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Master's thesis

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Stable rank of group C*-algebras

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Abstract

The concept of stable rank for a unital C^*-algebra was introduced by Rieffel in 1982. The case of stable rank one has been widely studied and is known to have interesting consequences for the C^*-algebra, especially for the K-theory of the C^*-algebra.

This thesis begins by introducing the concept of stable rank for unital C^*-algebras before moving on to the special case of stable rank one. The main focus of this thesis is to present results proving that the reduced group C^*-algebra $C_r^*(G)$ for certain discrete groups has stable rank one. The main result discussed is that of Dykema, Haagerup, and Rørdam which shows that $C_r^*(G_1 * G_2)$ has stable rank one for $|G_1| \geq 2$ and $|G_2| \geq 3$. The proofs use the reduced free product, a concept introduced by Voiculescu during his work with free probability theory, which is considered non-commutative probability theory. This thesis introduces and proves the existence of the reduced free product of a family $(A_i, \varphi_i)_{i \in I}$ of unital C^*-algebras A_i each equipped with faithful state φ_i. Finally, we discuss more recent results of Gerasimova and Osin, who showed that $C_r^*(G)$ has stable rank one for a class of acylindrically hyperbolic groups.

Contents

1 Introduction

Rieffel introduced in his 1982 paper the concept of topological stable rank for Banach algebras. It was firstly motivated by the density of certain functions for the Banach algebra $C_{\mathbb{R}}(X)$ for compact Hausdorff space X, and is in some sense a way to give a C^*-algebra a dimension. Rieffel shows in his original paper several connections to different kinds of stable ranks, such as the Bass stable rank, general stable rank and connected stable rank. This thesis begins with an introduction to the topological stable rank, which we will simply denote stable rank, and quickly focuses on examples and the particular case of having stable rank one. It turns out that a C^*-algebra having stable rank one gives several stability results for the topological K-theory. Most notable, it implies K_1 injectivity and the equivalences Murray-von Neumann equivalence, unitary equivalence and homotopy equivalence are the same. Moreover, $K_0(A)$ of a unital C^*-algebra with stable rank one has the cancellation property.

Rieffel asked the question: What is the stable rank of $C_r^*(F_n)$ for $n \geq 2$? It was later answered by Dykema, Haagerup and Rørdam who proved that the stable rank of $C_r^*(F_n)$ is one for $n \geq 2$. In fact, they proved that the reduced free product C^*-algebra of a family of C^*-algebras who have the Avitzour property has stable rank one. In particular, the reduced group C^*-algebra $C_r^*(G)$ has stable rank one when $G = G_1 * G_2$ and $|G_1| \geq 2$, $|G_2| \geq 3$ for discrete groups G_1, G_2. The proofs rely heavily on the properties of the reduced free product, a concept introduced by Voiculescu during his work with free probability theory. This thesis presents the construction of the reduced free product of a family $(A_i, \varphi_i)_{i \in I}$ of unital C^*-algebras each equipped with a faithful state φ_i. The main result of Dykema, Haagerup and Rørdam needs a result of Rørdam concerning the distance to invertible elements in the case of unital C^*-algebras whose stable rank is not one. The techniques of Dykema, Haagerup and Rørdam inspired firstly Dykema and de la Harpe, and then Gerasimova and Osin to further explore for which groups $C_r^*(G)$ has stable rank one, with the acylindrically hyperbolic groups with trivial finite radical being most notable. It is an open problem if there exists a group G such that $C_r^*(G)$ is simple and $\mathrm{sr}(C_r^*(G)) \neq 1$.

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2 Group C^*-algebras

This section is a short introduction to group C^*-algebras for discrete groups. The reader is referred to [5, 3] for proofs and further reading.

Let G be a discrete group and consider the Hilbert space $\ell^2(G)$. For $g \in G$, define $\delta_g : G \to \mathbb{C}$ by $\delta_g(g) = 1$ and $\delta_g(h) = 0$ for $h \neq g$. See that for any $g, h \in G$

$$\delta_g \delta_h = \delta_{gh}, \qquad \delta_g^* = \delta_{g^{-1}}.$$

The set $\{\delta_g \mid g \in G\}$ is an orthonormal basis of unitaries for $\ell^2(G)$. The left regular representation $\lambda : G \to \mathcal{U}(\ell^2(G))$ is defined by

$$\lambda_g(\delta_h) = \delta_{gh},$$

for all $g, h \in G$.

The group $*$-algebra $\mathbb{C}G$ is the set of formal sums $\sum_{g\in G} \alpha_g g$ with only finitely many $\alpha_g \in \mathbb{C}$ nonzero, where multiplication is defined by

$$\left(\sum_{g\in G} \alpha_g g\right)\left(\sum_{h\in G} \beta_h h\right) = \sum_{g,h\in G} \alpha_g \beta_h gh.$$

and involution defined by

$$\left(\sum_{g\in G} \alpha_g g\right)^* = \sum_{g\in G} \overline{\alpha_g} g^{-1}.$$

The left regular representation extends to an injective $*$-homomorphism $\mathbb{C}G \to B(\ell^2(G))$, also denoted by λ.

Definition 2.1. Let G be a discrete group. The reduced group C^*-algebra $C_r^*(G)$ of G is the norm-closure of $\lambda(\mathbb{C}G)$ in $B(\ell^2(G))$.

In this way, with slight abuse of notation, one may view $\mathbb{C}G$ as the dense $*$-subalgebra of $C_r^*(G)$ consisting of sums $\sum_{g\in G} \alpha_g \delta_g$ with only finitely many $\alpha_g \neq 0$. It follows by definition that $C_r^*(G)$ is isomorphic to the completion of $\mathbb{C}G$ with respect to the norm $\|s\|_\lambda = \|\lambda(s)\|$ for $s \in \mathbb{C}G$. A discrete group G is called C^*-simple if the reduced group C^*-algebra $C_r^*(G)$ is simple.

An important property of $C_r^*(G)$ is that it is equipped with a faithful trace, defined by $\tau(x) = \langle x\delta_e, \delta_e\rangle$ for $x \in C_r^*(G)$, where e is the identity in G. In this way, $\{\delta_g \mid g \in G\}$ is an orthonormal set of unitaries with respect to τ in $C_r^*(G)$.

Definition 2.2. The universal group C^*-algebra $C^*(G)$ of G is the completion of $\mathbb{C}G$ with respect to the norm

$$\|x\|_u = \sup\{\|\pi(x)\| \mid \pi : \mathbb{C}G \to B(H) \text{ cyclic } * -\text{representations}\}.$$

The universal group C^*-algebra has the following universal property: Let $u : G \to B(H)$ be any unitary representation of G. Then there is a unique $*$-homomorphism $\pi_u \colon C^*(G) \to B(H)$ such that $\pi_u(g) = u_g$ for all $g \in G$. It follows that $C^*(G)$ always has a one-dimensional representation from the trivial representation $G \to \mathbb{C}$.

By definition $\|s\|_\lambda \leq \|s\|_u$ for any $s \in \mathbb{C}G$, so there exists a natural $*$-homomorphism $C^*(G) \to C^*_r(G)$. This $*$-homomorphism is an isomorphism if and only if G is an amenable group. For a discrete abelian group G the Pontryagin dual $\widehat{G}$ of G is the set of all continuous homomorphisms from G to the circle $\mathbb{T}$. For such groups $C^*_r(G) \simeq C(\widehat{G})$. It is known that any abelian group is amenable, so for any discrete abelian group G,

$$C^*(G) = C^*_r(G) \simeq C(\widehat{G}).$$

3 Stable rank

The notion of topological stable rank was introduced by Rieffel in [17] as a concept of dimension for Banach algebras which generalizes the classical concept of dimension for compact spaces. Rieffel introduced the concept of topological stable rank during his work with the irrational rotation C^*-algebra, wishing to determine wether two projections in an irrational rotation C^*-algebra which have the same trace are unitarily equivalent. He describes it as non-stable K-theory, and we will see a connection to the topological K_0 and K_1 groups of a unital C^*-algebra. The results of this section can be found in [17, 18].

Definition 3.1. Let A be a unital Banach algebra. Let $\mathrm{Lg}_n(A)$, respectively $\mathrm{Rg}_n(A)$, be the set of n-tuples of elements of A which generate A as a left, respectively right, ideal.

Note that $(a_1, \dots, a_n) \in \mathrm{Lg}_n(A)$ if and only if there exists $(b_1, \dots, b_n) \in A^n$ such that $\sum_{j=1}^n b_j a_j = 1$. Similarly, $(a_1, \dots, a_n) \in \mathrm{Rg}_n(A)$ if and only if there exists $(c_1, \dots, c_n) \in A^n$ such that $\sum_{j=1}^n a_j c_j = 1$.

Definition 3.2. Let A be a unital Banach algebra. The *left topological stable rank* of A, denoted $\mathrm{lsr}(A)$, is the least integer n such that $\mathrm{Lg}_n(A)$ is dense in A^n equipped with the product topology. If no such n exists, we set $lsr(A) = \infty$. The *right topological stable rank*, denoted $rsr(A)$, is defined in an analogous way for $\mathrm{Rg}_n(A)$.

Remark 3.3. If A is a Banach algebra without unit, let $\tilde{A}$ be the unital Banach algebra obtained from adjoining a unit. Then the topological stable ranks of A are defined to be the topological stable ranks of $\tilde{A}$. If A has identity element, then A is a direct summand of $\tilde{A}$, so $\mathrm{ltsr}(\tilde{A}) = \mathrm{ltsr}(A)$. This thesis considers only the case of unital Banach- or C^*-algebras.

Rieffel was led was to the notion of topological stable rank after noticing a connection between the left topological stable rank of the Banach algebra $C_{\mathbb{R}}(X)$, for a compact Hausdorff space X, and the dimension of X. We recall the definition of Lebesgue covering dimension.

Let (X, d) be a metric space and let $S \subset X$. A covering of S is a finite collection $(U_j)_{j=1}^r$ of open subsets of X such that

$$S \subset \bigcup_{j=1}^{r} U_j.$$

The order of the covering is the largest integer n such that there are $n+1$ members in the covering which have non-empty intersection. A covering $(V_i)_{i=1}^k$ is a refinement of $(U_j)_{j=1}^r$ if every V_i is contained in some U_j.

Definition 3.4. Let (X, d) be a metric space and let $S \subset X$. Then $\dim(S) \leq n$ if every covering of S has a refinement of order less than or equal to n. We say that S has dimension n and write $\dim(S) = n$, if $\dim(S) \leq n$ and it does not hold that $\dim(S) \leq n-1$.

The following is a useful equivalence of the Lebesgue covering dimension of a compact Hausdorff space X. See [16, Proposition 3.3.2].

Theorem 3.5. *Let X be a compact Hausdorff space. The dimension of X is the least integer n such that the set $C(X, \mathbb{R}^{n+1} \setminus \{0\})$ is norm dense in $C(X, \mathbb{R}^{n+1})$.*

One can identify $f\colon X \to \mathbb{R}^{n+1}$ with an $(n+1)$-tuple of continuous functions $f_i\colon X \to \mathbb{R}$. The assumption that $f(X)$ does not contain 0 is then equivalent to saying that there does not exist $x_0 \in X$ such that $f_i(x_0) = 0$ for all $1 \leq i \leq n+1$.

Theorem 3.6. *Let X be a compact Hausdorff space. The dimension of X is the least integer n such that $Lg_{n+1}(C_{\mathbb{R}}(X))$ is norm dense in $C_{\mathbb{R}}(X)^{n+1}$.*

Proof. Let $f_1, \ldots, f_{n+1} \in C_{\mathbb{R}}(X)$. We claim that $C_{\mathbb{R}}(X) = \langle f_1, \ldots, f_{n+1} \rangle$ if and only if $(f_1(x), \ldots, f_{n+1}(x)) \neq 0$ for all $x \in X$. Suppose first that $C_{\mathbb{R}}(X) = \langle f_1, \ldots, f_{n+1} \rangle$. Choose $c > 0$ and let f be the constant function such that $f(x) = c$ for all $x \in X$. Clearly $f \in C_{\mathbb{R}}(X)$, and so by assumption there exists $g_1, \ldots, g_{n+1} \in C_{\mathbb{R}}(X)$ such that $f = \sum_{j=1}^{n+1} g_j f_j$. Assume for contradiction that there exists $x_0 \in X$ such that $f_j(x_0) = 0$ for all $1 \leq j \leq n+1$. Then $f(x_0) = 0 \neq c$, which cannot be.

Suppose now, conversely, that $(f_1(x), \ldots, f_{n+1}(x)) \neq 0$ for all $x \in X$. Set $h = \sum_{j=1}^{n+1} f_j^2$, and see that $h \in \langle f_1, \ldots, f_{n+1} \rangle$. By definition, $h(x) > 0$, hence h is invertible, and in particular $h^{-1}h = 1 \in \langle f_1, \ldots, f_{n+1} \rangle$, implying that $\langle f_1, \ldots, f_{n+1} \rangle = C_{\mathbb{R}}(X)$, which proves the claim.

Assume n is the least integer such that $\mathrm{Lg}_{n+1}(C_{\mathbb{R}}(X))$ is norm dense in $C_{\mathbb{R}}(X)^n$. Identify an $(n+1)$-tuple of continuous functions such that $(f_1(x), \ldots, f_{n+1}(x)) \neq 0$ for all $x \in X$ with a continuous function $f\colon X \to \mathbb{R}^{n+1} \setminus \{0\}$. The wanted follows from the claim.

Suppose, conversely, that $\dim X = n$. Identifying continuous $f\colon X \to \mathbb{R}^{n+1} \setminus \{0\}$ with an $(n+1)$-tuple of continuous functions such that $(f_1(x), \ldots, f_{n+1}(x)) \neq 0$ for all $x \in X$ gives the wanted. □

The following is an immediate consequence of Theorem 3.6.

Lemma 3.7. *Let X be a compact Hausdorff space. Then*

$$lsr(C_{\mathbb{R}}(X)) = \dim(X) + 1.$$

For compact Hausdorff space X consider the C^*-algebra $C(X)$ of continuous functions $f\colon X \to \mathbb{C}$. Any function $f\colon X \to \mathbb{C}$ can be identified with a pair (f_1, f_2) of functions $f_i\colon X \to \mathbb{R}$. Therefore, any n-tuple of elements in $C(X)$ can be identified with a continuous function $h\colon X \to \mathbb{R}^{2n}$.

Proposition 3.8. *Let X be a compact Hausdorff space. Then*

$$lsr(C(X)) = \left\lfloor \frac{\dim(X)}{2} \right\rfloor + 1.$$

Proof. By similar argument as in the proof of Theorem 3.6 we see that density of $\mathrm{Lg}_n(C(X))$ in $C(X)^n$ implies density of $C(X, \mathbb{R}^{2n} \setminus \{0\})$ in $C(X, \mathbb{R}^{2n})$. In particular, $\dim(X) < 2n$ and n is the least such integer, i.e. $n = \lfloor \dim(X)/2 \rfloor + 1$. □

Example 3.9. Let $\mathbb{T}^n$ be the unit sphere in $\mathbb{R}^{n+1}$. It is known that $\mathbb{T}^n$ is a compact Hausdorff space with dimension n. Moreover, $\mathbb{Z}^n$ is a discrete abelian amenable group with $\widehat{\mathbb{Z}^n} \simeq \mathbb{T}^n$, and so

$$C^*(\mathbb{Z}^n) = C^*_r(\mathbb{Z}^n) \simeq C(\mathbb{T}^n),$$

implying that $\mathrm{sr}(C^*(\mathbb{Z}^n)) = \lfloor n/2 \rfloor + 1$ by Proposition 3.8.

Proposition 3.10. *If A is a unital C^*-algebra, then $lsr(A) = rsr(A)$.*

Proof. See first that if $a = (a_1, \dots, a_n) \in \mathrm{Lg}_n(A)$ then $a^* = (a_1^*, \dots, a_n^*) \in \mathrm{Rg}_n(A)$, as there exists $(b_1, \dots, b_n) \in A^n$ such that $\sum_{i=1}^n b_i a_i = 1$, implying that

$$\sum_{i=1}^n a_i^* b_i^* = 1.$$

Assume $\mathrm{lsr}(A) = n$. For $c \in A^n$ and $\varepsilon > 0$, there exists $a \in \mathrm{Lg}_n(A)$ such that

$$\|c - a\| = \|c^* - a^*\| < \varepsilon,$$

meaning $a^* \in \mathrm{Rg}_n(A)$ approximates $c^* \in A^n$ arbitrarily close, proving the wanted. □

As a consequence of Proposition 3.10 we simple say the *stable rank* of A when A is a unital C^*-algebra and denote it $\mathrm{sr}(A)$.

Before moving on to the special case of stable rank one, we consider examples of unital C^*-algebras with stable rank ∞. The proofs need the notion of the Bass stable rank.

Definition 3.11. Let A be a ring with identity element. The *Bass stable rank* of A, denoted by Bsr(A), is the least integer n such that for any $(a_1, \dots, a_{n+1}) \in \mathrm{Lg}_{n+1}(A)$ there is $(b_1, \dots, b_n) \in A^n$ such that

$$(a_1 + b_1 a_{n+1}, a_2 + b_2 a_{n+1}, \dots, a_n + b_n a_{n+1}) \in \mathrm{Lg}_n(A).$$

If no such integer exists, we set $\mathrm{Bsr}(A) = \infty$.

The Bass table rank and the right topological stable rank are connected in the following way. For a proof, see [17, Theorem 2.3].

Theorem 3.12. *Let A be a unital Banach algebra. Then*

$$rtsr(A) \geq Bsr(A).$$

Proposition 3.13. *If a unital C^*-algebra A contains two isometries with orthogonal range projections, then $sr(A) = \infty$.*

Proof. We prove that $\mathrm{Bsr}(A) = \infty$, as then $\mathrm{sr}(A) = \infty$ by Theorem 3.12.

Let $s_1, s_2 \in A$ be isometries with orthogonal range projections. Define $t_n = s_2^n s_1$ for $n \geq 1$ and see that each t_n is an isometry as the s_i are isometries. If $i > j$ then, as s_1, s_2 have orthogonal range projections,

$$t_i^* t_j = s_1^* (s_2^*)^i s_2^j s_1 = s_1^* (s_2^*)^{i-j} s_1 = 0.$$

Similarly, $t_i^* t_j = 0$ for $i < j$, showing that $(t_n)_{n \geq 1}$ is a sequence of isometries with pairwise orthogonal ranges. Thus, for any $k \geq 1$, one can from the two isometries produce k isometries $v_1, \ldots, v_k$ with pairwise orthogonal range projections such that

$$p_0 = 1 - \sum_{i=1}^{k} v_i v_i^* \neq 0.$$

As the v_i's have orthogonal ranges p_0 is a projection, and $(v_1^*, \ldots, v_k^*, p_0) \in \mathrm{Lg}_{k+1}(A)$, as

$$\sum_{i=1}^{k} v_i v_i^* + p_0 p_0 = 1.$$

Assume for contradiction that $(v_1^*, \ldots, v_k^*, p_0)$ can be contracted as in the definition of the Bass stable rank and let $w_1, \ldots, w_k \in A$ be such that $(v_1^* + w_1 p_0, \ldots, v_k^* + w_k p_0) \in \mathrm{Lg}_k(A)$. Then there exists $z_1, \ldots, z_k \in A$ such that

$$\sum_{i=1}^{k} z_i (v_i^* + w_i p_0) = 1.$$

For any $1 \leq j \leq k$

$$p_0 v_j = v_j - \sum_{i=1}^{k} v_i v_i^* v_j = v_j - v_j = 0,$$

and so

$$\sum_{i=1}^{k} z_i (v_i^* + w_i p_0) v_j = \sum_{i=1}^{k} z_i (v_i^* v_j + w_i p_0 v_j) = z_j,$$

implying that $z_j = v_j$ for all $1 \leq j \leq k$. Now

$$\sum_{i=1}^{k} v_i(v_i^* + w_i p_0) = 1,$$

and multiplying from the left by p_0 implies $p_0 = 0$, a contradiction. We conclude that $\mathrm{Bsr}(A) = \infty$, hence $\mathrm{tsr}(A) = \infty$. □

The *Cuntz* algebra $\mathcal{O}_n$ for $n \geq 2$ is the C^*-algebra generated by n isometries $s_1, \dots, s_n$ such that $\sum_{i=1}^{n} s_i s_i^* = 1$, which implies that the range projections are pairwise orthogonal. In particular, $\mathrm{sr}(\mathcal{O}_n) = \infty$ for all $n \geq 2$ as a consequence of Proposition 3.13. In fact, any properly infinite C^*-algebra A contains two isometries with orthogonal ranges, meaning that $\mathrm{sr}(A) = \infty$ for all properly infinite C^*-algebras.

We will show that the reduced group C^*-algebra of the free group on $n \geq 1$ generators, $C_r^*(F_n)$, has stable rank one for all $n \geq 1$. This result was shown by Dykema, Haagerup and Rørdam in [9], an answer to the question asked by Rieffel in his original paper [17]. Interestingly, Anderson proved the following result for the full group C^*-algebra of the free group F_n. The original proof of Anderson's can be found in [17, Theorem 6.7]

Theorem 3.14. *Let $C^*(F_n)$ denote the full group C^*-algebra of the free group of $n \geq 2$ generators. Then sr$(C^*(F_n)) = \infty$.*

Proof. We give a proof for the case of $C^*(F_n)$ for $n \geq 4$.

By definition, $\mathcal{O}_2$ is generated by 2 isometries. This implies that $\mathcal{O}_2$ is generated by 4 self-adjoint elements, in turn implying that $\mathcal{O}_4$ is generated by 4 unitaries $v_1, \dots, v_4$. The full group C^*-algebra $C^*(F_n)$ is the universal C^*-algebra generated by n unitaries $u_1, \dots u_n$. Define $*$-homomorphism $\pi : C^*(F_n) \to \mathcal{O}_2$ by letting $\pi(u_i) = v_i$ for $1 \leq i \leq 4$ and $\pi(u_i) = 1$ for $i \geq 5$. It is straightforward to see that π indeed defines a $*$-homomorphism which moreover is surjective, proving that $\mathcal{O}_2$ is contained as a quotient of $C^*(F_n)$. It follows from [17, Theorem 4.3] that $C^*(F_n)$ has stable rank ∞. □

One can, using that $\mathcal{O}_n$ is singly generated, give a proof for the case of F_2 and F_3.

3.1 The special case of stable rank one

This thesis is primarily concerned with determining when certain C^*-algebras have stable rank one, as it has several different consequences, some related to the K-groups of the unital C^*-algebra. The first is an equivalent characterization for when a unital C^*-algebra A has stable rank one. Denote the group of invertible elements in A by $\mathrm{GL}(A)$.

Proposition 3.15. *A unital C^*-algebra A has stable rank one if and only if the invertible elements is a norm dense subset of A.*

Proof. Note that $\mathrm{Lg}_1(A)$ is the set of left invertible elements in A. Thus, if $\mathrm{GL}(A) \subset \mathrm{Lg}_1(A)$ is dense in A, then so is $\mathrm{Lg}_1(A)$.

Suppose, conversely, that $\mathrm{sr}(A) = 1$, meaning $\mathrm{Lg}_1(A)$ is dense in A. Let $a \in \mathrm{Lg}_1(A)$ and let $b \in A$ be the element such that $ba = 1$. As $\mathrm{Lg}_1(A)$ is dense, there exists $c \in \mathrm{Lg}_1(A)$ such that $\|b - c\| < \|a\|^{-1}$ and so

$$\|ca - 1\| = \|ca - ba\| \leq \|c - b\| \, \|a\| < 1.$$

Using the Neumann Series, ca is invertible, implying that c, hence also a, is invertible. In particular $a \in \mathrm{GL}(A)$ and the wanted follows. □

For $n \geq 1$, let $M_n(A)$ denote the matrix algebra of $n \times n$ matrices with entries in A and let 1_n denote the unit in $M_n(A)$. For $n, k \geq 1$, let $M_{n,k}(A)$ denote the set of matrices $n \times k$ matrices with entries in A.

Proposition 3.16. *Let A be a unital C^*-algebra and let $n \geq 1$ be given. Let $k, m \geq 0$ such that $k + m = n$. For $x \in M_n(A)$, write*

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a \in M_k(A)$, $b \in M_{k,m}(A)$, $c \in M_{m,k}(A)$ and $d \in M_m(A)$. Assume d is invertible. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1_k & bd^{-1} \\ 0 & 1_m \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1_k & 0 \\ d^{-1}c & 1_m \end{pmatrix}.$$

and x is invertible if and only if $a - bd^{-1}c \in M_k(A)$ is invertible.

Proof. That x has the wanted decomposition is a simple calculation.

Suppose first that $a - bd^{-1}c$ is invertible. Then

$$\begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix}$$

is invertible, and x is a product of invertible matrices, meaning x itself is invertible.

Suppose now, conversely, that x is invertible and see that

$$\begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1_k & -bd^{-1} \\ 0 & 1_m \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1_k & 0 \\ -d^{-1}c & 1_m \end{pmatrix}$$

And so

$$\begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix}$$

is invertible. Therefore, there exists $y \in M_k(A)$ such that

$$\begin{pmatrix} y & 0 \\ 0 & d^{-1} \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1_k & 0 \\ 0 & 1_n \end{pmatrix} = \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & d^{-1} \end{pmatrix}$$

so $y^{-1} = a - bd^{-1}c$, as wanted. □

We are now ready to prove the following theorem, which has interesting consequences for unital C^*-algebras with stable rank one.

Theorem 3.17. *Let A be a unital C^*-algebra. Then $sr(A) = 1$ if and only if there exists a $n \geq 1$ such that $sr(M_n(A)) = 1$ if and only if $sr(M_n(A)) = 1$ for all $n \geq 1$.*

Proof. Assume first that $\mathrm{sr}\,(M_{n+1}(A)) = 1$ for some $n \geq 1$. Let $0 < \varepsilon < 1$ be given. For $a \in A$, consider

$$\begin{pmatrix} a & 0 \\ 0 & 1_n \end{pmatrix} \in M_{n+1}(A),$$

where 1_n is the identity in $M_n(A)$. By assumption, there exists invertible element

$$x = \begin{pmatrix} a_0 & b \\ c & d \end{pmatrix} \in M_{n+1}(A)$$

with $a_0 \in A$, $b \in M_{1,n}(A)$, $c \in M_{n,1}(A)$ and $d \in M_n(A)$ such that

$$\left\| \begin{pmatrix} a_0 & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & 1_n \end{pmatrix} \right\| < \varepsilon,$$

implying that $\|d - 1_n\| < \varepsilon < 1$, ensuring that d is invertible with $\|d^{-1}\| < (1-\varepsilon)^{-1}$. It follows from Proposition 3.16 that

$$\begin{pmatrix} a_0 & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} a_0 - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d^{-1}c & 1_n \end{pmatrix}.$$

and $a_0 - bd^{-1}c$ is invertible, as x is invertible. See now that

$$\|a - (a_0 - bd^{-1}c)\| \leq \|t - a\| + \|bd^{-1}c\| < \varepsilon + \varepsilon^2(1-\varepsilon)^{-1} = \varepsilon(1-\varepsilon)^{-1},$$

which shows the wanted.

Suppose now, conversely, that $\mathrm{sr}(A) = 1$. We prove the wanted by induction. It is clear for $n = 1$. Let $n \geq 1$ be given and consider

$$y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{n+1}(A)$$

with $a \in A$, $b \in M_{1,n}(A)$, $c \in M_{n,1}(A)$ and $d \in M_n(A)$. For $\varepsilon > 0$, there exists invertible $a_0 \in A$ such that $\|a_0 - a\| < \varepsilon$ and by induction, there exists $d_0 \in M_n(A)$ such that $\|d - d_0\| < \varepsilon$ and $d_0 - ca_0^{-1}b$ is invertible. It follows by calculations as in the proof of Proposition 3.16 that

$$z = \begin{pmatrix} a_0 & b \\ c & d_0 \end{pmatrix}$$

is invertible. Furthermore, $\|y - z\| \leq 2\varepsilon$, which proves the wanted. □

Example 3.18. Clearly $\mathrm{GL}(\mathbb{C}) = \mathbb{C} \setminus \{0\}$ is dense in $\mathbb{C}$ and so $\mathrm{GL}(M_n(\mathbb{C}))$ is dense in $M_n(\mathbb{C})$ for all $n \geq 1$, cf. Theorem 3.17. It is known that each finite dimensional C^*-algebra is isomorphic to a direct sum of matrix algebras over $\mathbb{C}$ [22, Proposition 7.1.5]. Therefore, any finite dimensional unital C^*-algebra will have stable rank one.

Denote the set of projections in a unital C^*-algebra A by $\mathcal{P}(A)$. Two projections $p, q \in \mathcal{P}(A)$ are *Murray-von Neumann equivalent*, written $p \sim q$, if there exists $v \in A$ such that $p = v^*v$ and $q = vv^*$. If there exists unitary $u \in \mathcal{U}(A)$ such that $q = upu^*$, then p, q are *unitarily equivalent*, written $p \sim_u q$. It is well-known that the two are well-defined equivalence relations on $\mathcal{P}(A)$.

Definition 3.19. Let A be a unital C^*-algebra. A projection $p \in A$ is *finite* if $p \sim q \leq p$ implies $q = p$, and p is *infinite* if it is not finite.

Definition 3.20. A unital C^*-algebra A is *finite* if the unit 1 in A is finite, and A is *infinite* if 1 is infinite. A is *stably finite* if $M_n(A)$ is finite for all integers $n \geq 1$.

Proposition 3.21. *Let A be a unital C^*-algebra. If $sr(A) = 1$, then A is stably finite.*

Proof. We claim that a non-unitary isometry has distance 1 to the set of invertible elements. Let $v \in A$ be a non-unitary isometry, i.e. $v^*v = 1$ and $vv^* \neq 1$. See first that

$$\mathrm{dist}(v, \mathrm{GL}(A)) = \inf_{y \in \mathrm{GL}(A)} \|v - y\| \leq \|v\| + \inf_{y \in \mathrm{GL}(A)} \|y\| = \|v\| = 1.$$

For the opposite inequality, see first that if $w \in A$ such that $\|v - w\| < 1$, then w is not invertible. Indeed,

$$\|1 - v^*w\| = \|v^*v - v^*w\| \leq \|v - w\| \, \|v^*\| < 1,$$

meaning v^*w is invertible. If w was invertible, then v^* would be invertible, in contradiction to v being a non-unitary isometry. Therefore, for $z \in \mathrm{GL}(A)$ it holds that $\|v - z\| \geq 1$, implying that $\mathrm{dist}(v, \mathrm{GL}(A)) \geq 1$, and so $\mathrm{dist}(v, \mathrm{GL}(A)) = 1$.

As $\mathrm{sr}(A) = 1$, it follows from Theorem 3.17 that the invertible elements are dense in $M_n(A)$ for all $n \geq 1$. It now follows from the claim that $M_n(A)$ cannot have non-unitary isometries, meaning all isometries of $M_n(A)$ are unitary. Suppose now that $1_n \sim p$, so there exists $v \in M_n(A)$ such that $1_n = v^*v$ and $p = vv^*$. Then v is an isometry and therefore unitary, so $p = vv^* = 1_n$, meaning 1_n is finite, hence $M_n(A)$ is finite for all $n \geq 1$. $\square$

Proposition 3.21 gives a possible obstruction to having stable rank one, as any unital C^*-algebra which is not stably finite cannot have stable rank one.

For $n \geq 1$, let $\mathrm{GL}_n(A) = \mathrm{GL}(M_n(A))$. Two elements $x, y \in \mathrm{GL}_n(A)$ are *homotopic* in $\mathrm{GL}_n(A)$ if there exists continuous function $f\colon [0,1] \to \mathrm{GL}_n(A)$ such that $f(0) = x$ and $f(1) = y$, and if so we write $x \sim_h y$. It is well-known that $\sim_h$ is an equivalence relation.

Lemma 3.22. *Let A be a unital C^*-algebra with $sr(A) = 1$. For $n \geq 1$, if $x \in GL_n(A)$, then there exists $x_1, \ldots, x_n \in GL(A)$ such that*

$$x \sim_h \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}.$$

Proof. The proof is by induction over n. It is clear for $n = 1$. Assume now the claim holds for some $n \geq 1$. For $x \in \mathrm{GL}_{n+1}(A)$ write

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a \in M_n(A)$, $b \in M_{n,1}(A)$, $c \in M_{1,n}(A)$ and $d \in A$. As $\mathrm{sr}(A) = 1$, we can find $d_0 \in \mathrm{GL}(A)$ such that $\|d - d_0\| < \left\|x^{-1}\right\|^{-1}$. Let

$$x_0 = \begin{pmatrix} a & b \\ c & d_0 \end{pmatrix},$$

with a, b, c as previously. Then $\|x - x_0\| < \left\|x^{-1}\right\|^{-1}$, implying that $x_0 \in \mathrm{GL}_{n+1}(A)$ and $x \sim_h x_0$. As d_0 is invertible, we can, as previously, write

$$\begin{pmatrix} a & b \\ c & d_0 \end{pmatrix} = \begin{pmatrix} 1_n & bd_0^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a - bd_0^{-1}c & 0 \\ 0 & d_0 \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ d_0^{-1}c & 1 \end{pmatrix}.$$

and x_0 invertible forces $a - bd_0^{-1}c$ to be invertible, cf. Proposition 3.16. Moreover, as

$$\begin{pmatrix} 1_n & bd_0^{-1} \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} 1_n & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1_n & 0 \\ d_0^{-1}c & 1 \end{pmatrix} \sim_h \begin{pmatrix} 1_n & 0 \\ 0 & 1 \end{pmatrix},$$

we get

$$x_0 \sim_h \begin{pmatrix} a - bd_0^{-1}c & 0 \\ 0 & d_0 \end{pmatrix}.$$

Since $a - bd_0^{-1}c \in \mathrm{GL}_n(A)$, we can, by assumption, find diagonal matrix

$$y = \begin{pmatrix} y_1 & & 0 \\ & \ddots & \\ 0 & & y_n \end{pmatrix}$$

with $y_1, \ldots, y_n \in \mathrm{GL}(A)$ such that $a - bd_0^{-1}c \sim_h y$. We now have

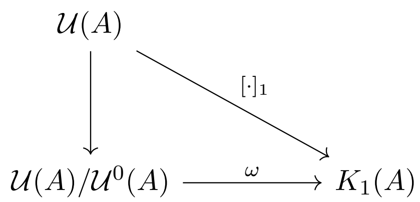
$$x \sim_h x_0 \sim_h \begin{pmatrix} y & 0 \\ 0 & d_0 \end{pmatrix},$$

as wanted. □

Let A be a unital C^*-algebra and let $\mathcal{U}(A)$ denote the group of unitary elements in A. Set $\mathcal{U}_n(A) = \mathcal{U}(M_n(A))$ for $n \geq 1$ and $\mathcal{U}_\infty(A) = \bigcup_{n=1}^\infty \mathcal{U}_n(A)$. Let $n, m \geq 1$, $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$. Define binary operation $u \oplus v = \mathrm{diag}(a, b)$ on $\mathcal{U}_\infty(A)$. Write $u \sim_1 v$ if there exists integer $k \geq \max\{m, n\}$ such that $u \oplus 1_{k-n} \sim_h v \oplus 1_{k-m}$ in $\mathcal{U}_k(A)$. It is well-known that $\sim_1$ defines an equivalence relation. Let $[u]_1$ denote the equivalence class containing $u \in \mathcal{U}_\infty(A)$. The K_1 group of A is then

$$K_1(A) = \{[u]_1 : u \in \mathcal{U}_\infty(A)\}.$$

Let $\mathcal{U}^0(A)$ be the connected component of the identity element in $\mathcal{U}(A)$. There is a group homomorphism $\omega : \mathcal{U}(A)/\mathcal{U}^0(A) \to K_1(A)$ making the following diagram commutative



The map ω is in general neither injective nor surjective. The unital C^*-algebra A is called K_1-*injective* if the map ω is injective, and K_1-*surjective* if ω is surjective. Rieffel proved that it is a consequence of having stable rank one that ω is an isomorphism.

Let $\mathcal{U}_n^0(A)$ be the connected component of the identity element in $\mathcal{U}_n(A)$.

Theorem 3.23 (Rieffel)**.** *Let A be a unital C^*-algebra. For all $n \geq sr(A)$ the map from $\mathcal{U}_n(A)/\mathcal{U}_n^0(A)$ to $\mathcal{U}_{n+1}(A)/\mathcal{U}_{n+1}^0(A)$ is an isomorphism, and in particular*

$$\mathcal{U}_n(A)/\mathcal{U}_n^0(A) \cong K_1(A).$$

Remark 3.24. Theorem 3.23 is stated and proven in [18, Theorem 2.10] in a slightly different way, such that for any $n \geq \mathrm{sr}(A)$

$$\mathrm{GL}_n(A)/\mathrm{GL}_n^0(A) \cong K_1(A)$$

where $\mathrm{GL}_n^0(A)$ is the connected component of the identity in $\mathrm{GL}_n(A)$. That the two are equivalent for a unital C^*-algebra follows as $\mathcal{U}(A)$ is a retract of $\mathrm{GL}(A)$.

We include the proof of K_1-surjectivity for a unital C^*-algebra A with stable rank one. The reader is referred to [18] for a complete proof of Theorem 3.23.

Recall that any invertible element $x \in A$ has a unique polar decomposition $x = u|x|$, where $|x| = (x^*x)^{\frac{1}{2}}$ and $u \in A$ is unitary.

Proof of K_1-surjectivity. Let $u \in \mathcal{U}_n(A)$ for some $n \geq 1$. Using Lemma 3.22, there exists $x_1, \ldots, x_n \in \mathrm{GL}_n(A)$ such that

$$u \sim_h \begin{pmatrix} x_i & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}.$$

Consider the polar decomposition $x_j = u_j|x_j|$ for each $1 \leq j \leq n$ and as each x_j is invertible, $u_j \in \mathcal{U}(A)$. Using [22, Lemma 2.1.5], we now get

$$\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \sim_h \begin{pmatrix} u_1 & & 0 \\ & \ddots & \\ 0 & & u_n \end{pmatrix} \sim_h \begin{pmatrix} u_1 \cdots u_n & 0 \\ 0 & 1_{n-1} \end{pmatrix},$$

By the standard picture of $K_1(A)$, [22, Proposition 8.1.4],

$$[u]_1 = \left[\begin{pmatrix} u_1 \cdots u_n & 0 \\ 0 & 1_{n-1} \end{pmatrix} \right]_1 = [u_1 \cdots u_n]_1,$$

proving surjectivity. □

Let A be a unital C^*-algebra and $I \subset A$ a closed, two-sided ideal, giving the following short exact sequence

$$0 \longrightarrow I \hookrightarrow A \xrightarrow{\pi} A/I \longrightarrow 0$$

To any such short exact sequence is the six-term exact sequence

$$\begin{array}{ccccc} K_1(I) & \xrightarrow{K_1(\iota)} & K_1(A) & \xrightarrow{K_1(\pi)} & K_1(A/I) \\ \uparrow{\scriptstyle \delta_0} & & & & \downarrow{\scriptstyle \delta_1} \\ K_0(A/I) & \xleftarrow[K_0(\pi)]{} & K_0(A) & \xleftarrow[K_0(\iota)]{} & K_0(I) \end{array}$$

where $\delta_0 : K_0(A/I) \to K_1(I)$ is the exponential map and $\delta_1 : K_1(A/I) \to K_0(I)$ is the index map.

Proposition 3.25. *Let A be a unital C^*-algebra. Assume there exists a closed, two-sided ideal $I \subset A$ with short exact sequence*

$$0 \longrightarrow I \hookrightarrow A \xrightarrow{\pi} A/I \longrightarrow 0.$$

If $sr(A) = 1$, then the index map $\delta_1 : K_1(A/I) \to K_0(I)$ is the zero map.

Proof. Let $u \in \mathcal{U}_n(A/I)$ and let $v \in M_n(A)$ such that $\pi(v) = u$. As $\mathrm{sr}(A) = 1$, there is invertible $w \in M_n(A)$ such that $\|v - w\| < 1$, cf. Theorem 3.17. In particular,

$$\|u - \pi(w)\| = \|\pi(v) - \pi(w)\| < 1,$$

implying that $u \sim_h \pi(w)$ in $\mathrm{GL}_n(A/I)$. Consider polar decomposition $w = x|w|$ and note that $x \in \mathcal{U}_n(A)$. Then $\pi(w) = \pi(x)\pi(|x|)$ with $\pi(x)$ unitary, and moreover $\pi(w) \sim_h \pi(x)$, implying $u \sim_h \pi(x)$, so $[u]_1 = [\pi(x)]_1 = K_1(\pi)([x]_1)$. Therefore, $K_1(\pi)$ is surjective and it follows, using the six-term exact sequence, that $\delta_1 : K_1(A/I) \to K_0(I)$ is the zero map. □

Example 3.26. Consider $C(\mathbb{D})$, where $\mathbb{D} \subset \mathbb{R}^2$ is the unit disk. It is known that $C(\mathbb{D})$ contains $C_0(\mathbb{R}^2)$ as an ideal with $C(\mathbb{D})/C_0(\mathbb{R}^2) = C(\mathbb{T})$. Therefore, we have the following short exact sequence

$$0 \longrightarrow C_0(\mathbb{R}^2) \hookrightarrow C(\mathbb{D}) \xrightarrow{\pi} C(\mathbb{T}) \longrightarrow 0,$$

with corresponding six-term exact sequence

$$\begin{array}{ccccc} K_1(C_0(\mathbb{R}^2)) & \xrightarrow{K_1(\iota)} & K_1(C(\mathbb{D})) & \xrightarrow{K_1(\pi)} & K_1(C(\mathbb{T})) \\ \uparrow{\scriptstyle \delta_0} & & & & \downarrow{\scriptstyle \delta_1} \\ K_0(C(\mathbb{T})) & \xleftarrow[K_0(\pi)]{} & K_0(C(\mathbb{D})) & \xleftarrow[K_0(\iota)]{} & K_0(C_0(\mathbb{R}^2)) \end{array}$$

One can show that $K_1(C(\mathbb{T})) = \mathbb{Z}$ and $K_1(C(\mathbb{D})) = 0$, implying that $K_1(\pi)$ cannot be surjective, and equivalently δ_1 is not the zero-map, hence $\mathrm{sr}(C(\mathbb{D}))$ cannot be one. In fact, $\mathrm{sr}(C(\mathbb{D})) = 2$ from Proposition 3.8.

We now have two obstructions to a unital C^*-algebra A having stable rank one: δ_1 not being the zero-map and A not being stably finite. It was an open question for some time whether every stably finite, simple C^*-algebra has stable rank 1. However, Villadsen gave a negative answer to this question, as he constructed stably finite, simple C^*-algebras with stable rank n for all $n \geq 2$, see [23]. It is currently an open question if there exists a group G such that $C_r^*(G)$ is simple and has stable rank $n \geq 2$. Note that the non-trivial amenable groups considered in Example 3.9 are not C^*-simple.

The next consequence for a unital C^*-algebra A with stable rank one is in regard to the different equivalence relations on projections, which in turn will be used to prove that A has the cancellation property.

Lemma 3.27. *Let A be a unital C^*-algebra with stable rank one. If p, q are projections in A with $p \sim q$, then $p \sim_u q$ and $1 - p \sim 1 - q$.*

Proof. Suppose that p, q are projections in A such that $p \sim q$. Let $u \in A$ such that $p = u^*u$ and $q = uu^*$. As $\mathrm{sr}(A) = 1$, there exists invertible $x \in A$ which approximates u closely and so that

$$\begin{aligned} \|x^*x - p\| = \|x^*x - u^*u\| &= \|x^*x - x^*u + x^*u - u^*u\| \\ &\leq \|x^*(x - u)\| + \|(x^* - u^*)u\| \\ &\leq \|x^*\|\,\|x - u\| + \|x^* - u^*\| < 1 \end{aligned}$$

and similarly $\|xx^* - q\| < 1$. Consider the polar decomposition $x = v|x|$, and as x is invertible, $v \in \mathcal{U}(A)$. Then $xx^* = v(x^*x)^{\frac{1}{2}}(x^*x)^{\frac{1}{2}}v^* = v(x^*x)v^*$, and

$$\|v(x^*x)v^* - vpv^*\| = \|v(x^*x - p)v^*\| \leq \|x^*x - p\| < 1.$$

It now follows using [22, Proposition 2.2.4] that $vpv^* \sim_h q$, implying that $vpv^* \sim_u q$, hence

$$p \sim_u vpv^* \sim_u q.$$

Using [22, Proposition 2.2.2], we now have $1 - p \sim 1 - q$. □

Remark 3.28. It is a result of Brown in [4] that unitarily equivalent projections in a C^*-algebra A with stable rank one are in fact homotopic. This combined with Lemma 3.27 shows that the three equivalences on projections in A are the same when $\mathrm{sr}(A) = 1$.

For $n \geq 1$, let $\mathcal{P}_n(A) = \mathcal{P}(M_n(A))$ and $\mathcal{P}_\infty(A) = \bigcup_{n=1}^\infty \mathcal{P}_n(A)$. Suppose $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$, then $p \sim_0 q$ if there is an element $v \in M_{m,n}(A)$ such that $p = v^*v$ and $q = vv^*$. Note that $\sim_0$ defines an equivalence relation on $\mathcal{P}_\infty(A)$ and for $p, q \in \mathcal{P}_n(A)$, $p \sim_0 q$ if and only if $p \sim q$. One can use the equivalence relation $\sim_0$ to define an abelian semigroup $\mathcal{D}(A) = \mathcal{P}_\infty / \sim_0$, which in turn is used to describe the K_0 group of A via the Grothendieck map, see [22, Definition 3.1.4]. For a unital C^*-algebra A,

$$K_0(A) = \{[p]_0 - [q]_0 \mid p, q \in \mathcal{P}_n(A), n \in \mathbb{N}\}.$$

The C^*-algebra A is said to have the *cancellation property* if and only if for every pair of projections p, q in $\mathcal{P}_\infty$, $[p]_0 = [q]_0$ if and only if $p \sim_0 q$.

As previously, define binary operation on $\mathcal{P}_\infty(A)$ by $p \oplus q = \mathrm{diag}(p, q)$. Define relation $\sim_s$ on $\mathcal{P}_\infty(A)$ by the following: If $p, q \in \mathcal{P}_\infty(A)$, then $p \sim_s q$ if and only if there exists $r \in \mathcal{P}_\infty(A)$ such that $p \oplus r \sim_0 q \oplus r$. Equivalently, $p \sim_s q$ if and only if $p \oplus 1_n \sim_0 q \oplus 1_n$ for some $n \geq 1$. Note that $\sim_s$ defines an equivalence relation on $\mathcal{P}_\infty(A)$ called *stable equivalence.*

Proposition 3.29. *Let A be a unital C^*-algebra with stable rank one. Then A has the cancellation property.*

Proof. See first that if $p \sim_0 q$ then $[p]_\mathcal{D} = [q]_\mathcal{D}$ in $\mathcal{D}(A)$, and so $[p]_0 = [q]_0$ by the defintion of $K_0(A)$.

For the converse implication, let $p, q \in \mathcal{P}_n(A)$ such that $[p]_0 = [q]_0$. Then, using the standard picture of $K_0(A)$ [22, Proposition 3.1.7], $p \sim_s q$, so there is $r \geq 1$ such that $p \oplus 1_r \sim_0 q \oplus 1_r$. Note that $p \oplus 1_r, q \oplus 1_r \in \mathcal{P}_{n+r}(A)$, and so we have $p \oplus 1_r \sim q \oplus 1_r$. Using Lemma 3.27, we get

$$\begin{pmatrix} 1_n - p & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1_n - q & 0 \\ 0 & 0 \end{pmatrix},$$

implying that $1_n - p \sim 1_n - q$, and Lemma 3.27 implies $p \sim q$, equivalent to $p \sim_0 q$. □

4 Free probability theory

Free probability theory was introduced by Dan Voiculescu in the 1980's during his work with the free group factors and is regarded as non-commutative probability theory [24, 25].

Definition 4.1. Let A be an algebra with unit and let φ be a linear functional on A such that $\varphi(1_A) = 1$. We call (A, φ) a non-commutative probability space.

We are particularly interested in the case of A being a unital C^*-algebra. In this case we consider a state φ on A and (A, φ) is called a *C^*-probability space.*

Just like in the classical case of probability theory, elements a of (A, φ) are called non-commutative random variables and $\varphi(a)$ the expectation of a. The notion of independence in classical probability theory relies heavily on commutativity and is therefore not of much use in our non-commutative scenario. Therefore, Voiculescu introduced the notion of being *free* as an analogue to independence in classical probability theory.

Definition 4.2. Let (A, φ) be a a C^*-probability space and let $(A_i)_{i\in I}$ be a family of unital C^*-subalgebras for some index set I. We say that $(A_i)_{i\in I}$ is *free* with respect to φ if

$$\varphi(a_{i_1} \dots a_{i_k}) = 0,$$

whenever, for $k \geq 1$, $a_{i_j} \in A_{i_j}$, $i_j \neq i_{j+1}$ with $\varphi(a_{i_j}) = 0$ for $j = 1, \dots, k$. A family $(a_i)_{i\in I}$ of random variables is free if the family of unital C^*-algebras $(C^*(a_i))_{i\in I}$ is free.

4.1 The reduced free product

Let I be an index set and consider a family $(A_i, \varphi_i)_{i\in I}$ of unital C^*-algebras A_i each equipped with a faithful state φ_i. The reduced free product C^*-algebra corresponding to each such family $(A_i, \varphi_i)_{i\in I}$ was introduced by Voiculescu in [24] and is denoted by

$$(A, \varphi) = *_{i\in I}(A_i, \varphi_i).$$

It is constructed in such a way that $(A_i)_{i\in I}$ is free in A with respect to φ.

We will need the definition of a standard orthonormal basis for a unital C^*-algebra. Let A be a unital C^*-algebra. Given a faithful state φ on A, define the following Euclidean structure corresponding to φ

$$\langle a, b \rangle_\varphi = \varphi(b^* a), \qquad \|a\|_\varphi = \langle a, a \rangle_\varphi^{1/2}, \qquad \text{for } a, b \in A.$$

Definition 4.3. Let A be a unital C^*-algebra equipped with a faithful state φ. A subset $X \subset A$ is called a *standard orthonormal basis* for A if $1 \in X$ and X moreover satisfies

(i) X is an orthonormal set with respect to the Euclidean structure corresponding to φ on A,

(ii) $\operatorname{span} X$ is a norm-dense *-subalgebra of A.

We will denote $X \setminus \{1\}$ by X°.

Lemma 4.4. *Let A be a separable unital C^*-algebra and assume that $F \subset A$ is a finite, orthonormal set containing* 1*. Then there exists a countable standard orthonormal basis X for A which contains F.*

Proof. Choose a dense subset $\{a_1, a_2, a_3, \ldots\}$ of A. We will, using this dense subset, inductively construct finite orthonormal sets $X_0 \subset X_1 \subset X_2 \subset \ldots$ satisfying the following

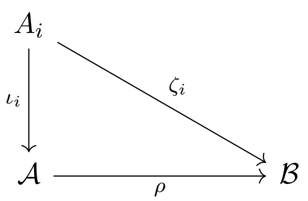
(i) $a_n \in \operatorname{span} X_n$,

(ii) $\operatorname{span} X_n$ is self-adjoint,

(iii) If $x, y \in X_{n-1}$ then $xy \in \operatorname{span} X_n$,

for all $n \geq 1$. Set $X_0 = F$. Suppose now X_{n-1} has been constructed and let V_n be the finite dimensional subspace of A spanned by $a_n, X_{n-1}, X_{n-1} \cdot X_{n-1}$ along with the adjoints of said elements. Let X_n be an orthonormal basis for V_n which extends X_{n-1}. Set now $X = \bigcup_{n=0}^{\infty} X_n$. It is clear that $\operatorname{span} X$ is a *-subalgebra. By construction, $1 \in X$ and $\{a_1, a_2, a_3, \ldots\} \subset X$, ensuring that X is in fact a norm-dense *-subalgebra in X. Lastly, each X_n is chosen to be an orthonormal extension of X_{n-1}, ensuring that X will be orthonormal, proving that X is a countable standard orthonormal basis for A containing F. □

The algebraic unital free product of unital C^*-algebras A_i, denoted by $\mathcal{A} = *^{\text{alg}}_{i \in I} A_i$, is the unital $*$-algebra equipped with unital inclusions $\iota_i : A_i \to \mathcal{A}$ for each $i \in I$ with the following universal property: If $\mathcal{B}$ is another unital $*$-algebra equipped with unital inclusions $\zeta_i : A_i \to \mathcal{B}$ for each $i \in I$, then there exists a unique $*$-homomorphism $\rho : \mathcal{A} \to \mathcal{B}$ such that the following diagram commutes for all $i \in I$:



A state on a unital $*$-algebra A is a linear map $\varphi : A \to \mathbb{C}$ such that $\varphi(1_A) = 1$ and $\varphi(a^*a) \geq 0$ for all $a \in A$. We say that φ is faithful if $\varphi(a^*a) = 0$ implies $a = 0$.

Elements in $\mathcal{A} = *^{\text{alg}}_{i\in I} A_i$ of the form

$$w = a_{i_1} a_{i_2} \dots a_{i_k},$$

are called words and a_{i_j} the letters of w. If no a_{i_j} is followed by $a_{i_j}^{-1}$, then w is a *reduced word.* If $\mathcal{A}$ has a state φ and we further assume that for each $1 \leq j \leq k$, $a_{i_j} \in A_{i_j}$, $\varphi(a_{i_j}) = 0$ and $i_j \neq i_{j+1}$ for all $1 \leq j \leq k-1$, then w is said to be a *centered reduced word of length k* with the convention that $1 \in \mathcal{A}$ is the reduced word of length 0. Each word $w \in \mathcal{A}$ can be written as a finite linear combination of centered reduced words by writing $a'_{i_j} = a_{i_j} - \varphi(a_{i_j})1$, which will be in A_{i_j} with $\varphi(a'_{i_j}) = 0$. Therefore, the elements of $\mathcal{A}$ can be expressed as finite linear combinations of centered reduced words.
The following lemma constructs orthonormal set in $\mathcal{A}$ from orthonormal sets X_i of A_i.

Lemma 4.5. *Let $(A_i, \varphi_i)_{i\in I}$ be a family of unital C^*-algebras each with state φ_i. Let $\mathcal{A} = *^{alg}_{i\in I} A_i$ and assume that $\mathcal{A}$ has a state φ_0 such that $\varphi_0(w) = 0$ for all centered reduced words $w \in \mathcal{A}$. Given orthonormal sets $X_i \subset A_i$ for $i \in I$, let $Y_0 = \{1\}$ and for $n \geq 1$ let Y_n be the set of words*

$$w = w_{i_1} w_{i_2} \cdots w_{i_n},$$

such that $w_{i_j} \in X^\circ_{i_j}$, $\varphi(w_{i_j}) = 0$ for all $1 \leq j \leq n$ and $i_j \neq i_{j+1}$ for all $1 \leq j \leq n-1$. Define $*_{i\in I} X_i = \bigcup_{n=0}^\infty Y_n$. Then $*_{i\in I} X_i$ is an orthonormal set with respect to $\varphi_0(b^*a) = \langle a, b\rangle_{\varphi_0}$.

Proof. Consider first $w \in X^\circ_i$, $z \in X^\circ_j$ for some $i, j \in I$ with $\varphi_0(w) = \varphi_0(z) = 0$. If $i \neq j$, then $\varphi_0(z^*w) = 0$ as z^*w is a centered reduced word. If $i = j$, then $\varphi_0(z^*w) = \varphi_i(z^*w) = 0$ for $z \neq w$ and $\varphi_0(w^*w) = \varphi_i(w^*w) = 1$, as X_i is an orthonormal set with respect to the Euclidean structure from φ_i on A_i.

Assume now $w, z \in *_{i\in I} X_i$ of length $n, m \geq 1$. Assume without loss of generality that $n \geq m$ and write

$$w = w_{i_1} \cdots w_{i_n}, \qquad z = z_{j_1} \cdots z_{j_m}$$

where for all l, k, $w_{i_k} \in X^\circ_{i_k}$, $z_{j_l} \in X^\circ_{j_l}$, $\varphi_0(w_{i_k}) = \varphi_0(z_{j_l}) = 0$ and $i_k \neq i_{k+1}$, $j_l \neq j_{l+1}$ for all $1 \leq k \leq n-1$, respectively $1 \leq l \leq m-1$. We claim that

$$\varphi_0(z^*w) = \delta_{n,m} \delta_{i_1,j_1} \cdots \delta_{i_n,j_n} \varphi_0(z^*_{j_1} w_{i_1}) \cdots \varphi_0(z^*_{j_m} w_{i_n}).$$

If $i_1 \neq j_1$, then z^*w is a centered reduced word and $\varphi_0(z^*w) = 0$ by assumption. If $i_1 = j_1$, write

$$(z^*_{j_1} w_{i_1})' = z^*_{j_1} w_{i_1} - \varphi_0(z^*_{j_1} w_{i_1}).$$

Then

$$\varphi_0(z^*w) = \varphi_0(z_{j_1}^* w_{i_1})\varphi_0(z_{j_m}^* \cdots z_{j_2}^* w_{i_2} \cdots w_{i_n}) + \varphi_0(z_{j_m}^* \cdots z_{j_2}^*(z_{j_1}^* w_{i_1})' w_{i_2} \cdots w_{i_n}).$$

By definition $\varphi_0((z_{j_1}^* w_{i_1})') = 0$, meaning $z_{j_m}^* \cdots z_{j_2}^*(z_{j_1}^* w_{i_1})' w_{i_2} \cdots w_{i_n}$ is a centered reduced word, so

$$\varphi_0(z^*w) = \varphi_0(z_{j_1}^* w_{i_1})\varphi_0(z_{j_m}^* \cdots z_{j_2}^* w_{i_2} \cdots w_{i_n}).$$

Therefore,

$$\varphi_0(z^*w) = \delta_{i_1,j_1}\varphi_0(z^*w) = \delta_{i_1,j_1}\varphi_0(z_{j_1}^* w_{i_1})\varphi_0(z_{j_m}^* \cdots z_{j_2}^* w_{i_2} \cdots w_{i_n}).$$

Iterating this process,

$$\varphi_0(z^*w) = \delta_{i_1,j_1}\delta_{i_2,j_2}\cdots\delta_{i_m,j_m}\varphi_0(z_{j_1}^* w_{i_1})\varphi_0(z_{j_2}^* w_{i_2})\cdots\varphi_0(z_{j_m}^* w_{i_m})\varphi_0(w_{i_{m+1}} \cdots w_{i_n}).$$

Note that $\varphi_0(w_{i_{m+1}} \cdots w_{i_n}) = 0$, as $w_{i_{m+1}} \cdots w_{i_n}$ is a centered reduced word. In particular, $\varphi_0(z^*w) = 0$ for $m \neq n$, proving that

$$\varphi_0(z^*w) = \delta_{n,m}\delta_{i_1,j_1}\cdots\delta_{i_n,j_n}\varphi_0(z_{j_1}^* w_{i_1})\cdots\varphi_0(z_{j_n}^* w_{i_n}).$$

See now that if $i_k \neq j_k$ for some k, then $z \neq w$ and the claim shows that $\varphi_0(z^*w) = 0$. If $i_k = j_k$ for all k, we note that each X_{i_k} is an orthonormal set, hence $\varphi_0(z_{i_k}^* w_{i_k}) = 0$ for $z_{i_k} \neq w_{i_k}$ and $\varphi_0(w_{i_k}^* w_{i_k}) = 1$. Therefore, $\varphi_0(z^*w) = 0$ for $z \neq w$ and $\varphi_0(w^*w) = 1$. We conclude that $*_{i\in I}X_i$ is an orthonormal set. □

Definition 4.6 (The Hilbert space free product)**.** Let I be an index set. Consider a family $(H_i, \xi_i)_{i\in I}$ of Hilbert spaces H_i each with unit vector $\xi_i \in H_i$. Define

$$H = \mathbb{C}\xi \oplus \bigoplus_{n\geq 1}\left(\bigoplus_{(i_1,\ldots,i_n)\in D_n(I)} \mathring{H}_{i_1} \otimes \ldots \otimes \mathring{H}_{i_n}\right),$$

where $\mathring{H}_{i_j}$ is the orthogonal complement of $\mathbb{C}\xi_{i_j}$ in H_{i_j}, and $D_n(I)$ is the set of n-tuples $(i_1, \ldots, i_n)$ such that $i_j \neq i_{j+1}$ for $1 \leq j \leq n-1$ and each $i_j \in I$. Moreover, ξ is a distinguished unit vector in $\mathbb{C}$, corresponding to $n = 0$. We denote the *Hilbert space free product* by $(H, \xi) = *_{i\in I}(H_i, \xi_i)$.

We will, for notational purposes, omit $D_n(I)$. For $j \in I$, set

$$H(j) = \mathbb{C}\xi \oplus \bigoplus_{n\geq 1}\left(\bigoplus_{\substack{(i_1,\ldots,i_n)\\ i_1\neq j}} \mathring{H}_{i_1} \otimes \ldots \otimes \mathring{H}_{i_n}\right).$$

Define unitary operators $V_j : H_j \otimes H(j) \to H$ by the following

$$\xi_j \otimes \xi \mapsto \xi, \qquad h \otimes \xi \mapsto h, \qquad \xi_j \otimes (h_{i_1} \otimes \ldots \otimes h_{i_n}) \mapsto h_{i_1} \otimes \ldots \otimes h_{i_n},$$
$$h \otimes (h_{i_1} \otimes \ldots \otimes h_{i_n}) \mapsto h \otimes h_{i_1} \otimes \ldots \otimes h_{i_n}.$$

for $h \in \overset{\circ}{H}_j$, $h_{i_k} \in \overset{\circ}{H}_{i_k}$ and $(i_1, \ldots, i_n) \in D_n(I)$ with $i_1 \neq j$. Note that the unitary V_j gives isomorphism $H_j \otimes H(j) \simeq H$ for all $j \in I$.

Theorem 4.7. *Let (A_i, φ_i) be a family of unital C^*-algebras A_i each with a faithful state φ_i and let $\mathcal{A} = *_{i\in I}^{alg} A_i$ be the algebraic free product. Then there exists a unique state φ_0 on $\mathcal{A}$ such that*

(i) $\varphi_0 \circ \iota_i = \varphi_i$ for all $i \in I$,

(ii) $\varphi_0(w) = 0$ for all centered reduced words $w \in \mathcal{A}$ of length $k \geq 1$.

Moreover, φ_0 is faithful on $\mathcal{A}$.

Proof. Uniqueness is clear as $\mathcal{A}$ is spanned by the set of centered reduced words.

Let (H_i, ξ_i, π_i) be the GNS-triple associated to φ_i for each $i \in I$. Let $(H, \xi) = *_{i\in I}(H_i, \xi_i)$ be as in Definition 4.6 and define representation λ_i of A_i on H by

$$\lambda_i(a) = V_i(\pi_i(a) \otimes I_{H(i)})V_i^*, \quad a \in A_i. \tag{4.1}$$

It is straightforward to check that each λ_i is a representation. By the universal property of the algebraic free product, there exists a unique $*$-homomorphism $\pi : \mathcal{A} \to B(H)$ such that $\lambda_i = \pi \circ \iota_i$ for all $i \in I$. Define $\varphi_0 : \mathcal{A} \to \mathbb{C}$ by

$$\varphi_0(a) = \langle \pi(a)\xi, \xi \rangle, \qquad a \in \mathcal{A}.$$

It is clear that φ_0 is a state on $\mathcal{A}$. We show that φ_0 has the wanted properties. Let $a_i \in A_i$. Then

$$(\varphi_0 \circ \iota)(a_i) = \langle \pi(\iota_i(a_i))\xi, \xi \rangle = \langle \lambda_i(a_i)\xi, \xi \rangle = \left\langle V_i(\pi_i(a_i) \otimes I_{H(i)})V_i^*\xi, \xi \right\rangle.$$

As $H_i = \overset{\circ}{H}_i \oplus \mathbb{C}\xi_i$ it suffices to check that $\varphi_0 \circ \iota_i = \varphi_i$ on each component. Consider first $\pi_i(a_i)\xi_i \in \overset{\circ}{H}_i$. Then $\varphi_i(a_i) = \langle \pi_i(a_i)\xi_i, \xi_i \rangle = 0$ and by the definition of the unitary V_i,

$$\lambda_i(a_i)\xi = V_i(\pi_i(a_i) \otimes I_{H(i)})V_i^*\xi = V_i(\pi_i(a_i)\xi_i \otimes \xi) = \pi_i(a_i)\xi_i,$$

implying $\varphi_0(\iota_i(a_i)) = 0 = \varphi_i(a_i)$. If $\pi_i(a_i)\xi_i = \alpha\xi_i$ for some $\alpha \in \mathbb{C}$, then $\varphi_i(a_i) = \alpha$ and

$$\lambda_i(a_i)\xi = V_i(\alpha\xi_i \otimes \xi) = \alpha\xi,$$

hence $\varphi_0(\iota_i(a_i)) = \alpha = \varphi_i(a_i)$. This shows (i).

Consider now centered reduced word $w = a_{i_1}a_{i_2}\dots a_{i_k}$ in $\mathcal{A}$. Then

$$\varphi_{i_j}(a_{i_j}) = \left\langle \pi_{i_j}(a_{i_j})\xi_{i_j}, \xi_{i_j} \right\rangle = 0,$$

implying that $\pi_{i_j}(a_{i_j})\xi_{i_j} \in \overset{\circ}{H}_{i_j}$ for all $1 \le j \le k$. By definition,

$$\pi(a_{i_j})\xi = \lambda_{i_j}(a_{i_j})\xi = V_{i_j}(\pi_{i_j}(a_{i_j}) \otimes I_{H(i_j)})V_{i_j}^*\xi = V_{i_j}(\pi_{i_j}(a_{i_j})\xi_{i_j} \otimes \xi) = \pi_{i_j}(a_{i_j})\xi_{i_j}.$$

It is now a simple calculation to see that

$$\pi(a_{i_1}a_{i_2}\dots a_{i_k})\xi = \pi_{i_1}(a_{i_1})\xi_{i_1} \otimes \pi_{i_2}(a_{i_2})\xi_{i_2} \otimes \dots \otimes \pi_{i_k}(a_{i_k})\xi_{i_k} \in \overset{\circ}{H}_{i_1} \otimes \dots \otimes \overset{\circ}{H}_{i_k} \qquad (4.2)$$

whenever $\varphi_{i_j}(a_{i_j}) = 0$, $a_{i_j} \in A_{i_j}$ and $j \neq j+1$ for all $1 \le j \le n-1$, implying that $\varphi_0(w) = 0$, as wanted.

It remains to show that φ_0 is faithful. Let $x \in \mathcal{A}$ and assume that $\varphi_0(x^*x) = 0$. Write $x = \sum_{k=1}^{n} \alpha_k w_k$ where each $w_k \in \mathcal{A}$ is a centered reduced word

$$w_k = a^k_{i_1} a^k_{i_2} \dots a^k_{i_{r_k}},$$

with $a^k_{i_j} \in A^k_{i_j}$ and $\varphi_0(a^k_{i_j}) = 0$ for $1 \le j \le r_k$. Let I_0 be the set of all indices $i \in I$ such that i appears as an index i_j in a word w_k in the decomposition of x. For each $i \in I_0$, let F_i be the set of elements from A_i which appear in a word w_k in the decomposition of x. Set $V_i = \operatorname{span}\{F_i \cup 1\} \subset A_i$. For each i, φ_i is faithful on A_i, meaning we can equip A_i with the previously mentioned Euclidean structure corresponding to φ_i. As V_i is finite dimensional we can find an orthonormal basis X_i of V_i which contains 1 and spans V_i. Define orthonormal set $*_{i\in I} X_i$ as in Lemma 4.5. For each k, $w_k \in \operatorname{span} *_{i\in I} X_i$, so we can write $x = \sum_{j=1}^{m} \beta_j z_j$ where each $z_j \in *_{i\in I} X_i$. Thus,

$$\varphi_0(x^*x) = \sum_{j=1}^{m}\sum_{k=1}^{m} \overline{\beta_j}\beta_k \varphi_0(z_j^* z_k) = \sum_{j=1}^{m} |\beta_j|^2 = 0$$

implying $\beta_j = 0$ for all $1 \le j \le m$, hence $x = 0$. $\square$

As φ_0 is faithful, we can define the Euclidean structure

$$\varphi_0(b^*a) = \langle a, b \rangle_{\varphi_0}, \qquad \|a\|_{\varphi_0} = \langle a, a \rangle_{\varphi_0}^{\frac{1}{2}} \qquad \text{for } a, b \in \mathcal{A}.$$

As in the case of C^*-algebras, one can to a state ψ on a unital $*$-algebra $\mathcal{A}$ associate a GNS-triple $(H_\psi, \pi_\psi, \xi_\psi)$ where H_ψ is the Hilbert space completion of $\mathcal{A}$ with inner product

$$\langle a, b \rangle_\psi = \psi(b^* a)$$

for $a, b \in \mathcal{A}$, $\pi_\psi : \mathcal{A} \to B(H_\psi)$ is a faithful $*$-representation and $\xi_\psi \in H_\psi$ a cyclic unit vector such that

$$\psi(a) = \langle \pi(a)\xi_\psi, \xi_\psi \rangle_\psi .$$

Any triple (H', π', ξ') with the same properties is unitarily equivalent to $(H_\varphi, \pi_\varphi, \xi_\varphi)$.

Note that (4.2) implies that $\pi(\mathcal{A})\xi$ is dense in H, as each $\pi_i(A_i)\xi_i$ is dense in H_i. Therefore, we may take (H, π, ξ) to be the GNS-triple for φ_0. As φ_0 is faithful, $\mathcal{A}$ embeds into a C^*-algebra $A = \overline{\pi(\mathcal{A})} \subset B(H)$. Extend φ_0 to φ on A by $\varphi(a) = \langle a\xi, \xi \rangle$ for $a \in A$. Then (A, φ) is said to be the C^*-completion of $(\mathcal{A}, \varphi_0)$. Note that for $a \in A$, $b \in \mathcal{A}$,

$$\|ab\|_\varphi^2 = \langle a\pi(b)\xi, a\pi(b)\xi \rangle = \|a\pi(b)\xi\|^2 .$$

As $\pi(\mathcal{A})\xi$ is norm-dense in H,

$$\|a\| = \sup\{\|ab\|_\varphi : b \in \mathcal{A}, \|b\|_\varphi \le 1\}, \qquad a \in A.$$

In this way, $\mathcal{A}$ embeds into A as a dense $*$-subalgebra via π. However, by slight abuse of notation, we will identify $a \in \mathcal{A}$ by a in A via π.

Proposition 4.8. *Let $(\mathcal{A}, \varphi_0)$ be a unital $*$-algebra equipped with a faithful state φ_0 and let (A, φ) be the C^*-completion of $(\mathcal{A}, \varphi_0)$. Suppose that (B, ψ) is another unital C^*-algebra with state ψ and $*$-homomorphism $\rho : \mathcal{A} \to B$ such that $\rho(\mathcal{A})$ is a dense subalgebra of B and $\psi \circ \rho = \varphi_0$. Then ρ extends to a $*$-isomorphism $\eta : A \to B$ satisfying $\psi \circ \eta = \varphi$.*

Proof. By construction, $\varphi(a) = \varphi_0(a)$ for $a \in \mathcal{A}$, so $\|a\|_\varphi = \|a\|_{\varphi_0}$. Moreover, for $b \in \mathcal{A}$,

$$\|b\|_{\varphi_0}^2 = \varphi_0(b^*b) = \psi(\rho(b^*)\rho(b)) = \|\rho(b)\|_\psi^2 .$$

Hence, for $a \in \mathcal{A}$,

$$\|a\| = \sup\{\|ab\|_{\varphi_0} : b \in \mathcal{A}, \|b\|_{\varphi_0} \le 1\} = \sup\{\|\rho(a)b\|_\psi : b \in \rho(\mathcal{A}), \|b\|_\psi \le 1\} = \|\rho(a)\| .$$

Then ρ extends to a $*$-isomorphism $\eta : A \to B$ with the wanted properties. □

With the uniqueness of the C^*-completion (A, φ), we are ready to define the reduced free product.

Definition 4.9. Let (A_i, φ_i) be a family of unital C^*-algebras A_i each with a faithful state φ_i. The *reduced free product* $(A, \varphi) = *_{i\in I}(A_i, \varphi_i)$ is the C^*-completion of $(\mathcal{A}, \varphi_0)$, with φ_0 as in Theorem 4.7.

We will be particularly interested in the case where each state is tracial.

Proposition 4.10. *Let $(A_i, \tau_i)_{i\in I}$ be a family of unital C^*-algebras each equipped with a normalized trace τ_i and let $(A, \tau) = *_{i\in I}(A_i, \tau_i)$ be the reduced free product. Then τ is a faithful normalized trace on A.*

Proof. Let $\mathcal{A} = *^{\text{alg}}_{i\in I} A_i$. Consider first $w, z \in \mathcal{A}$ centered reduced words and write

$$w = a_{i_1} \cdots a_{i_k}, \qquad z = b_{j_m} \cdots b_{j_1},$$

where for all k, l, $a_{i_k} \in A_{i_k}$, $b_{j_l} \in A_{j_l}$, $\tau(a_{i_k}) = \tau(b_{j_l}) = 0$ and $i_k \neq i_{k+1}$, $j_l \neq j_{l+1}$ for all $1 \le k \le n-1$, respectively $1 \le l \le m-1$. By the claim in the proof of Lemma 4.5,

$$\tau(wz) = \delta_{k,m}\delta_{i_k,j_m}\delta_{i_{k-1},j_{m-1}} \cdots \delta_{i_1,j_1}\tau(a_{i_k}b_{j_m}) \cdots \tau(a_{i_1}b_{j_1}).$$

Recall that $\tau_{|A_j} = \tau_j$, which are all assumed tracial, so

$$\tau(wz) = \delta_{k,m}\delta_{i_k,j_m}\delta_{i_{k-1},j_{m-1}} \cdots \delta_{i_1,j_1}\tau(b_{j_m}a_{i_k}) \cdots \tau(b_{j_1}a_{i_1}) = \tau(zw).$$

Consider now $x, y \in \mathcal{A}$ and write

$$x = \sum_{j=1}^{k} \alpha_j w_j, \qquad y = \sum_{i=1}^{m} \beta_i z_i,$$

with w_j, z_i centered reduced words for all i, j. Then

$$\tau(xy) = \sum_{j=1}^{k}\sum_{i=1}^{m} \alpha_j\beta_i\tau(w_j z_i) = \sum_{j=1}^{k}\sum_{i=1}^{m} \alpha_j\beta_i\tau(z_i w_j) = \tau(yx).$$

For $x, y \in A$, let $(x_n)_n, (y_n)_n \subset \mathcal{A}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then

$$\tau(xy) = \lim_{n\to\infty} \tau(x_n y_n) = \lim_{n\to\infty} \tau(y_n x_n) = \tau(yx),$$

showing that τ is a trace. That τ is normalized follows from τ_i normalized.

Let $x \in A$ and assume $\tau(x^*x) = 0$. For $b \in \mathcal{A}$, we have

$$\|xb\|_2^2 = \tau(b^*x^*xb) = \tau(xbb^*x^*) \le \|b\|^2 \tau(xx^*) = \|b\|^2 \tau(x^*x) = 0,$$

using that bb^* is self-adjoint, hence $bb^* \leq \|bb^*\| 1 = \|b\|^2 1$. As

$$\|x\| = \sup\{\|xb\|_2 \mid \|b\|_2 \leq 1, b \in \mathcal{A}\} = 0,$$

we get $x = 0$, showing that τ is faithful. □

Remark 4.11. A more generalized statement than the above, in which the assumption of the faithful states φ_i being tracial is omitted, was proven by Dykema in [10].

The reduced free product has the following universal property.

Theorem 4.12. *Let $(A, \varphi_i)_{i \in I}$ be a family of unital C^*-algebras A_i each equipped with a faithful state φ_i.*

(1) The reduced free product (A, φ) satisfies:

(i) There is a unital inclusion $\lambda_i : A_i \to A$ for each $i \in I$,

(ii) $\varphi \circ \lambda_i = \varphi_i$ for all $i \in I$,

(iii) $\varphi(w) = 0$ for all centered reduced words $w \in A$,

(iv) $A = C^* \left(\bigcup_{i \in I} \lambda_i(A_i) \right)$.

(2) If (B, ψ) is another unital C^*-algebra with state ψ which satisfies:

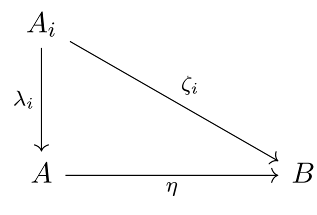
(a) There is a unital inclusion $\zeta_i : A_i \to B$ for each $i \in I$,

(b) $\psi \circ \zeta_i = \varphi_i$ for all $i \in I$,

(c) $\psi(w) = 0$ for all centered reduced words $w \in B$,

(d) $B = C^* \left(\bigcup_{i \in I} \zeta_i(A_i) \right)$,

then there exists a unique $*$-isomorphism $\eta : A \to B$ such that $\psi \circ \eta = \varphi$, and



commutes for all $i \in I$.

(3) The state φ is faithful and the canonical map $\rho : \mathcal{A} \to A$ is injective.

We understand a centered reduced word w in A by $\lambda_{i_1}(a_{i_1}) \cdots \lambda_{i_n}(a_{i_n})$, where $i_j \neq i_{j+1}$ for all $1 \leq j \leq n-1$ and $a_{i_j} \in A_{i_j}$ with $\varphi(a_{i_j}) = 0$ for all j. Similarly for B.

Proof of Theorem 4.12. We begin by proving (1). Let $\mathcal{A} = *_{i\in I}^{\text{alg}} A_i$ and equip $\mathcal{A}$ with state φ_0 from Theorem 4.7. Let (A, φ) be the C^*-completion of $(\mathcal{A}, \varphi_0)$, and let λ_i and (H, π, ξ) be as in the proof of Theorem 4.7. Then A is the norm closure of $\pi(\mathcal{A})$, and $\lambda_i = \pi \circ \iota_i$ therefore embeds A_i into A. Moreover, $\varphi = \varphi_0$ on $\mathcal{A}$, so (ii) and (iii) follow from Theorem 4.7. For (iv), note that the universal property of the algebraic free product ensures that $\mathcal{A}$ is the $*$-algebra generated by $(A_i)_{i\in I}$, and so the $*$-algebra generated by $\bigcup_{i\in I} \lambda_i(A_i) = \bigcup_{i\in I} \pi(\iota_i(A_i))$ will be equal to the $*$-algebra $\pi(\mathcal{A})$, and (iv) follows. This proves (1).
For (3), see that φ faithful follows from Proposition 4.10 and the subsequent remark. By the universal property of the algebraic free product, $\pi : \mathcal{A} \to A$ as before is the unique $*$-homomorphism such that

$$\begin{array}{ccc} A_i & & \\ {\scriptstyle \iota_i}\downarrow & \searrow^{\lambda_i} & \\ \mathcal{A} & \xrightarrow[\pi]{} & A \end{array}$$

commutes for all $i \in I$, and π is faithful, as wanted.

For (2), assume there is a unital C^*-algebra B equipped with state ψ satisfying $(a) - (d)$. It follows from the universal property of the algebraic free product that there exists unique $*$-homomorphism $\rho : \mathcal{A} \to B$ such that

$$\begin{array}{ccc} A_i & & \\ {\scriptstyle \iota_i}\downarrow & \searrow^{\zeta_i} & \\ \mathcal{A} & \xrightarrow[\rho]{} & B \end{array}$$

commutes for all $i \in I$. By argument as above, since $\zeta_i = \rho \circ \iota_i$, $\rho(\mathcal{A})$ will be dense in B. Moreover, $\varphi_i = \psi \circ \zeta_i = \psi \circ \rho \circ \iota_i$, with $\psi \circ \rho(w) = 0$ for all centered reduced words w in $\mathcal{A}$. It follows from Theorem 4.7 that $\psi \circ \rho = \varphi_0$ on $\mathcal{A}$. Using Proposition 4.8, ρ extends to a unique $*$-isomorphism $\eta : A \to B$ satisfying $\psi \circ \eta = \varphi$, and which makes

$$\begin{array}{ccc} A_i & & \\ {\scriptstyle \iota_i}\downarrow & \searrow^{\zeta_i} & \\ A & \xrightarrow[\eta]{} & B \end{array}$$

commutative for all $i \in I$. This proves (2). □

5 The distance to the invertible elements in a C^*-algebra

The main result of this section is Theorem 5.4, which shows that unital C^*-algebras whose stable rank is not one has an element with norm one and distance one to the invertible elements of A. It plays a crucial role in proofs later on. The results of this section are the work of Rørdam in [21].

Let A be a unital C^*-algebra and let H be the Hilbert space such that A embeds into $B(H)$ faithfully. Any $x \in B(H)$ has a polar decomposition $x = v|x|$, where $v \in B(H)$ is a partial isometry and $|x| = (x^*x)^{\frac{1}{2}}$. The partial isometry v has initial space $\ker(|x|)^\perp$, meaning v^*v is a projection on $\ker(|x|)^\perp$. We consider first some consequences of the polar decomposition.

Firstly, $x^* = |x|v^*$ so $xx^* = vx^*xv^*$. Moreover

$$(v(x^*x)^{1/2}v^*)^2 = v(x^*x)^{1/2}v^*v(x^*x)^{1/2}v^* = v(x^*x)v^* = xx^*.$$

Now

$$x = v|x| = v(x^*x)^{1/2} = v(x^*x)^{1/2}v^*v = (xx^*)^{1/2}v = |x^*|v,$$

implying $vp(|x|) = p(|x^*|)v$ for all polynomials p. It is now a simple consequence of Stone-Weierstrass that $vf(|x|) = f(|x^*|)v$ for all continuous functions f. If f is further assumed to be such that $f(0) = 0$, then $s = vf(|x|)$ is in A. Indeed, for polynomial g such that $g(0) = 0$ there exists polynomial h such that $g(t) = th(t)$. It now follows from Stone-Weierstrass that f can be approximated by such polynomials, hence $vf(|x|) = v|x|h(|x|) = xh(|x|) \in A$ for some continuous function h. If f in addition is positive, $|s| = f(|x|)$ and $|s^*| = f(|x^*|)$. To see this, note that

$$s^*s = f(|x|)v^*vf(|x|) = f(|x|)^2$$

and as $f \geq 0$ the positive square root of s^*s is $f(|x|)$, i.e. $|s| = f(|x|)$. Similar calculation for $|s^*| = f(|x^*|)$. Let $\lambda > 0$ and let p_λ, q_λ be the spectral projections on $[0, \lambda]$ for $|x|$, respectively $|x^*|$. Then $v(1 - p_\lambda) = (1 - q_\lambda)v$ and $v(1 - p_\lambda)v^* = (1 - q_\lambda)$.

For $x \in A$, let $\alpha(x)$ be the distance from x to the invertible elements of A,

$$\alpha(x) = \operatorname{dist}(a, \operatorname{GL}(A)) = \inf_{y \in \operatorname{GL}(A)} \|x - y\|.$$

For $0 < b < a$ define $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f(t) = \begin{cases} b^{-1} & t \leq b \\ t^{-1} & t > b \end{cases}, \qquad g(t) = \begin{cases} 0 & t \leq b \\ (a-b)^{-1}(t-b) & b < t \leq a \\ 1 & t > a \end{cases}.$$

Theorem 5.1. *Let A be a unital C^*-algebra and let $x \in A$ with polar decomposition $x = v|x|$. If $\alpha(x) < a$, then there exists $s \in GL(A)$ such that*

$$v(1 - p_a) = s(1 - p_a).$$

Proof. Let $x \in A$ and let $x = v|x|$ be the polar decomposition of x. Choose b such that $\alpha(x) < b < a$. Clearly $\alpha(x^*) = \alpha(x)$, so there exists $w \in \mathrm{GL}(A)$ such that $\|x^* - w\| < b$. Set $z = wf(|x^*|)$. As w is invertible and $f > 0$, it holds that $z \in \mathrm{GL}(A)$. Note that $1 - q_b$ is the spectral projection of $|x^*|$ on (b, ∞). Therefore, by the defintion of f,

$$|x^*|f(|x^*|)(1 - q_b) = 1 - q_b.$$

and moreover, $\|f(|x^*|)\| \leq b^{-1}$, hence $\|(x^* - w)f(|x^*|)\| \leq \|x^* - w\| \, \|f(|x^*|)\| < 1$.

Combining the above and using that v is a partial isometry,

$$\begin{aligned} 1 &> \|(x^* - w)f(|x^*|)\| = \|x^* f(|x^*|) - z\| = \|v^*|x^*|f(|x^*|) - z\| \\ &\geq \|(v^*|x^*|f(|x^*|) - z)(1 - q_b)\| = \|(v^* - z)(1 - q_b)\| = \|(v^* - z)(1 - q_b)v\| \\ &= \|(v^* - z)v(1 - p_b)\| = \|(1 - zv)(1 - p_b)\|. \end{aligned}$$

Set $y = (1 - zv)g(|x|)$ and note that $y \in A$ as $g(0) = 0$. By definition, $g(|x|) = (1 - p_b)g(|x|)$ and $\|g(|x|)\| \leq 1$, so

$$\|y\| = \|(1 - zv)(1 - p_b)g(|x|)\| \leq \|(1 - zv)(1 - p_b)\| \, \|g(|x|)\| < 1.$$

Using the Neumann series, the above implies that $1 - y$ is invertible. Furthermore, by the definition of g, $g(|x|)(1 - p_a) = 1 - p_a$, hence

$$y(1 - p_a) = (1 - zv)g(|x|)(1 - p_a) = (1 - zv)(1 - p_a),$$

and so

$$(1 - y)(1 - p_a) = 1 - p_a - (1 - zv)(1 - p_a) = zv(1 - p_a).$$

Setting $s = z^{-1}(1 - y) \in \mathrm{GL}(A)$ gives the wanted. ☐

Corollary 5.2. *Let A be a unital C^*-algebra. For each $x \in A$ there exists $s_0 \in \overline{GL(A)}^{\|\cdot\|}$ such that $\|x - s_0\| = \alpha(x)$ and $\|s_0\| = \|x\| - \alpha(x)$.*

Proof. Given $x \in A$ let $x = v|x|$ be the polar decomposition. For $0 \leq a \leq \|x\|$ define $f_a : \mathbb{R}_+ \to \mathbb{R}_+$ by $f_a(t) = \max\{t - a, 0\}$. Set $x_a = v f_a(|x|)$. Note that $f(0) = 0$, so $x_a \in A$. Moreover, using the continuous functional calculus,

$$\|x - x_a\| = \|v|x| - v f_a(|x|)\| = \||x| - f_a(|x|)\| = \sup_{t \in \sigma(|x|)} |t - f_a(t)| = a.$$

Furthermore,

$$\|x_a\| = \|v f_a(|x|)\| = \sup_{t \in \sigma(|x|)} |f_a(t)| = \sup_{\substack{t \in \sigma(|x|) \\ a \leq t}} |t - a| = \sup_{\substack{t \in \sigma(|x|) \\ a \leq t}} t - a = \|x\| - a.$$

Consider now $a > \alpha(x)$. By definition, $(1 - p_a) f_a(|x|) = f_a(|x|)$. Let $s \in \mathrm{GL}(A)$ from Theorem 5.1, and see that

$$x_a = v f_a(|x|) = v(1 - p_a) f_a(|x|) = s(1 - p_a) f_a(|x|) = s f_a(|x|).$$

For $\varepsilon > 0$, $s(f_a(|x|) + \varepsilon 1)$ is invertible, implying that x_a is a norm limit of invertible elements. Set $\alpha = \alpha(x)$, then x_α is the norm limit of x_a for $a > \alpha$, meaning $x_\alpha \in \overline{\mathrm{GL}(A)}^{\|\cdot\|}$. Letting $s_0 = x_\alpha$ gives the wanted. □

The above corollary shows that the distance from x to $\mathrm{GL}(A)$ is attained at some s_0 in the norm closure of $\mathrm{GL}(A)$ which furthermore have the least possible norm.

Corollary 5.3. *Let A be a unital C^*-algebra. For $x \in A$,*

$$\alpha(x) = \inf\{\lambda : v(1 - p_\lambda) \in GL(A)(1 - p_\lambda)\}.$$

Proof. Assume $v(1 - p_a) = s(1 - p_a)$ for some $a \geq 0$ and $s \in \mathrm{GL}(A)$. Arguing as in the proof of Corollary 5.2, we see that x_a is in the norm closure of $\mathrm{GL}(A)$. In particular,

$$a = \|x - x_a\| \geq \alpha(x),$$

which proves the wanted. □

This leads us to the main theorem of this section.

Theorem 5.4 (Rørdam)**.** *Let A be a unital C^*-algebra with $sr(A) \neq 1$. There exists $z \in A$ such that*

$$\|z\| = dist(z, GL(A)) = 1.$$

Proof. Let $x \in A$ such that $0 < \alpha(x) = \alpha$ and let $x = v|x|$ be the polar decomposition of x. Define continuous function $h : \mathbb{R}_+ \to \mathbb{R}_+$ by $h(t) = \min\{\alpha^{-1}t, 1\}$. Note that $h(0) = 0$, meaning $z = vh(|x|) \in A$. Moreover, by the continuous functional calculus,

$$\|z\| = \|vh(|x|)\| = \|h(|x|)\| = \sup_{t \in \sigma(|x|)} |h(t)| \leq 1.$$

Let $p_\lambda, \tilde{p}_\lambda$ be the spectral projections corresponding to the interval $[0, \lambda]$ for $|x|$, respectively $|z|$. As h is positive, $|z| = h(|x|)$ and $h(t) = 0$ for $t = 0$ only, hence $|z|$ and $h(|x|)$ have the same range projections, implying that $z = vh(|x|)$ is in fact the polar decomposition for z. For $\lambda < 1$, the definition of h implies that $\tilde{p}_\lambda = p_{\alpha\lambda}$. See now, that if

$$v(1 - \tilde{p}_\lambda) = s(1 - \tilde{p}_\lambda)$$

for some $s \in \mathrm{GL}(A)$ and $0 \leq \lambda < 1$, then

$$v(1 - p_{\alpha\lambda}) = s(1 - p_{\alpha\lambda}).$$

Using Corollary 5.3, we get

$$\alpha = \alpha(x) \leq \alpha\lambda < \alpha,$$

an obvious contradiction. Thus, such $s \in \mathrm{GL}(A)$ can only exist for $\lambda \geq 1$. It now follows from Theorem 5.1 that $\alpha(z) \geq 1$. Recall that $\alpha(z) \leq \|z\| \leq 1$, which finishes the proof. □

6 Stable rank of some reduced free product C^*-algebras

Let $(A_i, \tau_i)_{i\in I}$ be a family of unital C^*-algebras A_i each equipped with a faithful, normalized trace τ_i and let $(A, \tau) = *_{i\in I}(A_i, \tau_i)$ be the reduced free product. Equip A with the Euclidean structure corresponding to τ,

$$\langle a, b\rangle = \tau(b^*a), \quad \|a\|_2 = \langle a, a\rangle^{\frac{1}{2}},$$

for $a, b \in A$. Let X_i be a standard orthonormal basis for A_i and recall that $X_i^\circ = X_i \setminus \{1_i\}$. Define $Y = *_{i\in I} X_i$ as in Lemma 4.5. We claim Y is a standard orthonormal basis for A. Clearly $1 \in Y$ and is orthonormal by Lemma 4.5. Moreover, as $\operatorname{span} X_i$ is a *-algebra for all $i \in I$, $\operatorname{span} Y$ is a *-algebra. Lastly, by construction $\operatorname{span} Y$ is norm dense in A, proving that Y is a standard orthonormal basis for A. In this section, we understand by a reduced word a centered reduced word.

Let $a \in \operatorname{span} Y$, and write $a = \sum_{j=1}^n \alpha_j w_j$ for $w_j \in Y$. Then, as Y is an orthonormal set,

$$\|a\|_2^2 = \left\langle \sum_{j=1}^n \alpha_j w_j, \sum_{i=1}^n \alpha_i w_i \right\rangle = \sum_{j,i=1}^n \alpha_j \overline{\alpha_i} \langle w_j, w_i\rangle = \sum_{j=1}^n |\alpha_j|^2.$$

Define orthogonal projections $E_n : \operatorname{span} Y \to \operatorname{span} Y_n$ with Y_n defined as in Lemma 4.5. The first part of this section aims to bound the operator norm by the 2-norm. We start by showing the following lemma describing $E_n(vw)$ for $v \in Y_k$ and $w \in Y_l$.

Lemma 6.1. *Let $v \in Y_k$ and $w \in Y_l$ for some k, l and let $n \geq 0$ be given.*

(i) Assume $|k-l| < n \leq k+l$. Let $0 \leq q < \min\{k, l\}$ be the integer such that $k+l-n = 2q$ or $k+l-n = 2q+1$. Write

$$v = v_1 x v_2, \quad v_1 \in Y_{k-q-1}, \quad x \in X_i^\circ, \quad v_2 \in Y_q,$$
$$w = w_2 y w_1, \quad w_1 \in Y_{l-q-1}, \quad y \in X_j^\circ, \quad w_2 \in Y_q.$$

If $k+l-n = 2q$, then

$$E_n(vw) = \begin{cases} \langle v_2 w_2, 1\rangle\, v_1 x y w_1 & i \neq j \\ 0 & i = j \end{cases}, \tag{6.1}$$

and if $k+l-n = 2q+1$, then

$$E_n(vw) = \begin{cases} \sum_{u\in X_i^\circ} \langle v_2 w_2, 1\rangle \langle xy, u\rangle\, v_1 u w_1 & i = j \\ 0 & i \neq j \end{cases}.$$

Note that $\langle xy, u\rangle \neq 0$ *for at most finitely many* $u \in X_i^\circ$.

(ii) *Assume* $n = |k - l|$. *Set* $q = \min\{k, l\}$ *such that* $k + l - n = 2q$. *Write*

$$\begin{aligned} v &= v_1 v_2, \quad v_1 \in Y_{k-q}, \quad v_2 \in Y_q, \\ w &= w_2 w_1, \quad w_1 \in Y_{l-q}, \quad w_2 \in Y_q. \end{aligned}$$

Then $v_1 = 1$ *or* $w_1 = 1$ *and* $E_n(vw) = \langle v_2 w_2, 1\rangle\, v_1 w_1$.

(iii) *If* $n < |k - l|$ *or* $n > k + l$, *then* $E_n(vw) = 0$.

Proof. We prove the assertions (i), (ii) and (iii) simultaneously by induction over $\min\{k, l\}$. Assume first that $\min\{k, l\} = 0$, implying that $v = 1$ or $w = 1$, and moreover either $n = |k - l|$ and $q = 0$ or $n < |k - l|$, or $n > k + l$. In all cases the claims are clear. Assume now that $\min\{k, l\} \geq 1$. Write

$$\begin{aligned} v &= v'x', \quad v' \in Y_{k-1}, \quad x' \in X_s^\circ \\ w &= y'w', \quad w' \in Y_{l-1}, \quad y' \in X_t^\circ. \end{aligned}$$

Consider first the case of $s \neq t$. Then vw is a reduced word of length $k + l$, hence

$$E_n(vw) = \begin{cases} vw & n = k + l \\ 0 & n \neq k + l, \end{cases} \tag{6.2}$$

See first that this formula agrees with (iii). If $n = k + l$ or $n = k + l - 1$ then $q = 0$ in (i), implying that $v_2 = w_2 = 1$ so $\langle v_2 w_2, 1\rangle = 1$. Moreover $q = 0$ implies that $x' = x$ and $y' = y$ and $i \neq j$, so the formulae hold in both cases in (i). If $|k - l| \leq n < k + l - 1$, then $q \geq 1$ in either case of $k + l - n = 2q$ or $k + l - n = 2q + 1$. Note that, in the notation of (i) and (ii),

$$\begin{aligned} v_2 &= v_2' x', \quad v_2' \in Y_{q-1}, \\ w_2 &= y' w_2', \quad w_2' \in Y_{q-1}, \end{aligned}$$

meaning that $v_2 w_2 = v_2' x' y' w_2'$ is reduced of length strictly greater than 0. In particular, $\langle v_2 w_2, 1\rangle = 0$, and the expression for $E_n(vw)$ in (6.2) agrees with the formulae in (i) and (ii).

Consider now the case of $s = t$. As $\operatorname{span} X_s$ is a $*$-subalgebra, $x'y' \in \operatorname{span} X_s$, so we can write

$$vw = v'x'y'w' = \langle x'y', 1\rangle\, v'w' + \sum_{u \in X_s^\circ} \langle x'y', u\rangle\, v'uw',$$

with $\langle x'y', u\rangle \neq 0$ for only finitely many $u \in X_s^\circ$. Using this expression, the formulae for $E_n(vw)$ holds for $n \geq k + l - 1$. Consider now $|k - l| \leq n < k + l - 1$. Then $q \geq 1$ and we again write $v_2 = v_2'x', w_2 = y'w_2'$ with $v_2', w_2' \in Y_{q-1}$. Then $v' = v_1xv_2'$, $w' = w_2'yw_1$ and

$$E_n(vw) = \left\langle x'y', 1 \right\rangle E_n(v'w').$$

By the induction hypothesis $E_n(v'w')$ is given by the formulae in (i) and (ii). Note that

$$\begin{aligned}\langle v_2w_2, 1\rangle = \tau(v_2'x'y'w_2') &= \tau(v_2 \left\langle x'y', 1 \right\rangle 1w_2') + \tau(v_2(x'y' - \left\langle x'y', 1 \right\rangle 1)w_2') \\ &= \tau(v_2 \left\langle x'y', 1 \right\rangle 1w_2') = \left\langle x'y', 1 \right\rangle \left\langle v_2'w_2', 1 \right\rangle,\end{aligned}$$

so the formulae for $E_n(vw)$ in (i) and (ii) hold.

Lastly, if $n < |k-l|$, then $n < |(k-1)-(l-1)|$ and

$$E_n(vw) = \left\langle x'y', 1 \right\rangle E_n(v'w') = 0,$$

as wanted. ☐

Let $a \in \operatorname{span} Y$. The the support of a, denoted by $\operatorname{supp}(a)$, is the set of elements $w \in Y$ such that $\langle a, w\rangle \neq 0$. For $i \in I$ let $F_i(a)$ be the set of $x \in X_i^\circ$ which appear as letters in the words $w \in \operatorname{supp}(a)$. The support of a is finite, meaning that each $F_i(a)$ is finite and in particular, $F_i(a) \neq \emptyset$ for only finitely many $i \in I$. Define

$$K(a) = \max_{i\in I} \left(\sum_{x\in F_i(a)} \|x\|^2 \right)^{\frac{1}{2}}. \tag{6.3}$$

Let $a \in \operatorname{span} Y$ and let $k \geq 1$ be the length of the longest word w in the support of a. It is clear that $a \in \operatorname{span}\left(\bigcup_{j=1}^k Y_j\right)$.

Lemma 6.2. *Let $a \in \operatorname{span} Y_k$, $b \in \operatorname{span} Y_l$ and let $n \geq 0$ be given. If $|k-l| \leq n \leq k+l$, then*

$$\|E_n(ab)\|_2 \leq \begin{cases} \|a\|_2 \|b\|_2 & k+l-n \text{ even} \\ K(a) \|a\|_2 \|b\|_2 & k+l-n \text{ odd} \end{cases}.$$

If $n < |k-l|$ or $n > k+l$, then $E_n(ab) = 0$.

Proof. See first that it is an immediate consequence of Lemma 6.1 that $E_n(ab) = 0$ whenever $n < |k-l|$ and $n > k+l$. Assume that $|k-l| \leq n \leq k+l$ and consider first the case of $k+l-n = 2q$ for some integer $0 \leq q \leq \min\{k, l\}$. Write

$$a = \sum_{v_1, v_2} \alpha_{v_1v_2} v_1v_2, \qquad b = \sum_{w_1, w_2} \beta_{w_2w_1} w_2w_1,$$

summing over $v_1 \in Y_{k-q}$, $v_2 \in Y_q$ such that v_1v_2 is reduced, and over $w_1 \in Y_{l-q}$, $w_2 \in Y_q$

such that w_2w_1 is reduced. Note that only finitely many $\alpha_{v_1v_2}$, $\beta_{w_2w_1}$ are non-zero. Using Lemma 6.1,

$$E_n(ab) = \sum_{v_1,v_2}\sum_{w_1,w_2} \alpha_{v_1v_2}\beta_{w_2w_1} \langle v_2w_2, 1\rangle\, v_1w_1,$$

now summing over all $v_1 \in Y_{k-q}, w_1 \in Y_{l-q}$ and $v_2, w_2 \in Y_q$ such that v_1v_2, w_2w_1 and v_1w_1 are all reduced words. As Y is an orthonormal set,

$$\|E_n(ab)\|_2^2 \leq \sum_{v_1,w_1}\left|\sum_{v_2,w_2} \alpha_{v_1v_2}\beta_{w_2w_1} \langle v_2w_2, 1\rangle\right|^2,$$

with the right hand side being a sum over all $v_1 \in Y_{k-q}, w_1 \in Y_{l-q}$ and $v_2, w_2 \in Y_q$ such that v_1v_2 and w_2w_1 are reduced. Using the Cauchy-Schwartz inequality

$$\begin{aligned}\left|\sum_{v_2,w_2} \alpha_{v_1v_2}\beta_{w_2w_1} \langle v_2w_2, 1\rangle\right|^2 &= \left|\left\langle \sum_{w_2}\beta_{w_2w_1}w_2, \sum_{v_2}\overline{\alpha}_{v_1v_2}v_2^*\right\rangle\right|^2 \\ &\leq \left\|\sum_{w_2}\beta_{w_2w_1}w_2\right\|_2^2 \left\|\sum_{v_2}\overline{\alpha}_{v_1v_2}v_2^*\right\|_2^2 \\ &= \sum_{w_2}|\beta_{w_2w_1}|^2\sum_{v_2}|\alpha_{v_1v_2}|^2 .\end{aligned}$$

Thus,

$$\|E_n(ab)\|_2^2 \leq \sum_{v_1,w_1}\sum_{v_2}|\alpha_{v_1v_2}|^2\sum_{w_2}|\beta_{w_2w_1}|^2 = \sum_{v_1,v_2}|\alpha_{v_1v_2}|^2\sum_{w_1,w_2}|\beta_{w_2w_1}|^2 = \|a\|_2^2\,\|b\|_2^2,$$

as wanted.

Suppose now that $k + l - n = 2q + 1$ for some $0 \leq q < \min\{k, l\}$. Write

$$a = \sum_{i\in I}\sum_{v_1,x,v_2} \alpha_{v_1xv_2}v_1xv_2, \qquad \sum_{i\in I}\sum_{w_1,y,w_2} \beta_{w_2yw_1}w_2yw_1,$$

summing over $v_1 \in Y_{k-q-1}, x \in X_i^\circ$ and $v_2 \in Y_q$ such that v_1xv_2 is reduced, respectively $w_1 \in Y_{l-q-1}$, $y \in X_i^\circ$, $w_2 \in Y_q$ such that w_2yw_1 is reduced. Using Lemma 6.1 (i), we see that

$$E_n(ab) = \sum_{v_1,w_1}\sum_{i\in I}\sum_{u\in X_i^\circ}\sum_{x,y\in X_i^\circ}\sum_{v_2,w_2} \alpha_{v_1xv_2}\beta_{w_2yw_1} \langle v_2w_2, 1\rangle \langle xy, u\rangle\, v_1uw_1,$$

now summing over all $v_1 \in Y_{k-q-1}, w_1 \in Y_{l-q-1}$, $v_2, w_2 \in Y_q$ such that v_1xv_2 and w_2yw_1 are reduced.

Thus,

$$\|E_n(ab)\|_2^2 = \sum_{v_1,w_1} \sum_{i\in I} \sum_{u\in X_i^\circ} \left| \sum_{x,y\in X_i^\circ} \sum_{v_2,w_2} \alpha_{v_1xv_2}\beta_{w_2yw_1} \langle v_2w_2, 1\rangle \langle xy, u\rangle \right|^2 .$$

Fix v_1, w_1 and $i \in I$. Set

$$z = \sum_{x,y\in X_i^\circ} \left\langle \sum_{w_2} \beta_{w_2yw_1} w_2, \sum_{v_2} \overline{\alpha}_{v_1xv_2} v_2^* \right\rangle xy \in \operatorname{span} X_i.$$

It is a simple calculation to see that for $u \in X_i^\circ$,

$$|\langle z, u\rangle|^2 = \left| \sum_{x,y\in X_i^\circ} \sum_{v_2,w_2} \alpha_{v_1xv_2}\beta_{w_2yw_1} \langle v_2w_2, 1\rangle \langle xy, u\rangle \right|^2 .$$

As $\alpha_{v_1xv_2} = 0$ for $x \notin F_i(a)$, we get

$$\begin{aligned}
\|z\|_2^2 &= \left\| \sum_{x\in F_i(a)} x \sum_{y\in X_i^\circ} \left\langle \sum_{w_2} \beta_{w_2yw_1} w_2, \sum_{v_2} \overline{\alpha}_{v_1xv_2} v_2^* \right\rangle y \right\|_2^2 \\
&\le \left(\sum_{x\in F_i(a)} \left\| x \sum_{y\in X_i^\circ} \left\langle \sum_{w_2} \beta_{w_2yw_1} w_2, \sum_{v_2} \overline{\alpha}_{v_1xv_2} v_2^* \right\rangle y \right\|_2 \right)^2 \\
&\le \left(\sum_{x\in F_i(a)} \|x\| \left\| \sum_{y\in X_i^\circ} \left\langle \sum_{w_2} \beta_{w_2yw_1} w_2, \sum_{v_2} \overline{\alpha}_{v_1xv_2} v_2^* \right\rangle y \right\|_2 \right)^2 \\
&\le \left(\sum_{x\in F_i(a)} \|x\|^2 \right) \left(\sum_{x\in F_i(a)} \left\| \sum_{y\in X_i^\circ} \left\langle \sum_{w_2} \beta_{w_2yw_1} w_2, \sum_{v_2} \overline{\alpha}_{v_1xv_2} v_2^* \right\rangle y \right\|_2^2 \right) \\
&\le K(a)^2 \sum_{x\in F_i(a)} \sum_{y\in X_i^\circ} \left| \left\langle \sum_{w_2} \beta_{w_2yw_1} w_2, \sum_{v_2} \overline{\alpha}_{v_1xv_2} v_2^* \right\rangle \right|^2 \\
&\le K(a)^2 \sum_{x,y\in X_i^\circ} \left\| \sum_{w_2} \beta_{w_2yw_1} w_2 \right\|_2^2 \left\| \sum_{v_2} \alpha_{v_1xv_2} v_2 \right\|_2^2 \\
&= K(a)^2 \sum_{x,v_2} |\alpha_{v_1xv_2}|^2 \sum_{y,w_2} |\beta_{w_2yw_1}|^2 .
\end{aligned}$$

Now,

$$\sum_{u\in X_i^\circ}\left|\sum_{x,y\in X_i^\circ}\sum_{v_2,w_2}\alpha_{v_1xv_2}\beta_{w_2yw_1}\langle v_2w_2,1\rangle\langle xy,u\rangle\right|^2=\sum_{u\in X_i^\circ}|\langle z,u\rangle|^2\leq\|z\|_2^2$$
$$\leq K(a)^2\sum_{x,v_2}|\alpha_{v_1xv_2}|^2\sum_{y,w_2}|\beta_{w_2yw_1}|^2.$$

Thus,

$$\begin{aligned}\|E_n(ab)\|_2^2 &\leq \sum_{v_1,w_1}\sum_{i\in I}K(a)^2\sum_{x,v_2}|\alpha_{v_1xv_2}|^2\sum_{y,w_2}|\beta_{w_2yw_1}|^2\\ &= K(a)^2\left(\sum_{i\in I}\sum_{x,v_2}|\alpha_{v_1xv_2}|^2\right)\left(\sum_{i\in I}\sum_{y,w_2}|\beta_{w_2yw_1}|^2\right)\\ &= K(a)^2\|a\|_2^2\|b\|_2^2,\end{aligned}$$

as wanted. □

Lemma 6.3. *For $a\in\operatorname{span}Y_k$,*

$$\|a\|\leq(2k+1)K(a)\|a\|_2.$$

Proof. It follows from the definition of the reduced free product that it suffices to show that

$$\|ab\|_2^2\leq(2k+1)K(a)\|a\|_2\|b\|_2$$

for all $b\in\operatorname{span}Y$. Let $b\in\operatorname{span}Y$ and set $b_j=E_j(b)$. Using Lemma 6.2 for all $n\geq 0$,

$$\begin{aligned}\|E_n(ab)\|_2=\left\|\sum_{j=|n-k|}^{n+k}E_n(ab_j)\right\|_2 &\leq\sum_{j=|n-k|}^{n+k}\|E_n(ab_j)\|_2\\ &\leq\sum_{j=|n-k|}^{n+k}K(a)\|a\|_2\|b_j\|_2\\ &\leq K(a)\|a\|_2(2k+1)^{\frac12}\left(\sum_{j=|n-k|}^{n+k}\|b_j\|_2^2\right)^{\frac12},\end{aligned}$$

with the last inequality following from the Cauchy-Schwarz inequality. Now,

$$\begin{aligned}
\|ab\|_2^2 &\le \sum_{n=0}^{\infty} \|E_n(ab)\|_2^2 \\
&\le K(a)^2 \|a\|_2^2 (2k+1) \sum_{n=0}^{\infty} \sum_{j=|n-k|}^{n+k} \|b_j\|_2^2 \\
&\le K(a)^2 \|a\|_2^2 (2k+1)^2 \sum_{n=0}^{\infty} \|b_j\|_2^2 \\
&= (2k+1)^2 K(a)^2 \|a\|_2^2 \|b\|_2^2 .
\end{aligned}$$

□

Lemma 6.4. *For* $a \in \operatorname{span}\left(\bigcup_{j=0}^{k} Y_j\right)$,

$$\|a\| \le (2k+1)^{\frac{3}{2}} K(a) \|a\|_2 .$$

Proof. Let $a_j = E_j(a)$ and note that $a = \sum_{j=1}^{k} a_j$. Moreover, $K(a_j) \le K(a)$, which follows simply by the definition. Using Lemma 6.3,

$$\begin{aligned}
\|a\| \le \sum_{j=1}^{k} \|a_j\| &\le \sum_{j=1}^{k} (2j+1) K(a_j) \|a_j\|_2 \\
&\le (2k+1) K(a) \sum_{j=1}^{k} \|a_j\|_2 \\
&\le (2k+1) K(a) (k+1)^{\frac{1}{2}} \left(\sum_{j=1}^{k} \|a_j\|_2^2 \right)^{\frac{1}{2}} \\
&\le (2k+1)^{\frac{3}{2}} K(a) \|a\|_2 .
\end{aligned}$$

□

Lemma 6.4 plays a crucial role in the proof of the main theorem of this section. The strategy of the proofs of Lemma 6.4 and Lemma 6.3 follows the work of Uffe Haagerup in [13], in which he shows that the free group of finitely many generators, F_n for $1 \le n < \infty$, has the *rapid decay property*, which will be touched upon later.

Lemma 6.5. *Suppose*

$$v = a_1 a_2 \cdots ar, \qquad w = b_1 b_2 \cdots b_s \qquad z = c_t c_{t-1} \cdots c_1,$$

are reduced words in A of length r, s and t such that $s < \min\{r, t\}$. Then uwz is a linear combination of reduced words in A of the form

$$a_1 a_2 \cdots a_{r'-1} a_{r'} b_1' b_2' \cdots b_s' c_{t'} c_{t'-1} \cdots c_2 c_1$$

and of, possibly unreduced, words of the form

$$a_1 a_2 \cdots a_{r'-1} a_{r'} c_{t'} c_{t'-1} \cdots c_2 c_1,$$

where $t' \geq t - s$ and $r' \geq r - s$ in both cases.

Proof. We prove the wanted by induction over s. Consider first $s = 0$. Then $w = 1$ and

$$vwz = vz = a_1 a_2 \cdots a_r c_t c_{t-1} \cdots c_1,$$

which is of the wanted form. Let $s > 0$. Now

$$a_r \in A_i, \quad b_1 \in A_j, \quad b_s \in A_k, \quad c_t \in A_l$$

for some $i, j, k, l \in I$. We consider the following cases;

i) $i \neq j$ and $k \neq l$, *ii)* $i = j$ and $k \neq l$, *iii)* $i \neq j$ and $k = l$, *iv)* $i = j$ and $k = l$.
The case *i)* is clear as then vwz is reduced. Consider now the case of *iv)*. If $s = 1$, then $i = j = k = l$. Set $b_1' = a_r b_1 c_t - \langle a_r b_1 c_t, 1 \rangle 1 \in A_i$ and note that $\tau(b_1') = 0$. Then

$$uwz = a_1 a_2 \cdots a_{r-1} b_1' c_{t-1} c_{t-2} \ldots c_1 + \langle a_r b_1 c_t, 1 \rangle a_1 a_2 \cdots a_{r-1} c_{t-1} c_{t-2} \cdots c_1,$$

which is of the wanted form. If $s \geq 2$, set

$$b_1' = a_r b_1 - \langle a_r b_1, 1 \rangle 1 \in A_i, \qquad b_s' = b_s c_t - \langle b_s c_t, 1 \rangle 1 \in A_k.$$

and $\tau(b_1') = \tau(b_s') = 0$. Then

$$\begin{aligned} uwz &= a_1 a_2 \ldots a_{r-1} b_1' b_2 \cdots b_{s-1} b_s' c_{t-1} c_{t-2} \cdots c_1 \\ &\quad + \langle a_r b_1, 1 \rangle a_1 a_2 \cdots a_{r-1} b_2 \cdots b_{s-1} b_s' c_{t-1} c_{t-2} \cdots c_1 \\ &\quad + \langle b_s c_t, 1 \rangle a_1 a_2 \cdots a_{r-1} b_1' b_2 \cdots b_{s-1} c_{t-1} c_{t-2} \cdots c_1 \\ &\quad + \langle a_r b_1, 1 \rangle \langle b_s c_t, 1 \rangle a_1 a_2 \cdots a_{r-1} b_2 \cdots b_{s-1} c_{t-1} c_{t-2} \cdots c_1. \end{aligned}$$

We see that the first term is reduced and the rest are, by the induction hypothesis, linear combinations of words of the wanted form. The cases of *ii*) and *iii*) follow by similar argument. □

Lemma 6.6. *Assume for some distinct pair of indices $i_1, i_2 \in I$ that there exist at least one unitary $x \in X_{i_1}^\circ$ and at least two unitaries $y, z \in X_{i_2}^\circ$. Then for each $a \in \operatorname{span} Y$ there exist unitaries $u, v \in \operatorname{span} Y$ and a constant $K < \infty$ such that*

$$\|(uav)^n\|_2 = \|a\|_2^n, \qquad K((uav)^n) \leq K,$$

for all $n \geq 1$.

Proof. Let $x \in X_{i_1}^\circ$ and $y, z \in X_{i_2}^\circ$ be distinct unitary elements. Let $k \geq 1$ be the length of the longest word in the support of a, ensuring $a \in \operatorname{span}\left(\bigcup_{j=1}^k Y_j\right)$ and k is the least such integer. Choose integer l such that $l \geq (k+3)/2$. Define

$$u' = (xy^*)^l, \qquad v = (xz)^l,$$

and note that $u', v \in \operatorname{span} Y$ are both of length $2l$. We begin by showing that whenever $w \in Y_j$ and $j \leq k$, then $u'wv$ is a linear combination of reduced words in Y starting with x and ending with z. For $s, r \leq 2l$, let u'_s be the word consisting of the first s letters of u' and v_r the word consisting of the last r letters of v. It now follows from Lemma 6.5 that $u'wv$ is a linear combination of reduced words of the form $u'_s w' v_r$ and of possibly unreduced words of the form $u'_s v_r$, for $s, r \geq 2l - j \geq 3$ in either case. Letting $w' = w - \tau(w)1$ and noting that $w \in Y$, we get $w' \in \operatorname{span} Y$ hence $u'_s w' v_r \in \operatorname{span} Y$. Now, by construction, each $u'_s w' v_r$ is a linear combination of elements in Y starting with x and ending with z. It remains to show that the possibly unreduced words $u'_s v_r$ are of the same form whenever $s, r \geq 3$. Consider first the cases of s, r both even or s, r both odd. Then, by the choice of u' and v, $u'_s v_r$ will be a reduced word starting with x and ending with z. Assume now that s is even and r is odd, then

$$u'_s v_r = u'_{s-1} y^* z v_{r-1},$$

and if s is odd and r is even,

$$u'_s v_r = u'_{s-2} y^* x^2 z v_{r-2} = u'_{s-2} y^* (x^2 - \langle x^2, 1 \rangle 1) z v_{r-2} + \langle x^2, 1 \rangle u'_{s-2} y^* z v_{r-2}.$$

Hence, in all cases $u'_s v_r$ is a linear combination of words in Y beginning with x and ending with z. We now have

$$u'av = \sum_{j=1}^N \alpha_j w_j,$$

where $w_1, \ldots, w_N$ are distinct elements of Y starting with x and ending with z each of length no greater than $2l+k$. Choose now integer $m \geq {(2l+k+1)}/{2}$ and consider

$$r = (xy)(xz)^m(xy),$$

which is a unitary element in Y. For each $n \geq 1$ and choice of $j_1, \ldots, j_n \in \{1, \ldots, N\}$, $rw_{j_1}rw_{j_2}\ldots rw_{j_n}$ is a reduced element of Y. Furthermore, by the choice of r and properties of the reduced free product,

$$rw_{j_1}rw_{j_2}\ldots rw_{j_n} = rw_{i_1}rw_{i_2}\ldots rw_{i_n}$$

for some $n \geq 1$ implies that $i_1 = j_1, \ldots, i_n = j_n$.

Set $u = ru'$. For $n \geq 1$, we now have

$$(uav)^n = \sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\ldots\sum_{j_n=1}^{N} \alpha_{j_1}\alpha_{j_2}\ldots\alpha_{j_n} rw_{j_1}rw_{j_2}\ldots rw_{j_n},$$

where the words $rw_{j_1}rw_{j_2}\ldots rw_{j_n}$ are reduced and distinct elements of Y. Hence the above is the unique way to express $(uav)^n$ as a linear combination of basis elements in Y. By the defintion,

$$K((uav)^n) = K(uav),$$

for all $n \geq 1$. Therefore, we may set $K = K(uav)$. Moreover,

$$\|(uav)^n\|_2 = \sum_{j_1=1}^{N} |\alpha_{j_1}|^2 \sum_{j_2=1}^{N} |\alpha_{j_2}|^2 \ldots \sum_{j_n=1}^{N} |\alpha_{j_n}|^2 = \|a\|_2^n \,.$$

□

For each $a \in A$ let $r(a)$ denote the spectral radius of a. For $u, v \in \mathcal{U}(A)$,

$$\begin{aligned}\|(uav)^n\| = \|uavuav\ldots uav\| &= \|v(uav)v^*v(uav)(v^*v)\ldots v^*v(uav)v^*\| \\ &= \|(vuavv^*)^n\| = \|(vua)^n\|\,,\end{aligned}$$

meaning $r(uav) = r(vuavv*) = r(vua)$. Therefore,

$$\inf_{u,v\in U(A)} r(uav) = \inf_{\tilde{u}\in U(A)} r(\tilde{u}a). \tag{6.4}$$

For $a \in A$ and $u \in \mathcal{U}(A)$,

$$\operatorname{dist}(ua, \mathrm{GL}(A)) = \inf_{x\in \mathrm{GL}(A)} \|ua - x\| = \inf_{x\in \mathrm{GL}(A)} \|a - u^*x\| = \inf_{y\in \mathrm{GL}(A)} \|a - y\| = \operatorname{dist}(a, \mathrm{GL}(A)).$$

as $u^*x \in \mathrm{GL}(A)$ for any $x \in \mathrm{GL}(A)$. Moreover, for $a \in A$,

$$\begin{aligned}\operatorname{dist}(a, \mathrm{GL}(A)) &\leq \operatorname{dist}(a, \{a - \lambda 1 | \lambda \in \mathbb{C}\} \cap \mathrm{GL}(A)) \\ &= \inf_{\lambda \in \rho(a)} \|a - (a - \lambda 1)\| = \inf_{\lambda \in \rho(a)} |\lambda| \leq r(a),\end{aligned}$$

where $\rho(a)$ is the resolvent of a. Therefore,

$$r(ua) \geq \operatorname{dist}(ua, \mathrm{GL}(A)) = \operatorname{dist}(a, \mathrm{GL}(A)). \tag{6.5}$$

Theorem 6.7 (Dykema-Haagerup-Rørdam)**.** *Let $(A_i, \tau_i)_{i \in I}$ be a family of unital C^*-algebras A_i each equipped with a faithful normalized trace τ_i. Suppose that for some distinct pair of indices $i_1, i_2 \in I$ there exist unitary elements $x \in A_{i_1}$ and $y, z \in A_{i_2}$ such that*

$$0 = \tau_{i_1}(x) = \tau_{i_2}(y) = \tau_{i_2}(z) = \tau_{i_2}(z^* y).$$

Consider the reduced free product C^*-algebra $(A, \tau) = *_{i \in I}(A_i, \tau_i)$. Then $sr(A) = 1$.

Proof. For each A_i there exists an increasing separable net $(B_{i,\alpha})_\alpha$ whose union is dense in A_i. We may further assume that each $B_{i_1,\alpha}$ contains x and each $B_{i_2,\alpha}$ contains y, z. Consider the reduced free product $(A_\alpha, \phi) = *_\alpha(B_{i,\alpha}, \phi_i)$. Then $(A_\alpha)_\alpha$ is an increasing net in A whose union is dense in A. Furthermore, if each A_α has stable rank 1, then so does A. Therefore, we may assume that each A_i is separable. Using Lemma 4.4 on the orthogonal sets $\{1, x\}$ and $\{1, y, z\}$ in A_{i_1}, respectively A_{i_2}, we can find standard orthonormal basis X_i for each $i \in I$ such that $x \in X_{i_1}^\circ$ and $y, z \in X_{i_2}^\circ$. Let $Y = *_{i \in I} X_i$ as previously defined. We begin by proving the following inequality for $a \in \operatorname{span} Y$:

$$\inf_{u \in U(a)} r(ua) \leq \|a\|_2 .$$

Let $a \in \operatorname{span} Y$ and let $k \geq 1$ be such that $a \in \operatorname{span} \bigcup_{j=0}^k Y_j$. Let now $u, v \in \operatorname{span} Y$ be the unitaries and $K < \infty$ the constant as in Lemma 6.6 and choose integer $l \geq 1$ large enough so that $u, v \in \operatorname{span} \bigcup_{j=0}^l Y_j$. Then for each $n \geq 1$, $(uav)^n \in \operatorname{span} \bigcup_{j=0}^{n(k+2l)} Y_j$. It follows from Lemma 6.6 and Lemma 6.4 that

$$\|(uav)^n\| \leq (2n(k+2l)+1)^{\frac{3}{2}} K((uav)^n) \|(uav)^n\|_2 \leq (2n(k+2l)+1)^{\frac{3}{2}} K \|a\|_2^n .$$

Using (6.4) and (6.5), we see that

$$\begin{aligned}\operatorname{dist}(a, \mathrm{GL}(A)) \leq \inf_{u \in U(A)} r(ua) \leq r(uav) = \liminf_{n \to \infty} \|(uav)^n\|^{\frac{1}{n}} \\ \leq \liminf_{n \to \infty} (2n(k+2l)+1)^{\frac{3}{2n}} K^{\frac{1}{n}} \|a\|_2 = \|a\|_2 .\end{aligned}$$

Having established the desired inequality, we will now show that $\mathrm{sr}(A) = 1$. Assume for contradiction that $\mathrm{sr}(A) \neq 1$. It follows from Theorem 5.4 that there exists $b \in A$ such that

$$1 = \|b\| = \mathrm{dist}(b, \mathrm{GL}(A)).$$

As Y is a standard orthonormal basis for A, there exists sequence $(a_k)_{k\geq 1} \subset \mathrm{span}\, Y$ such that a_k converges to b in norm. Using the claim, we get

$$\mathrm{dist}(a_k, \mathrm{GL}(A)) \leq \|a_k\|_2$$

for all $k \geq 1$, which will in turn hold for b. It is an easy calculation to see that $\|a\|_2 \leq \|a\|$ for all $a \in A$, implying $\|b\|_2 = \|b\| = 1$. Moreover,

$$\tau(1 - b^*b) = \tau(1 - bb^*) = 1 - \|b\|_2 = 0.$$

Note that $1-b^*b, 1-bb^*$ are positive and τ is faithful, so $1 = b^*b = bb^*$, meaning b is unitary, hence also invertible, in contradiction to $\mathrm{dist}(b, \mathrm{GL}(A)) = 1$, proving that $\mathrm{sr}(A) = 1$. ☐

The condition on the family $(A_i, \tau_i)_{i\in I}$ of existence of unitaries with the properties in Theorem 6.7 is called the Avitzour condition and was considered by David Avitzour in [1]. One of the consequences of the Avitzour conditions is that A is a simple C^*-algebra.

Theorem 6.8. *Let A_1, A_2 be unital C^*-algebras with faithful tracial states φ_1 and φ_2. Suppose there exist unitary elements $x \in A_1$ and $y, z \in A_2$ such that*

$$0 = \varphi_1(x) = \varphi_2(y) = \varphi_2(z) = \varphi_2(z^*y).$$

Let $(A, \varphi) = (A_1, \varphi_1) * (A_2, \varphi_2)$ be the reduced free product. Then A is simple.

A proof of Theorem 6.8 can be found in [1, Proposition 3.1], although in a slightly more general setting than the above.

Corollary 6.9. *Let G be a discrete group. Suppose that $G = G_1 * G_2$ of two groups G_1, G_2 satisfying $|G_1| \geq 2$ and $|G_2| \geq 3$. Then the reduced group C^*-algebra $C_r^*(G)$ of G has stable rank one.*

Proof. It follows by the construction of the reduced free product and its universal property that

$$(C_r^*(G), \tau) \simeq (C_r^*(G_1), \tau_1) * (C_r^*(G_2), \tau_2),$$

where τ_1, τ_2 are the canonical traces on G_1, respectively G_2. Recall that $C_r^*(G_j)$ contains $\{\delta_{g_j} \mid g_j \in G_j\}$ as an orthonormal set of unitaries with respect to the Euclidean structure arising from the trace τ_j. Hence the conditions in Theorem 6.7 are satisfied, showing that $C_r^*(G)$ has stable rank 1. ☐

Let $n \geq 2$ and consider the free group of n generators F_n. It is known that $F_n \cong \mathbb{Z}^{*n}$, the n-fold free product of $\mathbb{Z}$, and so it follows from Corollary 6.9 that $\mathrm{sr}(C_r^*(F_n)) = 1$.

7 Stable rank of $C_r^*(G)$ for certain groups G

Taking inspiration from the proofs of Dykema, Haagerup and Rørdam in [9], Gerasimova and Osin expanded the results in [12] to show that $\mathrm{sr}(C_r^*(G)) = 1$ for acylindrically hyperbolic groups with trivial finite radical and finite direct products of such groups. The proof of Gerasimova and Osin rely on a result of Dykema and de la Harpe in [11]. The groups considered in this section are assumed discrete.

Definition 7.1. Let G be a group and let F be a non-empty subset of G. Then

$$\langle F \rangle = \{f_1 \cdots f_n \mid n \in \mathbb{N},\ f_i \in F\}$$

is the subsemigroup of G generated by F. We say that F is *semifree* if the subsemigroup of G generated by F is free over F, i.e. if $n, m \in \mathbb{N}$, $f_1, \ldots, f_n, f_1', \ldots, f_m' \in F$ then $f_1 \cdots f_n = f_1' \cdots f_m'$ implies that $n = m$ and $f_i = f_i'$ for all $1 \leq i \leq n$.

Definition 7.2. A group G has the *free semigroup property* if for every finite subset F of G there exists $t \in G$ such that $tF = \{tf \mid f \in F\}$ is semifree.

Denote by $r_2(a)$ the spectral radius of $a \in C_r^*(G)$ corresponding to the 2-norm:

$$r_2(a) = \limsup_{k\to\infty} \sqrt[k]{\|a^k\|_2}.$$

Since $\|a\|_2 \leq \|a\|$,

$$r_2(a) \leq r(a),$$

where $r(a)$ denotes the spectral radius of a with respect to the operator norm.

Definition 7.3. Let G be a group and $F \subset G$. We say that F has the *ℓ^2-spectral radius property* if, for every $a \in \mathbb{C}G$ with $\mathrm{supp}(a) \subset F$, it holds that $r_2(a) = r(a)$.

The following theorem of Dykema and de la Harpe [11, Theorem 1.4] describes a different class of groups G with $\mathrm{sr}(G) = 1$. The strategy of the proof follows those of the proofs of Dykema, Haagerup and Rørdam presented in the previous section. It will play a crucial role later on.

Theorem 7.4 (Dykema-de la Harpe)**.** *Let G be a group and suppose for every finite subset $F \subset G$ there exists $t \in G$ such that tF is semifree and has the ℓ^2-spectral radius property. Then $sr(C_r^*(G)) = 1$.*

Proof. Recall that for any a in a unital C^*-algebra A we have

$$\mathrm{dist}(a, \mathrm{GL}(A)) \leq r(a).$$

Let G be a group and let $c = \sum_{g\in X} c_g\delta_g \in \mathbb{C}G$ for a semifree subset X of G. We claim that $r_2(c) = \|c\|_2$. Indeed, see that $c^n = \sum_{y\in X^n} c_y\delta_y$ with $c_y = c_{g_1}\cdots c_{g_n}$ for $y = g_1\cdots g_n \in X^n$. It now follows as X is semifree that

$$\|c^n\|_2^2 = \sum_{y\in X^n} |c_y|^2 = \sum_{g_1} |c_{g_1}|^2 \sum_{g_2} |c_{g_2}|^2 \ldots \sum_{g_n} |c_{g_n}|^2 = \|c\|_2^{2n},$$

for all $n \geq 1$. Hence

$$r_2(c) = \limsup_{k\to\infty} \sqrt[k]{\|c^k\|_2} = \|c\|_2 .$$

Suppose now that for every finite subset $F \subset G$ there exists $t \in G$ such that tF is semifree and has the ℓ^2-spectral radius property. Assume for contradiction that $\mathrm{sr}(C_r^*(G)) \neq 1$. Then there exists $a \in C_r^*(G)$ with $1 = \|a\| = \mathrm{dist}(a, \mathrm{GL}(C_r^*(G)))$, cf. Theorem 5.4. As in the proof of Theorem 6.7, $\|a\|_2 = 1$ would imply that a is unitary, in contradiction to $1 = \mathrm{dist}(a, \mathrm{GL}(C_r^*(G)))$. Hence, $0 < \|a\|_2 < 1$. Let $\varepsilon = 1 - \|a\|_2$ and see that $0 < \varepsilon < 1$. Let $b = \sum_{g\in X} \beta_g\delta_g \in \mathbb{C}G$ with $X = \mathrm{supp}(b)$ such that $\|b - a\| < \frac{\varepsilon}{3}$. Then

$$1 = \mathrm{dist}(a, \mathrm{GL}(C_r^*(G))) \leq \frac{\varepsilon}{3} + \mathrm{dist}(b, \mathrm{GL}(C_r^*(G))),$$

implying that

$$1 - \frac{\varepsilon}{3} \leq \mathrm{dist}(b, \mathrm{GL}(C_r^*(G))).$$

Moreover, using again that $\|b_a\|_2 \leq \|b - a\|$,

$$\|b\|_2 \leq \|a\|_2 + \|b - a\| < 1 - \varepsilon + \frac{\varepsilon}{3} < 1 - \frac{\varepsilon}{3} \leq \mathrm{dist}(b, \mathrm{GL}(C_r^*(G))).$$

By defintion of the support, X is a finite subset. Thus, by assumption, there exists $t \in G$ such that tX is semifree and has the ℓ^2-spectral radius property. Let now $c = \delta_\gamma b \in \mathbb{C}G$. Then, as δ_γ is unitary, $\|c\|_2 = \|b\|_2$ and $\mathrm{dist}(c, \mathrm{GL}(C_r^*(G))) = \mathrm{dist}(b, \mathrm{GL}(C_r^*(G)))$. Using the claim, $r_2(c) = \|c\|_2 = \|b\|_2$, and as tX has the ℓ^2-spectral radius property,

$$\|b\|_2 < \mathrm{dist}(b, \mathrm{GL}(C_r^*(G))) = \mathrm{dist}(c, \mathrm{GL}(C_r^*(G))) \leq r(c) = r_2(c) = \|b\|_2 ,$$

which cannot be. We conclude that $\mathrm{sr}(C_r^*(G)) = 1$. □

A *combing* on a group G generated by a set S is a map that to each pair of elements $g, h \in G$ assigns a path $\gamma_{g,h}$ from g to h in the Cayley graph of G with respect to the generating set S. A combing yields a map $G \times G \to \mathcal{P}(G)$ which associates a path $\gamma_{g,h}$ with its set of vertices.

Definition 7.5. Let G be a group. A *generalized combing* of G is a map $C\colon G\times G\to\mathcal{P}(G)$. The combing C is *symmetric* if $C(x,y)=C(y,x)$ for all $x,y\in G$, and is *G-equivariant* if $C(gx,gy)=gC(x,y)$ for all $x,y,g\in G$.

Definition 7.6. Let G be a group. A map $\ell\colon G\to[0,\infty)$ is a *pseudolength function* if

(i) $\ell(g^{-1})=\ell(g)$ for all $g\in G$,

(ii) $\ell(gh)\leq\ell(g)+\ell(h)$ for all $g,h\in G$.

We say that ℓ is a *length function* if ℓ is a pseudolength function which furthermore satisfies $\ell(g)=0$ if and only if $g=1$.

From now on, all pseudolength functions ℓ are assumed to take values only in $\mathbb{N}\cup\{0\}$.

Fix a group G with pseudolength function $\ell:G\to[0,\infty)$. For $n\in\mathbb{N}$, define

$$B(n)=\{g\in G|\ell(g)\leq n\}.$$

Let $C:G\times G\to\mathcal{P}(G)$ be a generalized combing. We associate to C two growth functions $\gamma,\rho:\mathbb{N}\to\mathbb{N}\cup\{\infty\}$ defined by

$$\gamma(n)=\sup_{g\in G}|C(1,g)\cap B(n)|,\qquad \rho(n)=\sup_{g\in B(n)}\sup_{x\in C(1,g)}\ell(x).$$

Note that γ and ρ can in general take infinite values.
As previously, we wish to bound the operator norm by the 2-norm.

Proposition 7.7. *Let G be a group with a pseudolength function $\ell:G\to[0,\infty)$ and let S be a subset of G. Suppose that there exists a symmetric G-equivariant generalized combing $C:G\times G\to\mathcal{P}(G)$ such that*

$$C(1,s)\cap C(s,g)\cap C(1,g)\neq\emptyset$$

for all $s\in S$ and $g\in G$, and the associated growth functions γ and ρ take only finite values. Then for every $a\in\mathbb{C}G$ and $n\in\mathbb{N}$ such that supp$(a)\subset S\cap B(n)$, it holds that

$$\|a\|\leq\gamma(\rho(n))^{\frac{3}{2}}\,\|a\|_2\,.$$

Further, if S is a subsemigroup of G and $\lim_{k\to\infty}\sqrt[k]{\gamma(\rho(k))}=1$, then $r(a)=r_2(a)$.

The following two lemmas are needed in order to prove Proposition 7.7

Lemma 7.8. *Let G be a group with a pseudolength function $\ell\colon G \to [0,\infty)$ and a generalized combing $C\colon G \times G \to \mathcal{P}(G)$ such that the associated growth functions γ and ρ only take finite values. Then, for any $n \geq 1$ and any $s \in B(n)$, $|C(1,s)| \leq \gamma(\rho(n))$.*

Proof. By definition of γ and ρ, which furthermore only take finite values, we get $C(1,s) \subset B(\rho(n))$ for every $s \in B(n)$. Thus

$$|C(1,s)| = |C(1,s) \cap B(\rho(n))| \leq \gamma(\rho(n)).$$

□

Denote by $\mathbb{R}_+G$ the subset of $\mathbb{C}G$ consisting of linear combinations $\sum_{g\in G} \alpha_g \delta_g$ with $\alpha_g \in \mathbb{R}_+$ and only finitely many $\alpha_g \neq 0$.

Lemma 7.9. *Under the assumptions of Proposition 7.7, suppose further that $a \in \mathbb{R}_+G$. Then, for every $b \in \mathbb{R}_+G$,*

$$\|ab\|_2 \leq \gamma(\rho(n))^{\frac{2}{3}} \|a\|_2 \|b\|_2 .$$

Proof. Fix integer $n \geq 1$ and $a \in \mathbb{R}_+G$ such that $\mathrm{supp}(a) \subset S \cap B(n)$. For notational purposes we define

$$X_g = C(1,g) \cap B(\rho(n))$$

for each $g \in G$. Furthermore, for $g \in G$ and $x \in X_g$ set

$$S_{g,x} = \{s \in \mathrm{supp}(a) | x \in C(1,g) \cap C(s,g)\}.$$

We claim that for any $g \in G$,

$$\mathrm{supp}(a) \subset \bigcup_{x\in X_g} S_{g,x}.$$

To see this, note that for $s \in \mathrm{supp}(a)$ we have $C(1,s) \subset B(\rho(n))$, as noted in the proof of Lemma 7.8. By assumption

$$C(1,s) \cap C(s,g) \cap C(1,g) \cap B(\rho(n)) = C(1,s) \cap C(s,g) \cap X_g \neq \emptyset.$$

Let $x \in C(1,s) \cap C(s,g) \cap X_g$. Then $s \in S_{g,x}$, implying $s \in \bigcup_{x\in X_g} S_{g,x}$, which proves the claim. We note that $|X_g| \leq \gamma(\rho(n))$ for any $g \in G$. Write $a = \sum_{s\in\mathrm{supp}(a)} \alpha_s \delta_s$ and $b = \sum_{g\in G} \beta_g \delta_g$, with only finitely many $\beta_g \neq 0$.

Using that $\alpha_s, \beta_g \geq 0$ and the Cauchy-Schwarz inequality, we see that

$$\begin{aligned}
\|ab\|_2^2 = \sum_{g\in G} |(ab)(g)|^2 &= \sum_{g\in G} \left(\sum_{s\in \mathrm{supp}(a)} \alpha_s \beta_{s^{-1}g} \right)^2 \leq \sum_{g\in G} \left(\sum_{x\in X_g} \sum_{s\in S_{g,x}} \alpha_s \beta_{s^{-1}g} \right)^2 \\
&\leq \sum_{g\in G} \left(\sum_{x\in X_g} 1^2 \right) \left(\sum_{x\in X_g} \left(\sum_{s\in S_{g,x}} \alpha_s \beta_{s^{-1}g} \right)^2 \right) \\
&\leq \gamma(\rho(n)) \sum_{g\in G} \sum_{x\in X_g} \left(\sum_{s\in S_{g,x}} \alpha_s \beta_{s^{-1}g} \right)^2 \\
&\leq \gamma(\rho(n)) \sum_{g\in G} \sum_{x\in X_g} \left(\sum_{s\in S_{g,x}} \alpha_s^2 \right) \left(\sum_{s\in S_{g,x}} \beta_{s^{-1}g}^2 \right)
\end{aligned}$$

For $x, g \in G$, let

$$T_{g,x} = \{s^{-1}g | s \in S_{g,x}\}.$$

Substituting $t = s^{-1}g$, we see that

$$\|ab\|_2^2 \leq \gamma(\rho(n)) \sum_{g\in G} \sum_{x\in X_g} \left(\sum_{s\in S_{g,x}} \alpha_s^2 \right) \left(\sum_{t\in T_{g,x}} \beta_t^2 \right) = \gamma(\rho(n)) \sum_{s\in G} \sum_{t\in G} C_{s,t} \alpha_s^2 \beta_t^2,$$

for some $C_{s,t} \geq 0$. We claim that $C_{s,t}$ is bounded from above by $\gamma(\rho(n))^2$. Note that each term $\alpha_s^2\beta_t^2$ appears at most once in the product $\left(\sum_{s\in S_{g,x}} \alpha_s^2\right)\left(\sum_{t\in T_{g,x}} \beta_t^2\right)$. For fixed s and t we see that $C_{s,t}$ is bounded by the number of pairs $(g, x) \in G\times G$ such that $x \in X_g$, $s \in S_{g,x}$ and $t \in T_{g,x}$. If $s \in S_{g,x}$, then $s \in \mathrm{supp}(a) \subset B(n)$ and $x \in C(1, s) \cap C(s, g) \subset C(1, s)$. Using Lemma 7.8, we see that $|C(1, s)| \leq \gamma(\rho(n))$. Hence, for fixed s, there are at most $\gamma(\rho(n))$ elements x such that $s \in S_{g,x}$.

Fix now x and t. See that $t \in T_{g,x}$ is equivalent to $gt^{-1} \in S_{g,x}$, hence $gt^{-1} \in \mathrm{supp}(a)$. Using that C is G-equivariant, we see that

$$x \in C(1, gt^{-1}) \cap C(gt^{-1}, g) = g(C(g^{-1}, t^{-1}) \cap C(t^{-1}, 1)).$$

As ℓ is a pseudolength function, $\ell(tg^{-1}) = \ell(gt^{-1}) \leq n$, implying that $C(1, tg^{-1}) \subset B(\rho(n))$. As C is G-equivariant and symmetric, we now have

$$\begin{aligned}
g^{-1}x \in C(g^{-1}, t^{-1}) \cap C(t^{-1}, 1) &= t^{-1}(C(tg^{-1}, 1) \cap C(1, t)) \\
&= t^{-1}(C(1, tg^{-1}) \cap C(t, 1)) \subset t^{-1}(B(\rho(n)) \cap C(1, t)).
\end{aligned}$$

By the definition of γ,

$$|B(\rho(n)) \cap C(1, t)| \leq \gamma(\rho(n)),$$

meaning that for any fixed x, t there exist at most $\gamma(\rho(n))$ elements g satisfying $t \in T_{g,x}$. All combined, we have $C_{s,t} \leq \gamma(\rho(n))^2$. In conclusion,

$$\|ab\|_2^2 \leq \gamma(\rho(n))^3 \sum_{s\in G} \alpha_s^2 \sum_{t\in G} \beta_t^2 = \gamma(\rho(n))^3 \|a\|_2^2 \|b\|_2^2 .$$

□

Proof of Proposition 7.7. Given $f = \sum_{g\in G} \eta_g \delta_g \in \mathbb{C}G$, we define $f^+ = \sum_{g\in G} |\eta_g|\delta_g \in \mathbb{R}_+G$. As previously, write $a = \sum_{s\in\mathrm{supp}(a)} \alpha_s\delta_s$ and $b = \sum_{g\in G} \beta_g\delta_g$, with only finitely many $\beta_g \neq 0$. See first that for any $f \in \mathbb{C}G$,

$$\|f\|_2 = \sum_{g\in G} |\eta_g|^2 = \left\|f^+\right\|_2 .$$

By similar calculations as in the previous proof,

$$\|ab\|_2 = \left\|(ab)^+\right\|_2 = \sum_{g\in G} \left| \sum_{s\in\mathrm{supp}(a)} \alpha_s \beta_{s^{-1}g} \right|^2 \leq \sum_{g\in G} \left(\sum_{s\in\mathrm{supp}(a)} |\alpha_s||\beta_{s^{-1}g}| \right)^2 = \left\|a^+b^+\right\|_2 .$$

Recall that $\mathbb{C}G$ is dense in $C_r^*(G)$, and as the above inequality holds for arbitrary $b \in \mathbb{C}G$, we obtain using Lemma 7.9 that

$$\begin{aligned} \|a\| = \sup_{b\in\mathbb{C}G\setminus\{0\}} \frac{\|ab\|_2}{\|b\|_2} &\leq \sup_{b\in\mathbb{C}G\setminus\{0\}} \frac{\|a^+b^+\|_2}{\|b^+\|_2} = \sup_{c\in\mathbb{R}_+G\setminus\{0\}} \frac{\|ac\|_2}{\|c\|_2} \\ &\leq \sup_{c\in\mathbb{R}_+G\setminus\{0\}} \frac{\gamma(\rho(n))^{\frac{3}{2}} \|a^+\|_2 \|c\|_2}{\|c\|_2} = \gamma(\rho(n))^{\frac{3}{2}} \|a\|_2 . \end{aligned}$$

Assume now that S is a subsemigroup of G and $\lim_{k\to\infty} \sqrt[k]{\gamma(\rho(k))} = 1$. Using that ℓ is a pseudolength function, $\mathrm{supp}(a^k) \subset S \cap B(nk)$. Then

$$\|a^k\| \leq \gamma(\rho(nk))^{\frac{3}{2}} \|a^k\|_2,$$

and in particular

$$r(a) = \lim_{k\to\infty} \sqrt[k]{\|a^k\|} \leq \limsup_{k\to\infty} \gamma(\rho(nk))^{\frac{3}{2k}} \|a^k\|_2^{\frac{1}{k}} = r_2(a).$$

As previously noted $r(a) \geq r_2(a)$, implying that $r_2(a) = r(a)$. □

As in [12], define a class $\mathcal{C}$ of groups in the following way:

Definition 7.10. Let $\mathcal{C}$ be the class of groups G with the following property: For any finite subset $F \subset G$ there exists pseudolength function ℓ on G, an element $t \in G$ and a symmetric G-equivariant generalized combing $C : G \times G \to \mathcal{P}(G)$ such that

(i) tF is semifree,

(ii) $C(1, s) \cap C(s, g) \cap C(1, g) \neq \emptyset$ for all $g \in G$ and $s \in S$,

(iii) The growth functions γ and ρ associated to C and computed with respect to ℓ are bounded from above by some polynomials in n.

Corollary 7.11. *Let $G \in \mathcal{C}$. Then $sr(C_r^*(G)) = 1$.*

Proof. Let $G \in \mathcal{C}$ and let F be a finite subset of G. By assumption, the growth functions are bounded from above by some polynomials, meaning

$$1 \leq \sqrt[k]{\gamma(\rho(k))} \leq \sqrt[k]{p(k)} \to 1, \quad k \to \infty,$$

for some polynomial p. Moreover, tF is by assumption semifree and has the ℓ^2-spectral property by Proposition 7.7. As F is arbitrary, it now follows from Theorem 7.4 that $\mathrm{sr}(C_r^*(G)) = 1$. □

Definition 7.12. A group G has the *rapid decay property*, written property (RD), if there exists a length function ℓ on G and constants $s, c \geq 0$ such that for all $a = \sum_{g \in G} \alpha_g \delta_g \in \mathbb{C}G$, it holds that

$$\|a\| \leq c \left(\sum_{g \in G} |(1 + \ell(g))^s \alpha_g|^2 \right)^{\frac{1}{2}}.$$

Rapid decay was, as previously mentioned, first established for free groups of finitely many generators by Haagerup in [13], and was formalized and studied as a concept by Jolissaint in [14]. Dykema and de la Harpe show in [11] that groups with property (RD) have the ℓ^2-spectral radius property:

Proposition 7.13. *Let G be a group with property (RD). For every $a \in \mathbb{C}G$,*

$$r(a) = \lim_{n \to \infty} \sqrt[n]{\|a^n\|_2} = r_2(a).$$

In particular, G has the ℓ^2-spectral radius property.

Proof. Let ℓ be the length function for which G has property (RD). Define $\ell(a) = \max_{g \in \mathrm{supp}(a)} \ell(g)$ and let $c, s \geq 0$ be the constants such that for every $a \in \mathbb{C}G$,

$$\|a\| \leq c \left(\sum_{g \in G} |(1 + \ell(g))^s \alpha_g|^2 \right)^{\frac{1}{2}} \leq c(1 + \ell(a))^s \|a\|_2 .$$

As $\|a^k\|_2 \leq \|a^k\|$ and $\ell(a^k) \leq k\ell(a)$ for all integers $k \geq 1$,

$$r_2(a) \leq r(a) = \lim_{k\to\infty} \sqrt[k]{\|a^k\|} \leq \limsup_{k\to\infty} \sqrt[k]{c(1+\ell(a^k))^s \|a^k\|_2}$$
$$\leq \limsup_{k\to\infty} \sqrt[k]{c(1+k\ell(a))^s \|a^k\|_2} = r_2(a),$$

with the last equality following from the fact that $\sqrt[k]{c(1+k\ell(a))^s} \to 1$ as $k \to \infty$. □

7.1 Acylindrically hyperbolic groups

This section presents the definition of acylindrically hyperbolic groups and the main theorem of Gersimova and Osin in [12]. For further reading on acylindrically hyperbolic groups, the reader is referred to [12, 15, 8].

Definition 7.14. Let $\delta > 0$. A metric space X is δ-hyperbolic if it is geodesic and if for any geodesic triangle Δ in X every side of Δ is contained in the union of the δ-neighborhoods of the the two other sides.

Definition 7.15. Let G be a group and X a metric space. An isometric action of G on X is *acylindrical* if for every $\varepsilon > 0$ there exist $R, N > 0$ such that for every $x, y \in S$ with $d(x, y) \geq R$, there are at most N elements $g \in G$ satisfying

$$d(x, gx) \leq \varepsilon, \qquad d(y, gy) \leq \varepsilon.$$

A subgroup is virtually cyclic if it contains a cyclic subgroup of finite index.

Definition 7.16. A group G is *acylindrically hyperbolic* if it admits an acylindrical action on a hyperbolic space X which has unbounded orbits and G is not virtually cyclic.

Examples of acylindrically hyperbolic groups can be found in [12] and [8].

It was shown by Dahmani, Guirardel and Osin in [8] that every acylindrically hyperbolic group G contains a unique maximal finite normal subgroup called the *finite radical* of G, which is denoted by $K(G)$. As noted in [12], the acylindrically hyperbolic groups with trivial finite radical and direct products of such groups are C^*-simple.

Theorem 7.17 (Gerasimova-Osin)**.** *Let $G_1, \ldots, G_k$ be acylindrically hyperbolic groups with $K(G_i) = \{1\}$ for all $1 \leq i \leq k$. Then $sr(C_r^*(G_1 \times \cdots \times G_k)) = 1$. In particular, $C_r^*(G)$ has stable rank one for any acylindrically hyperbolic group with trivial finite radical.*

The following proposition is the main result of Section 4 in [12], but the proof is beyond the scope of this thesis.

Proposition 7.18. *Let G be an acylindrically hyperbolic group with trivial finite radical $K(G)$ and let F be a non-empty finite subset of G. Then there exists length function ℓ on G, an element $t \in G$ and a symmetric G-equivariant generalized combing $C : G \times G \to \mathcal{P}(G)$ such that*

(i) tF *is semifree,*

(ii) $C(1, s) \cap C(s, g) \cap C(1, g) \neq \emptyset$ *for all* $g \in G$ *and* $s \in S$,

(iii) The growth functions γ and ρ associated to C and computed with respect to ℓ are bounded from above by a linear function.

It remains to show that the reduced group C^*-algebra of finite direct products of such groups also has stable rank one. In fact, $\mathcal{C}$ is closed under taking finite direct product.

Lemma 7.19. *$\mathcal{C}$ is closed under taking finite direct products.*

Proof. It suffices to prove that $G = G_1 \times G_2 \in \mathcal{C}$ for any $G_1, G_2 \in \mathcal{C}$.

Let finite $F \subset G$ be given. Let F_i be the projection of F to G_i for $i = 1, 2$ and see that $F \subset F_1 \times F_2$. As $G_1, G_2 \in \mathcal{C}$, there exist $t_1 \in G_1$ and $t_2 \in G_2$ such that t_1F_1, respectively t_2F_2, is semifree in G_1, respectively G_2. We claim that $(t_1, t_2)(F_1 \times F_2)$ is semifree in G, which in turn implies that $(t_1, t_2)F$ is semifree in G. To prove the claim, let $t_1F_1 = \{f_i\}_i$ and $t_2F_2 = \{g_j\}_j$. Assume that

$$(f_{i_1}, g_{j_1}) \ldots (f_{i_n}, g_{j_n}) = (f_{k_1}, g_{l_1}) \ldots (f_{k_m}, g_{l_m}),$$

equivalent to $f_{i_1} \ldots f_{i_n} = f_{k_1} \ldots f_{k_m}$ and $g_{j_1} \ldots g_{j_n} = g_{k_1} \ldots g_{k_m}$. As each t_iF_i is semifree, the equality implies that $n = m$ and $i_s = k_s$, $j_s = l_s$ for all $1 \leq s \leq n$, which proves the claim. Letting $t = (t_1, t_2)$ proves (i) in Definition 7.10 for G.

Let ℓ_i, C_i be the pseudolength functions and symmetric G_i-equivariant generalized combings on G_i for $i = 1, 2$ such that

$$C_i(1, s) \cap C_i(s, g) \cap C_i(g, 1) \neq \emptyset,$$

for all $g \in G_i$ and $s \in S_i$, where S_i is the subsemigroup generated by t_iF_i. Define generalized combing $C : G \times G \to \mathcal{P}(G)$ by

$$C((g_1, g_2), (h_1, h_2)) = C_1(g_1, h_1) \times C_2(g_2, h_2)$$

for all $(g_1, g_2), (h_1, h_2) \in G$. It follows immediately from the fact that each C_i is symmetric and G_i-equivariant that C is symmetric and G-equivariant, proving (ii) in Definition 7.10.

Define $\ell : G \to [0, \infty)$ by $\ell((g, h)) = \max\{\ell_1(g), \ell_2(h)\}$ for all $(g, h) \in G$. As ℓ_1, ℓ_2 are pseudolength functions, so is ℓ. It remains to show that the corresponding growth functions γ, ρ on G are bounded from above by some polynomials. Let $B_i(n)$, respectively $B(n)$, denote the balls of radius n centered at 1 in G_i, respectively G, with respect to the pseudolength functions ℓ_i, ℓ. Then $B(n) = B_1(n) \times B_2(n)$ for all $n \in \mathbb{N}$. For every $g = (g_1, g_2) \in G$,

$$\begin{aligned} B(n) \cap C(1, g) &= (B_1(n) \times B_2(n)) \cap (C_1(1, g_1) \times C_2(1, g_2)) \\ &= (B_1(n) \cap C_1(1, g_1)) \times (B_2(n) \cap C_2(1, g_2)). \end{aligned}$$

Thus

$$\begin{aligned} \gamma(n) = \sup_{g \in G} |B(n) \cap C(1, g)| &= \sup_{(g_1, g_2) \in G_1 \times G_2} |(B_1(n) \cap C_1(1, g_1)) \times (B_2(n) \cap C_2(1, g_2))| \\ &= \sup_{(g_1, g_2) \in G_1 \times G_2} |B_1(n) \cap C_1(1, g_1)||B_2(n) \cap C_2(1, g_2)| \\ &\leq \gamma_1(n)\gamma_2(n). \end{aligned}$$

Recall that for $g = (g_1, g_2) \in B(n)$ and $x = (x_1, x_2) \in C(1, g)$ we have $\ell_i(x_i) \leq \rho_i(n)$. Thus

$$\ell(x) \leq \max\{\rho_1(n), \rho_2(n)\},$$

and so

$$\rho(n) = \sup_{g \in B(n)} \sup_{x \in C(1, g)} \ell(x) \leq \rho_1(n)\rho_2(n),$$

using that $\rho_i(n) \geq 1$ for all $n \geq 1$. By assumption γ_i, ρ_i are bounded from above by polynomials, so the above implies that the same holds for γ, ρ. We conclude that $G \in \mathcal{C}$. □

Combining Proposition 7.18, Lemma 7.19 and Corollary 7.11 proves Theorem 7.17.

Examples 7.20. Consider the free group of two generators, F_2. We will show that $F_2 \in \mathcal{C}$. Let $\ell(g)$ for $g \in F_2$ be the word length of the reduced word g.

Let $F \subset F_2$ be a finite subset, and let n be the length of the longest word in F. We claim that the subsemigroup $S = \langle (ba)^{n+1} F \rangle$ is semifree. Note that elements in S begin with ba. Assume that $x, y \in S$ with $x = y$. Write

$$x = (ba)^{n+1} f_1 (ba)^{n+1} f_2 \cdots (ba)^{n+1} f_k, \qquad y = (ba)^{n+1} f'_1 (ba)^{n+1} f'_2 \cdots (ba)^{n+1} f'_r,$$

with $f_i, f'_j \in F$ for all i, j. After (possibly) reducing, write

$$x = (ba)^{n+1} x_1 (ba)^{n+1} x_2 \cdots (ba)^{n+1} x_k, \qquad y = (ba)^{n+1} y_1 (ba)^{n+1} y_2 \cdots (ba)^{n+1} y_2.$$

Thus, $x = y$ implies, using the properties of the free group, that $r = k$ and $x_i = y_i$ for all $1 \leq i \leq k$, in turn implying $f_i = f'_i$ for all $1 \leq i \leq k$.

Consider combing on F_2 which to each pair of elements in $g, h \in F_2$, assigns the shortest path $\gamma_{g,h}$ in the Cayley graph of F_2. One can do so, as the Cayley graph of F_2 is a tree. Let $C : F_2 \times F_2 \to \mathcal{P}(F_2)$ be the generalized combing corresponding to the described combing. That C is symmetric and equivariant follows simply as the Cayley graph is a tree. Note that if $g, h \in F_2$ start with different letters then $C(g, h)$ must contain 1. If g, h start with the same letter, $C(g, h)$ will contain said letter. In either case,

$$C(1, g) \cap C(g, h) \cap C(1, h) \neq \emptyset.$$

Let $n \geq 1$ and assume $g \in B(n)$. Then for all $x \in C(1, g)$ we have $\ell(x) \leq \ell(g) \leq n$, implying

$$\rho(n) = \sup_{g \in B(n)} \sup_{x \in C(1,g)} \ell(x) \leq n.$$

Moreover, see that $|C(1, g)|$ is equal $\ell(g) + 1$. Therefore,

$$\gamma(n) = \sup_{g \in G} |C(1, g) \cap B(n)| = n + 1,$$

which proves that $F_2 \in \mathcal{C}$.

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