ALMOST COMMUTING SELF-ADJOINT MATRICES — A SHORT PROOF OF HUAXIN LIN'S THEOREM

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Abstract. We give a self-contained and elementary proof of Huaxin Lin's theorem that pairs of almost commuting self-adjoint matrices are near commuting pairs of self-adjoint matrices (in a uniform way). As in Lin's proof, the result is obtained by showing that a certain corona C*-algebra has property (FN), i. e. any normal element can be approximated by a normal element with finite spectrum.

We prove a generalization of Lin's theorem, that almost commuting self-adjoint elements in any C*-algebra with property (IR) (a property weaker than stable rank one) are close to commuting self-adjoint elements (again in a uniform way).

Using similar methods, we give a necessary and sufficient condition for a C*-algebra to have property (FN), and from this it follows in particular that every C*-algebra of real rank zero, stable rank one and with trivial K_1-group has property (FN).

1. Introduction

The old problem whether pairs of almost commuting self-adjoint matrices are uniformly close to commuting pairs of self-adjoint matrices was recently solved affirmatively by Huaxin Lin. We refer the reader to [4] for an account of the history of this problem.

The precise statement is given by

1.1. Theorem (Huaxin Lin [4]). *For every $\varepsilon > 0$ there is a $\delta > 0$ such that for any n and any pair $a, b \in M_n(\mathbb{C})$ of self-adjoint matrices such that $\|a\|, \|b\| \leq 1$ and*

$$\|ab - ba\| < \delta,$$

there exists a commuting pair $a', b' \in M_n(\mathbb{C})$ of self-adjoint matrices with

$$\|a - a'\| + \|b - b'\| < \varepsilon.$$

It is a crucial part of the theorem that δ can be chosen independent of n.

Observing that $x^*x - xx^* = 2i(ab - ba)$ when $x = a + ib$ and a and b are self-adjoint elements, we see that asking for commuting self-adjoint approximants to a given almost commuting pair of self-adjoints is equivalent to asking for normal approximants to a given almost normal matrix. Hence Theorem 1.1 can be phrased "almost normal implies close to normal".

2. Proof of Theorem 1.1

We shall in this section present an elementary and self-contained proof of Theorem 1.1. The proof is given in the context of C*-algebras but it does not use any deep or technical theory. We only need to use the continuous function calculus in C*-algebra.

Following the line in Lin's original proof, consider a sequence (n_j) of natural numbers, and define the C*-algebras

$$\begin{aligned} M &= \{(a_j) \mid a_j \in M_{n_j}(\mathbb{C}),\ \sup \|a_j\| < \infty\}, \\ A &= \{(a_j) \mid a_j \in M_{n_j}(\mathbb{C}),\ a_j \to 0\}, \end{aligned}$$

of bounded sequences, respectively null sequences. Clearly A is an ideal of M, and so we can consider the quotient C*-algebra M/A and the quotient map $\pi\colon M \to M/A$. Actually M is the multiplier algebra of A (a fact we shall not use), and M is a finite von Neumann algebra.

2.1. Lemma. *For each normal element $x \in M/A$, for each finite or countably infinite subset F of $\mathbb{C}$ and for each $\varepsilon > 0$ there is a normal element $y \in M/A$ with $\|x - y\| \le \varepsilon$ and $\mathrm{sp}(y) \cap F = \varnothing$.*

Proof. Recall that every matrix $x \in M_{n_j}(\mathbb{C})$ has a unitary polar decomposition $x = u|x|$, where $|x| = (x^*x)^{1/2}$, and where u is a unitary in $M_{n_j}(\mathbb{C})$. Hence each element in M has such a unitary polar decomposition. If $x \in M/A$, then $x = \pi(y)$ for some $y \in M$ and $y = v|y|$ for some unitary $v \in M$, whence $x = \pi(v)\pi(|y|) = \pi(v)|x|$, and $\pi(v)$ is a unitary element of M/A. Therefore each element of M/A has a unitary polar decomposition.

Let x be a normal element in M/A and write $x = u|x|$. From the identity $x^*x = xx^*$ we see that u and $|x|$ commute. Hence for every $\varepsilon > 0$, the element $y = u(|x| + \varepsilon 1)$ is normal (and clearly invertible) and $\|x - y\| \le \varepsilon$. In other words, the set of invertible normal elements is dense in the set of normal elements. It is also relatively open, since the set of invertible elements in M/A is an open subset of M/A.

Upon applying the homeomorphism $x \mapsto x - \lambda 1$ on set of normal elements of M/A, we find that for any $\lambda \in \mathbb{C}$, the set of normal elements $x \in M/A$ with $\lambda \notin \mathrm{sp}(x)$ is dense and relatively open in the set of normal elements.

Now, by Baire's category theorem, the set of normal elements y with $\mathrm{sp}(y) \cap F = \varnothing$ is dense in the set of normal elements, and this completes the proof. $\square$

For each $\varepsilon > 0$ consider the ε-grid Γ_ε in $\mathbb{C}$ and the corresponding lattice Σ_ε of "center-points" in Γ_ε, defined by

$$\begin{aligned}\Gamma_\varepsilon &= \{x+iy \in \mathbb{C} \mid x \in \varepsilon\mathbb{Z} \quad \text{or} \quad y \in \varepsilon\mathbb{Z}\},\\ \Sigma_\varepsilon &= \{x+iy \in \mathbb{C} \mid x \in \varepsilon(\mathbb{Z}+\tfrac{1}{2}) \quad \text{and} \quad y \in \varepsilon(\mathbb{Z}+\tfrac{1}{2})\}.\end{aligned}$$

2.2. Lemma. *For each normal element x in M/A and for each $\varepsilon > 0$ there is a normal element y in M/A with $\mathrm{sp}(y) \subseteq \Gamma_\varepsilon$ and $\|x-y\| < \varepsilon$.*

Proof. By Lemma 2.1 there is a normal element x_1 in M/A with

$$\|x - x_1\| < (1-\tfrac{\sqrt{2}}{2})\varepsilon, \qquad \mathrm{sp}(x_1) \cap \Sigma_\varepsilon = \varnothing.$$

There is a continuous retraction $f\colon \mathbb{C}\backslash\Sigma_\varepsilon \to \Gamma_\varepsilon$ with $|f(z)-z| < \frac{\sqrt{2}}{2}\varepsilon$ for all $z \in \mathbb{C}\backslash\Sigma_\varepsilon$, and so we can take y to be $f(x_1)$. □

2.3. Lemma. *Let x be a normal element of M/A. Suppose V is a relatively open subset of $\mathrm{sp}(x)$, and that V is homeomorphic to the open unit interval. Then for each $\lambda_0 \in V$ and each $\varepsilon > 0$, there exists a normal element y in M/A such that $\mathrm{sp}(y) \subseteq \mathrm{sp}(x) \setminus \{\lambda_0\}$ and $\|x-y\| \le \varepsilon$.*

Proof. Set $X = \mathrm{sp}(x)$. Let U be a relatively open subset of V satisfying

$$\lambda_0 \in U \subseteq \overline{U} \subseteq V, \qquad \mathrm{diam}(U) \le \varepsilon.$$

Choose a homeomorphism f_0 from V onto $\mathbb{T} \setminus \{-1\}$, where $\mathbb{T}$ is the unit circle in the complex plane. Extend f_0 to a continuous function $f\colon X \to \mathbb{T}$ by setting $f(z) = -1$ for all $z \in X \setminus V$. Set $u = f(x)$, and observe that u is unitary. Let a be any element in M with $\pi(a) = u$, and let $a = v|a|$ be a polar decomposition for a where v is a unitary in M. Then, since $\pi(|a|) = |u| = 1$, we have $\pi(v) = u$.

Consider the open subset $W = f_0(U)$ of $\mathbb{T}$, and let 1_W be the characteristic function corresponding to this set. By the Borel function calculus in von Neumann algebras, $1_W(v)$ defines a projection in M. (One could obtain the projection $1_W(v)$ by less technical means by expressing v as a sequence (v_j) of unitary matrices.) Let $e \in M/A$ be the projection $\pi(1_W(v))$.

Suppose $\varphi\colon X \to \mathbb{C}$ is a continuous function which is zero on $X \setminus V$. Then

$$\hat{\varphi}(z) = \begin{cases} \varphi \circ f_0^{-1}(z), & \text{if } z \in \mathbb{T}\setminus\{-1\} \\ 0, & \text{if } z = -1 \end{cases},$$

defines a continuous function $\mathbb{T} \to \mathbb{C}$ satisfying $\varphi = \hat{\varphi} \circ f$, and hence $\varphi(x) = \hat{\varphi}(u)$. Since e commutes with u (because $1_W(v)$ commutes with v) we see that e commutes with $\varphi(x)$. If φ in addition is constant equal to 1 on U, then $\hat{\varphi}$ is constant equal to 1 on W, which implies that $\varphi(x)e = e\varphi(x) = e$. If φ vanishes on $X \setminus U$, then $\hat{\varphi}$ vanishes on $\mathbb{T} \setminus W$, and so $\varphi(x)e = e\varphi(x) = \varphi(x)$ in this case.

Let $h\colon X \to [0,1]$ be a continuous function with $h|_{\overline{U}} = 1$ and $h|_{X\setminus V} = 0$. By the argument above, $h(x)e = eh(x) = e$, and since the function $z \mapsto zh(z)$ vanishes on $X \setminus V$, we get

$$xe = xh(x)e = exh(x) = eh(x)x = ex.$$

We show next that

$$\mathrm{sp}_{e(M/A)e}(xe) \subseteq \overline{U}, \qquad \mathrm{sp}_{(1-e)(M/A)(1-e)}(x(1-e)) \subseteq X \setminus U.$$

It suffices to show that $\varphi(xe) = 0$ and $\psi(x(1-e)) = 0$ for every pair of continuous functions $\varphi, \psi\colon X \to \mathbb{C}$, where φ vanishes on $\overline{U}$ and ψ vanishes on $X \setminus U$ (and where the continuous functions operate in the respective corner algebras). We may assume that φ is equal to 1 on the set $X \setminus V$. From the argument in the previous paragraph we get

$$\varphi(xe) = \varphi(x)e = e - (1-\varphi(x))e = 0, \quad \psi(x(1-e)) = \psi(x)(1-e) = 0,$$

as desired.

Choose $\lambda_1 \in U \setminus \{\lambda_0\}$ and set

$$y = \lambda_1 e + (1-e)x.$$

Then y is normal,

$$\mathrm{sp}(y) \subseteq \{\lambda_1\} \cup (X \setminus U) \subseteq X \setminus \{\lambda_0\},$$

and $\|x - y\| = \|xe - \lambda_1 e\| \le \mathrm{diam}(U) \le \varepsilon$ as desired. □

2.4. Lemma. *Let $\varepsilon > 0$ and let x be a normal element of M/A whose spectrum is contained in Γ_δ for some $\delta > 0$. Then there is a normal element y in M/A with finite spectrum and with $\|x - y\| < \varepsilon$.*

Proof. Upon applying Lemma 2.3 a finite number of times, one obtains a normal element x_1 of M/A with $\|x - x_1\| < \varepsilon/2$ and with the property that each connected component of $\mathrm{sp}(x_1)$ has diameter less than $\varepsilon/2$. Now, each $\lambda \in \mathrm{sp}(x_1)$ is contained in some clopen subset of $\mathrm{sp}(x_1)$ of diameter less than $\varepsilon/2$. Hence $\mathrm{sp}(x_1)$ can be partitioned into a finite family $V_1, V_2, \dots, V_n$ of clopen (non-empty) subsets of $\mathrm{sp}(x_1)$ each with diameter less than $\varepsilon/2$. Choose any $\lambda_j \in V_j$, and observe that there is a continuous retraction $f\colon \mathrm{sp}(x_1) \to \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ satisfying $|f(z) - z| < \varepsilon/2$ for all $z \in \mathrm{sp}(x_1)$. Then $y = f(x_1)$ is a normal element with spectrum $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and with distance less than ε to x. □

The consequence of Lemmas 2.3 and 2.4, that any normal element in M/A with one-dimensional spectrum can be approximated by normal elements with finite spectra, also follows from a more general result of Huaxin Lin (see [2, Theorem 5.4]).

Proof of Theorem 1.1. Suppose Theorem 1.1 were false. Then by the remark below Theorem 1.1, there would exist a sequence (n_j) of natural numbers and a sequence (x_j) of matrices $x_j \in M_{n_j}(\mathbb{C})$ with $\|x_j\| \leq 1$ such that

$$\left\| x_j^* x_j - x_j x_j^* \right\| \to 0,$$

and yet for some $\varepsilon > 0$, every x_j would have distance at least ε to the set of normal matrices in $M_{n_j}(\mathbb{C})$. Let M and A be the C*-algebras of bounded sequences, respectively, null sequences corresponding to the sequence (n_j) as defined at the beginning of this section. Set $x = (x_j) \in M$ and set $y = \pi(x) \in M/A$. Then y is normal. By Lemma 2.4 there exists a normal element y' in M/A with finite spectrum and with $\|y - y'\| < \varepsilon/4$.

There is a normal element $x' = (x_j') \in M$ with $\pi(x') = y'$. To see this choose complex polynomials p and q in one variable so that $p(\mathrm{sp}(y')) \subseteq \mathbb{R}$ and $(q \circ p)(\lambda) = \lambda$ for all $\lambda \in \mathrm{sp}(y')$. Then $p(y')$ is self-adjoint and $y' = q(p(y'))$. Let z be any element in M with $\pi(z) = p(y')$. Then $(z + z^*)/2$ is self-adjoint and $\pi((z + z^*)/2) = p(y')$. Hence we may take $x' = q((z + z^*)/2)$. By the definition of the norm on the quotient algebra there exists $a = (a_j) \in A$ such that

$$\|x - x' - a\| = \|y - y'\| + \varepsilon/4 < \varepsilon/2.$$

(One can actually find a such that $\|x - x' - a\| = \|y - y'\|$.) Choose j such that $\|a_j\| < \varepsilon/2$. Then $\left\| x_j - x_j' \right\| < \varepsilon$, in contradiction with the choice of x_j. □

3. Normal elements with finite spectra

The method from Section 2 to approximate normal elements in the C*-algebra M/A by normal elements with one-dimensional spectra can be applied to give a characterization of the C*-algebras with property (FN) (see Theorem 3.5 below). Recall that a C*-algebra A has property (FN) if every normal element in A can be approximated by normal elements in A with finite spectrum. This property has been studied by Lin (among others) and has been established for certain simple AF-algebras (see [3]) and all purely infinite simple C*-algebras with trivial K_1-group (see [2]). Theorem 3.5 generalizes these results.

Recall (see [7]) that a unital C*-algebra A is said to have stable rank one if the group, $\mathrm{GL}(A)$, of invertible elements in A is a dense subset of A. A non-unital C*-algebra A has stable rank one if the C*-algebra $\tilde{A}$ obtained by adjoining a unit to A has stable rank one. A C*-algebra has real rank zero if every self-adjoint element in A can be approximated by self-adjoint elements with finite spectra (see [1]). In addition to these properties of a C*-algebra we shall also consider the following:

3.1. Definition. For each unital C*-algebra A denote by $\mathrm{R}(A)$ the set of elements $x \in A$ with the property that for no ideal I of A is $x + I$ one-sided and not two-sided invertible in A/I.

A unital C*-algebra A will be said to have property (IN) if every normal element in A belongs to the norm closure of $\mathrm{GL}(A)$, and A will be said to have property (IR) if all elements in $\mathrm{R}(A)$ belong to the norm closure of $\mathrm{GL}(A)$.

A non-unital C*-algebra A has property (IN), respectively property (IR), if the C*-algebra obtained by adjoining a unit to A has this property.

It follows immediately from the definition, that every C*-algebra of stable rank one also has property (IR). The set of normal element of a unital C*-algebra A is contained in $\mathrm{R}(A)$. Hence every C*-algebra with property (IR) also has property (IN).

Every von Neumann algebra has property (IN) (by the Borel function calculus) and every simple von Neumann algebra has property (IR) (see [9]).

It is shown in [9, Theorem 4.4] that all simple, purely infinite, unital C*-algebras have property (IR) (and hence also property (IN)). There are no known examples of simple C*-algebras which are neither purely infinite nor have stable rank one. It is therefore possible that all simple C*-algebras have properties (IR) and (IN).

The set $\mathrm{R}(A)$ is always closed, and it contains the closure of $\mathrm{GL}(A)$ (see [9, Proposition 2.1]). So if A is a unital C*-algebras for which $\mathrm{R}(A) \neq A$, then A cannot have stable rank one. We shall return with a more detailed discussion of the property (IR) in Section 4.

3.2. Lemma. *Let A be a unital C*-algebra with property* (IN). *Then for each normal element x in A and each $\varepsilon > 0$ there exists an invertible* normal *element y in A with $\|x - y\| \leq \varepsilon$.*

If A is a general C-algebra with property* (IN) *(unital or not), then for each normal element x in A, for each non-zero $\lambda \in \mathbb{C}$ and for each $\varepsilon > 0$ there exists a normal element y in A with $\|x - y\| \leq \varepsilon$ and with $\lambda \notin \mathrm{sp}(y)$.*

Proof. Define the continuous function $f_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ to be $f_\varepsilon(t) = \max\{t - \varepsilon, 0\}$. Let $x = v|x|$ be the polar decomposition of x, where $|x| = (x^*x)^{1/2} \in A$ and where v is a partial isometry in the bidual A^{**} of A. The partial isometry v will commute with $|x|$ because x is normal. Since x belongs to the closure of $\mathrm{GL}(A)$ it follows from [8, Theorem 2.2], that there is a unitary element u in A such that $vf_\varepsilon(|x|) = uf_\varepsilon(|x|)$. Because v commutes with $|x|$, we conclude that v and $f_\varepsilon(|x|)$ commute, and this in turn implies that $uf_\varepsilon(|x|)$ is normal. Hence u commutes with $f_\varepsilon(|x|)$. It follows that $y = u(f_\varepsilon(|x|) + \varepsilon)$ is a normal (and clearly invertible) element in A, satisfying

$$\|x - y\| \leq \|v(|x| - f_\varepsilon(|x|)) + (uf_\varepsilon(|x|) - y)\| \leq 2\varepsilon.$$

If A is unital, then we see that the normal element x is close to a normal element, which does not have a given λ in it spectrum, by applying the argument given above to the normal element $x - \lambda 1$.

Suppose now that A is non-unital, and let $\tilde{A}$ be the C*-algebra obtained by adjoining a unit to A. We may assume that $\varepsilon < |\lambda|$. By assumption, $x - \lambda 1$ belongs to the closure of $\mathrm{GL}(\tilde{A})$, so if $x - \lambda 1 = v|x - \lambda 1|$ is the polar decomposition of $x - \lambda 1$, with v a partial isometry in $\tilde{A}^{**}$ (as above), then $vf_\varepsilon(|x - \lambda 1|) = uf_\varepsilon(|x - \lambda 1|)$ for some unitary u in $\tilde{A}$. Set $y = u(f_\varepsilon(|x - \lambda 1|) + \varepsilon 1) + \lambda 1$. Then, arguing as above, we see that y is normal, $\|x - y\| \le 2\varepsilon$ and that $y - \lambda 1$ is invertible. We must also show that $y \in A$.

With $s \colon \tilde{A} \to \mathbb{C}$ the scalar map, we have

$$s(f_\varepsilon(|x - \lambda 1|)) = |\lambda| - \varepsilon, \quad s(vf_\varepsilon(|x - \lambda 1|)) = -\lambda|\lambda|^{-1}(|\lambda| - \varepsilon),$$

which implies that $s(u) = -\lambda/|\lambda|$. Hence $s(y) = (-\lambda/|\lambda|)|\lambda| + \lambda = 0$, so that $y \in A$. □

The proof of Lemma 2.1 (together with Lemma 3.2) yields the following:

3.3. Lemma. *Let A be a C*-algebra with property* (IN). *For each normal element $x \in A$, for each finite or countably infinite subset F of $\mathbb{C}$, such that $0 \notin F$ if A is non-unital, and for each $\varepsilon > 0$ there is a normal element $y \in A$ with $\|x - y\| \le \varepsilon$ and $\mathrm{sp}(y) \cap F = \varnothing$.*

Let Γ_ε and Σ_ε be as defined in Section 2. (Note that $0 \in \Gamma_\varepsilon$ and that $0 \notin \Sigma_\varepsilon$.) We shall in the following refer to a topological space as being *one-dimensional* if it has covering dimension ≤ 1. Every closed subset of Γ_ε is one-dimensional in this sense.

Copying the proof of Lemma 2.2 we obtain:

3.4. Proposition. *Let A be a C*-algebra with property* (IN). *Then for each normal element $x \in A$ and for each $\varepsilon > 0$ there is a normal element $y \in A$ with $\mathrm{sp}(y) \subseteq \Gamma_\varepsilon$ and $\|x - y\| < \varepsilon$. Hence each normal element in A can be approximated by normal elements in A with one-dimensional spectrum.*

3.5. Theorem. *The following two conditions are equivalent for every C*-algebra A:*

(i) *A has property* (FN).

(ii) *A has property* (IN), *the real rank of A is zero, and the unitary group of A (or of $\tilde{A}$ if A is non-unital) is connected.*

Recall from [2, Lemma 2.2] that if A is a C*-algebra of real rank zero, then the unitary group of A (or of $\tilde{A}$) is connected if $K_1(A) = 0$.

Proof. The implication (i) $\implies$ (ii) is trivial. Huaxin Lin proved in [2, Theorem 5.4] that every normal element in A with one-dimensional spectrum can be approximated by normal elements with finite spectra provided the unitary group in A (or in $\tilde{A}$) is connected. Combining this with Proposition 3.4 yields the theorem. □

4. A generalization of Lin's Theorem

Having read a preliminary version of this paper, George Elliott pointed out to us that Lin's theorem, as well as a generalization thereof, where one can replace the C*-algebra $M_n(\mathbb{C})$ in Theorem 1.1 with any C*-algebra of stable rank one, can be derived from Proposition 3.4 using Terry Loring's result that $\mathrm{C}(\Gamma)$ is semiprojective for every finite one-dimensional CW-complex Γ (see [5, Theorem 5.1]). We prove below a generalization of Lin's theorem that includes all C*-algebras with property (IR) (see Theorem 4.4 below).

From Loring's result we get:

4.1. Lemma. *For any sequence (A_n) of C*-algebras and any normal element x in*

$$\prod A_n \Big/ \bigoplus A_n,$$

with $\mathrm{sp}(x)$ contained in some one-dimensional finite CW-complex Γ, there exists a normal element y in $\prod A_n$ such that $x = \pi(y)$.

Proof. By semiprojectivity the $*$-homomorphism $\mathrm{C}(\Gamma) \to \prod A_n / \bigoplus A_n$ given by $f \mapsto f(x)$ factors through $\prod A_n / \bigoplus_{n=1}^k A_n$ for some k. Hence x lifts to a normal element of $\prod A_n / \bigoplus_{n=1}^k A_n$, and every normal element of $\prod A_n / \bigoplus_{n=1}^k A_n$ lifts to a normal element of $\prod A_n$. $\square$

We shall in the proposition and in the remark below illustrate some connections between almost commuting self-adjoint elements in a C*-algebra A and the property (IR) and the set $\mathrm{R}(A)$ defined in 3.1.

4.2. Proposition. *For every C*-algebra A, the following holds:*

(i) *For every $x \in A$,*

$$\mathrm{dist}(x, \mathrm{R}(A)) \le \big\| |x| - |x^*| \big\| .$$

(ii) *Suppose I is a closed two-sided ideal in A, and let $\pi \colon A \to A/I$ be the quotient mapping. Let x be a normal element in A/I. Then there is a sequence (a_n) of elements in A with $\pi(a_n) = x$, $\|a_n\| = \|x\|$, and $\|a_n^* a_n - a_n a_n^*\| \to 0$.*

(iii) *If A has property (IR) then every quotient A/I of A has property (IN).*

Proof. (i). Set $\delta = \big\| |x| - |x^*| \big\|$, let $x = v|x|$ be the polar decomposition of x (with v a partial isometry in A^{**}), and set $x_0 = v f_\delta(|x|)$, where $f_\delta \colon \mathbb{R}^+ \to \mathbb{R}^+$ is as in Lemma 3.2. Then $\|x - x_0\| \le \delta$. We show that $x_0 \in \mathrm{R}(A)$.

Suppose I is a closed two-sided ideal in A and suppose that $(y =)\ \pi(x)$ is non-invertible. Then either $0 \in \mathrm{sp}(|y|)$ or $0 \in \mathrm{sp}(|y^*|)$. Assume with no loss of generality the former. Since $\big\| |y| - |y^*| \big\| \le \big\| |x| - |x^*| \big\| = \delta$, it follows that $\mathrm{sp}(|y^*|) \cap [0, \delta] \neq \emptyset$. Hence 0 is in the spectrum of both $|\pi(x_0)| = f_\delta(|y|)$ and of $|\pi(x_0^*)| = f_\delta(|y^*|)$. Accordingly, $\pi(x_0)$ is neither left nor right invertible.

(ii). Let $\varepsilon > 0$ be given, and let $a \in A$ be any lift of x with $\|b\| = \|x\|$. Then $a^*a - aa^* \in I$. Using Arveson's quasi-central approximate unit for I (see [6, 3.12.14]), we can find an element $e \in I$ satisfying

$$0 \leq e \leq 1, \quad \|ea - ae\| \leq \varepsilon/3, \quad \big\|(1-e)(a^*a - aa^*)(1-e)\big\| \leq \varepsilon/3.$$

It follows that the element $b = (1-e)a \in A$ is a lift of x with $\|b\| = \|x\|$ and that $\big\|b^*b - bb^*\big\| \leq \varepsilon$.

(iii). Upon adjoining a unit to A if necessary, we may assume that A is unital. Let x be a normal element of A/I and let $\varepsilon > 0$. By (ii) there is an element $a \in A$ with $\pi(a) = x$ and with $\big\||a| - |a^*|\big\| < \varepsilon$. By (i), $\mathrm{dist}(a, \mathrm{R}(A)) < \varepsilon$, which combined with the assumption that A has property (IR) implies that $\|a - b\| < \varepsilon$ for some invertible element $b \in A$. Now, $\pi(b)$ is clearly invertible in A/I and $\|x - \pi(b)\| < \varepsilon$. □

Remark. Suppose A is a unital C*-algebra which has an ideal I so that the map $K_1(A) \to K_1(A/I)$ induced by the quotient mapping is not surjective and, moreover, does not have $[u]_1$ in its image for some unitary $u \in A/I$. Then u does not lift to a unitary element of A, and no invertible element in the connected component of $\mathrm{GL}(A/I)$ containing u can be lifted to an invertible element of A. Combining this Proposition 4.2 (iii) (and its proof) we see that A cannot have property (IR).

4.3. Lemma. *Let A_n be a sequence of C*-algebras with property* (IR). *Then the C*-algebra $\prod A_n$ has property* (IR), *and hence the quotient $\prod A_n / \bigoplus A_n$ has property* (IN).

Proof. By Proposition 4.2 it suffices to show that the C*-algebra $A = \prod A_n$ has property (IR). Consider first the case where all A_n are unital. Let $x = (x_n)$ belong to $\mathrm{R}(A)$. Then x_n is the image of x under the canonical surjection $A \to A_n$, whence $x_n \in \mathrm{R}(A_n)$ by the definition of $\mathrm{R}(\cdot)$. By the assumption on A_n, each x_n belongs to the closure of $\mathrm{GL}(A_n)$.

As in the proof of Lemma 3.2, if $x_n = v_n|x_n|$ is the polar decomposition for x_n (with $v_n \in A_n^{**}$), then for each $\varepsilon > 0$ there is a unitary $u_n \in A_n$ such that $v_n f_\varepsilon(|x_n|) = u_n f_\varepsilon(|x_n|)$, and $y_n = u_n(f_\varepsilon(|x_n|) + \varepsilon 1)$ is an invertible element of A_n with $\|x_n - y_n\| \leq 2\varepsilon$. Set $u = (u_n) \in A$ and $y = (y_n) = u(f_\varepsilon(|x|) + \varepsilon 1)$. Then y is an invertible element of A and $\|x - y\| \leq 2\varepsilon$, which proves that x belongs to the closure of $\mathrm{GL}(A)$.

Suppose I is a closed two-sided ideal in some C*-algebra A. Then $\tilde{I}$ is a unital sub-C*-algebra of $\tilde{A}$. By the proof of [7, Theorem 4.4] every element of $\tilde{I}$, that belongs to the closure of $\mathrm{GL}(\tilde{A})$ actually belongs to the closure of $\mathrm{GL}(\tilde{I})$. Since $\mathrm{R}(\tilde{I}) \subseteq \mathrm{R}(\tilde{A})$, this shows that I has property (IR) if A has this property.

Consider now the general case where the C*-algebras A_n are not assumed to be unital. Then $\tilde{A}_n$ has property (IR), and hence $\prod \tilde{A}_n$ has property (IR) by the already established part of the lemma. Now, $\prod A_n$ must have property (IR) being a closed two-sided ideal in $\prod \tilde{A}_n$. □

4.4. Theorem. *For every $\varepsilon > 0$ there is a $\delta > 0$ such that for any C*-algebra A with property* (IR) *and any pair $a, b \in A$ of self-adjoint elements such that $\|a\|, \|b\| \leq 1$ and*

$$\|ab - ba\| < \delta,$$

there exists a commuting pair $a', b' \in A$ of self-adjoint elements with

$$\|a - a'\| + \|b - b'\| < \varepsilon.$$

Proof. Exactly as in the proof (in Section 2) of Theorem 1.1, it suffices to show that for every sequence (A_n) of C*-algebras with property (IR), for each normal element x in

$$\prod A_n \Big/ \bigoplus A_n,$$

and for each $\varepsilon > 0$, there is a normal element $y \in \prod A_n$ such that $\|\pi(y) - x\| < \varepsilon$. By Proposition 3.4 (which applies because of Lemma 4.3), x is within ε of a normal element x_1 with one-dimensional spectrum, and by Lemma 4.1, $x_1 = \pi(y)$ for some normal element $y \in \prod A_n$. ☐

4.5. Corollary. *Suppose that A is a C*-algebra with property* (IR)*, let I be a closed two-sided ideal in A, and let $\pi \colon A \to A/I$ be the quotient mapping. Then for every normal element $x \in A/I$ and for every $\varepsilon > 0$ there exists a normal element $a \in A$ with $\|\pi(a) - x\| < \varepsilon$.*

Proof. We may without loss of generality assume that $\|x\| \leq 1$. Choose the $\delta > 0$ corresponding to our given $\varepsilon > 0$ in Theorem 4.4. By Proposition 4.2 (ii) x lifts to an element $b \in A$ with $\|b^*b - bb^*\| < \delta$ and $\|b\| \leq 1$. By Theorem 4.4 there is a normal element $a \in A$ with $\|a - b\| < \varepsilon$. ☐

Remark. Theorem 4.4 in the special case where we require real rank zero and trivial K_1-group in addition to property (IR), also follows using 3.5: As in the proof of Theorem 4.4 it suffices to show that for any sequence (A_n) of C*-algebras with the three given properties, the C*-algebra $\prod A_n / \bigoplus A_n$ has property (FN). By Lemma 4.3 and Theorem 3.5 all that remains to be shown is that $\prod A_n / \bigoplus A_n$ has real rank zero. Let us show that $\prod A_n$ has real rank zero (real rank zero passes to quotients):

Let (x_n) be a self-adjoint element of $\prod A_n$, let $\varepsilon > 0$ and choose, as we may, a sequence (y_n) of invertible self-adjoints such that $\sup \|x_n - y_n\| \leq \varepsilon$. Let f be any continuous map $\mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus (-\varepsilon, \varepsilon)$ such that $|f(t) - t| \leq \varepsilon$ for all t. Put $z_n = f(y_n)$ and note that (z_n) is invertible in $\prod A_n$ and that $\sup \|x_n - z_n\| \leq 2\varepsilon$.

Remark. The proof of Theorem 4.4 also yields the following result: For every C*-algebra A, if the C*-algebra

$$\prod_{n=1}^{\infty} A \Big/ \bigoplus_{n=1}^{\infty} A \quad (= \ell^{\infty}(A)/c_0(A))$$

has property (IN), then for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any pair $a, b \in A$ of self-adjoint elements with $\|a\|, \|b\| \leq 1$ and $\|ab - ba\| < \delta$, there exists a commuting pair $a', b' \in A$ of self-adjoint elements with $\|a - a'\| + \|b - b'\| < \varepsilon$.

The converse does not hold. For instance $\ell^\infty(A)/c_0(A)$ does not have property (IN) if $A = \mathrm{C}(X)$ and $\dim(X) \geq 2$.

4.6. Example. The C*-algebra $\ell^\infty(\mathrm{B}(\ell^2))/c_0(\mathrm{B}(\ell^2))$ has real rank zero and trivial K_1-group (being a quotient of the von Neumann algebra $\ell^\infty(\mathrm{B}(\ell^2))$. However, it does not have property (IN). If it had, then by the remark above, $\mathrm{B}(\ell^2)$ would have the property "almost normal implies close to normal", which is well known not to be the case:

Let (ξ_k) be an orthonormal basis for the Hibert space ℓ^2 and define for each $n \in \mathbb{N}$ the tapered unilateral shift s_n to be

$$s_n \xi_k = \min\{k/n, 1\} \xi_{k+1}.$$

Then $\|s_n^* s_n - s_n s_n^*\| \to 0$, yet every s_n has distance ≥ 1 to the set of normal operators (being an essential unitary with non-zero Fredholm index). This was noted by Man-Duen Choi among others. (Observe also, that a sequence with properties similar to that of (s_n) could have been constructed using Proposition 4.2 (ii)).

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