(a) u)\,dt$$
$$= \int_0^{||a||} D_\varphi(f_t(u^*au))\,dt = \varphi(u^*au)$$

for every $a \in A^+$ and every unitary $u \in A$. But then φ would be a trace. $\square$

LEMMA 5.5. *Let A be a unital C^*-algebra and φ a faithful state on A. Suppose that (A,φ) is eigenfree by the endomorphism ρ and unitary v, and suppose that φ is not a trace. Then for every n large enough, the identity is a properly infinite projection in $A \otimes M_n(\mathbf{C})$.*

Proof. By Lemma 5.4 there are $a \in A^+$ and a unitary $u \in A$ such that $D_\varphi(a) < D_\varphi(u^*au)$. Let $(e_{ij})_{1\le i,j\le n}$ be a system of matrix units for $M_n(\mathbf{C})$ and for $1 \le l \le n$ let $f_l = \sum_{j=1}^{l} e_{jj}$. Let $k,n \in \mathbf{N}$ be such that

$$D_\varphi(a) < \frac{k}{n} < \frac{k+1}{n} < D_\varphi(u^*au)\,.$$

Then in $A \otimes M_n(\mathbf{C})$ and for every $m \in \mathbf{N}$

$$\begin{aligned} D_{\varphi\otimes\mathrm{tr}_n}(\rho^m(a)\otimes 1_n) &= D_\varphi(a) < (\varphi\otimes\mathrm{tr}_n)(1\otimes f_k) \\ &< (\varphi\otimes\mathrm{tr}_n)(1\otimes f_{k+1}) < D_\varphi(u^*au) \\ &= D_{\varphi\otimes\mathrm{tr}_n}(\rho^m(u^*au)\otimes 1_n)\,. \end{aligned} \tag{12}$$

By Proposition 3.3, $(A\otimes M_n(\mathbf{C}), \varphi\otimes\mathrm{tr}_n)$ is eigenfree by the endomorphism $\rho^{n^2}\otimes\mathrm{id}_n$ and a unitary $w \in A\otimes M_n(\mathbf{C})$. Thus, by Lemma 1.2, $w^*(\rho^{n^2}(A)\otimes M_n(\mathbf{C}))w$ and $\rho^{n^2}(A)\otimes M_n(\mathbf{C})$ are free. Hence, from (12) and Lemma 5.3, $w^*(1\otimes f_{k+1})w \precsim \rho^{n^2}(u^*au)\otimes 1_n$ and $\rho^{n^2}(a)\otimes 1_n \precsim w^*(1\otimes f_k)w$, which shows that in $A\otimes M_n(\mathbf{C})$

$$1\otimes f_{k+1} \precsim \rho^{n^2}(a)\otimes 1_n \precsim 1\otimes f_k\,.$$

Iterating, one easily obtains that $1\otimes f_m \precsim 1\otimes f_l$ in $A\otimes M_r(\mathbf{C})$ whenever $k \le m \le r$ and $k \le l \le r$. Thus, if $n \ge k$ then $1\otimes f_{2n} \precsim 1\otimes f_n$ and hence the identity of $A\otimes M_n(\mathbf{C})$ is properly infinite. □

LEMMA 5.6. *Let A be a unital C^*-algebra and φ a faithful state on A. Suppose that (A,φ) is eigenfree and that φ is not a trace. Then the unit, 1, of A is properly infinite.*

Proof. Let ρ be an endomorphism making (A,φ) eigenfree. Using Lemma 5.5, let $n \ge 2$ be such that $1\otimes 1_n$ is properly infinite. Identify the unit, $1 \in A$, with $e = 1\otimes e_{11} \in A\otimes M_n(\mathbf{C})$, and let $f = 1\otimes(e_{11}+e_{22})$, where $e_{11}, e_{22} \in M_n(\mathbf{C})$ are orthogonal minimal projections. Since $1\otimes 1_n$ is properly infinite, there is a sequence $(r_j)_{j=1}^\infty$ of mutually orthogonal projections in $A\otimes M_n(\mathbf{C})$, each equivalent to $1\otimes 1_n$. Let $\rho_n = \rho^{n^2}\otimes\mathrm{id}_n$. Since $\rho_n(1\otimes 1_n) = 1\otimes 1_n$, and $\rho_n(f) = f$, we find a sequence, $(p_j)_{j=1}^\infty$ of projections in $\rho^{n^2}(A)\otimes M_n(\mathbf{C})$, each equivalent to f and such that $p_j \le \rho_n(r_j)$. Fix j such that $\varphi\otimes\mathrm{tr}_n(p_j) < 1/n$. By Proposition 3.3 there is a Haar unitary $w \in A \otimes M_n(\mathbf{C})$ such that $w^* p_j w$ and e are free. Then, by Proposition 1.1, $w^* p_j w$ is equivalent to a subprojection of e. Therefore, f is equivalent to a subprojection of e, and thus e is properly infinite. □

Having proved that A is properly infinite, (5) now follows from [C3, 1.4], since the set, $\mathcal{P}$, of all properly infinite projections in A is seen to satisfy the conditions (Π_1), (Π_2), (Π_3) and (Π_4) of [C3]. Therefore, Theorem 4 follows from Proposition 3.2 and Lemma 5.6.

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