Solution by power series

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1 Introduction

We consider the second order homogeneous linear equation

$$A(t)x'' + B(t)x' + C(t)x = 0 \tag{1}$$

with the aim of constructing solutions in the form of power series. The method we shall use works equally well for equations of order $n = 1$ or $n > 2$, but the second-order case is the most important for applications.

The basic method consists of substituting a power series

$$x(t) = \sum_{n=0}^{\infty} c_n t^n \tag{2}$$

into the differential equation, and then sort out what conditions the coefficients $c_0, c_1, \ldots$ must satisfy in order for the equation to be satisfied. If we succeed the outcome will thus be a solution given by an infinite series, rather than a solution expressed in closed form with elementary functions.

2 Power series

Let us first review (from your real analysis course) a few basic facts about power series. A general power series has the form

$$\sum_{n=0}^{\infty} c_n (t-a)^n$$

with coefficients $c_0, c_1, \dots \in \mathbb{C}$ and $a \in \mathbb{R}$. For simplicity we confine ourselves to the case $a = 0$, as the general case is easily converted to this situation by replacement of the translated variable $t - a$ with t.

Recall that for every power series (2) there exists $\rho \in [0, +\infty]$, called the *radius of convergence*, such that the series converges absolutely for every $t \in I = (-\rho, \rho)$, and diverges whenever $|t| > \rho$. It follows from the ratio test that if the limit $\lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|}$ exists in $[0, +\infty]$, then it equals ρ.

If $\rho > 0$ then the sum $x(t)$ belongs to $C^\infty(I)$, and its derivatives are the sums of the power series obtained by termwise differentiations,

$$x'(t) = \sum_{n=1}^{\infty} n c_n t^{n-1}, \quad x''(t) = \sum_{n=2}^{\infty} n(n-1) c_n t^{n-2}, \quad \text{etc.} \tag{3}$$

In particular, $x^{(k)}(0) = k! c_k$ for $k = 0, 1, \ldots$. It follows that when the power series converges, it is the Taylor series at 0 of its sum. The fact that all the coefficients c_k are uniquely determined by the sum $x(t)$ is called the *identity principle* for power series.

Recall also that power series can be manipulated algebraically termwise. If $x(t) = \sum_{n=0}^{\infty} a_n t^n$ and $y(t) = \sum_{n=0}^{\infty} b_n t^n$, with intervals of convergence I and J, respectively, then

$$x(t) + y(t) = \sum_{n=0}^{\infty} (a_n + b_n) t^n, \quad \lambda x(t) = \sum_{n=0}^{\infty} \lambda a_n t^n \tag{4}$$

for all $\lambda \in \mathbb{R}$, and

$$x(t)y(t) = \sum_{n=0}^{\infty} c_n t^n, \tag{5}$$

where $c_n = \sum_{i+j=n} a_i b_j$ for each n. Both series converge for $t \in I \cap J$ (at least).

In the application to differential equations we shall use the technique of *shift of summation index*, according to which the power series $\sum_{n=0}^{\infty} c_n t^n$ can be rewritten as $\sum_{n=1}^{\infty} c_{n-1} t^{n-1}$. More generally

$$\sum_{n=N}^{\infty} c_n t^n = \sum_{n=N+k}^{\infty} c_{n-k} t^{n-k},$$

where the summation index n is shifted by k and its initial value N is shifted also by k, but in the opposite direction. The identity is valid simply because both expressions expand to the same sum $c_N t^N + c_{N+1} t^{N+1} + \dots$.

3 Example

We are now ready to present the solution method in a simple example, the equation

$$x'' + x = 0.$$

With the Ansatz that $x(t)$ is given by (2) with a positive radius of convergence, it follows from (3) that

$$\sum_{n=2}^{\infty} n(n-1) c_n t^{n-2} + \sum_{n=0}^{\infty} c_n t^n = 0.$$

We shift the first summation by 2 in order to obtain the same power of t in both sums:

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} t^n + \sum_{n=0}^{\infty} c_n t^n = 0.$$

Next we apply (4) and obtain

$$\sum_{n=0}^{\infty} ((n+2)(n+1) c_{n+2} + c_n) t^n = 0.$$

By the identity principle we conclude that $(n+2)(n+1)c_{n+2} + c_n = 0$ for all n, and hence

$$c_{n+2} = -\frac{c_n}{(n+1)(n+2)}.$$

A relation of this type is called a *recursion formula*, and it shows that we can determine all the coefficients c_n recursively from the first two. In fact, by a simple induction we derive

$$c_{2k} = \frac{(-1)^k c_0}{(2k)!}, \quad c_{2k+1} = \frac{(-1)^k c_1}{(2k+1)!}$$

for all k, and hence

$$x(t) = c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} t^{2k} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1}.$$

Here we recognize the Taylor series of $\cos t$ and $\sin t$, but in a general situation we will obtain power series that do not necessarily represent known functions. Note that since c_0 and c_1 can vary freely, we have obtained a 2-dimensional linear solution space, spanned by the functions represented by the two series. This is in accordance with the general principle for a linear homogeneous equation.

4 Main theorems

The main theorem about the power series method deals with an important issue which was not addressed in the example above, namely the actual convergence of the series solutions produced by the method. We shall call a power series, which satisfies (1), by substitution into the equation and by manipulations using (3), (4), (5) and index shifts as above, a *formal solution*, regardless of whether it converges or not.

We first state a result for the special case where $A(t) = 1$. Then the equation reads

$$x'' + P(t)x' + Q(t)x = 0 \tag{6}$$

for some functions $P(t)$ and $Q(t)$. The original equation can obviously be brought to this form by division with $A(t)$. In the following, we say that a function f is *sum of a power series* on $]-R, R[$ if its Taylor series at 0 converges on the entire interval with sum f.

4.1 Theorem. *Let $R > 0$ and assume $P(t)$ and $Q(t)$ are sums of power series on $I =]-R, R[$. Then every formal power series solution to* (6) *converges to a solution on I, and all solutions on this interval can be obtained in this fashion.*

Most important applications are of the form

$$A(t)x'' + B(t)x' + C(t)x = 0 \tag{7}$$

with functions A, B and C that are polynomials. In the simplified form (6) we then have rational functions $P(t) = B(t)/A(t)$ and $Q(t) = C(t)/A(t)$, but when deriving the formal solutions it is more convenient *not* to simplify, but instead use the original form (7). The next theorem concerns the equation (7).

4.2 Theorem. *Suppose $A(t)$, $B(t)$ and $C(t)$ are polynomials and that $A(0) \neq 0$. Then every formal power series solution to* (7) *has a radius of convergence $R > 0$, which is at least the distance from* 0 *to the nearest zero of A in the complex plane. Its sum is a solution on $I =]-R, R[$, and all solutions on this interval can be obtained in this fashion.*

Proof of Theorems 4.1 and 4.2. Once convergence of a formal power series solution is known, then all the manipulations mentioned for the concept of a formal solution (see above) correspond to actual operations on the sum. In that case the formal solution converges to an actual solution. Hence it suffices to prove the convergence stated in the theorems.

We first prove Theorem 4.1. We begin by deriving the general form of the recursion formula for the power series solutions to (6). Let $\sum_{k=0}^{\infty} p_k t^k$ and $\sum_{k=0}^{\infty} q_k t^k$ be the Taylor series at 0 for P and Q, known to converge with sum $P(t)$ and $Q(t)$ for $|t| < R$. Substituting (2) and (3) into (6) yields

$$\sum_{n=2}^{\infty} n(n-1)c_n t^{n-2} + \left(\sum_{k=0}^{\infty} p_k t^k\right)\left(\sum_{n=1}^{\infty} n c_n t^{n-1}\right) + \left(\sum_{k=0}^{\infty} q_k t^k\right)\left(\sum_{n=0}^{\infty} c_n t^n\right) = 0.$$

In the first two terms we shift the sums so that they become

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}t^n \quad \text{and} \quad \left(\sum_{k=0}^{\infty} p_k t^k\right)\left(\sum_{n=0}^{\infty}(n+1)c_{n+1}t^n\right).$$

Applying (5) to the products we then obtain

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}t^n + \sum_{n=0}^{\infty}\sum_{i+j=n} p_i(j+1)c_{j+1}t^n + \sum_{n=0}^{\infty}\sum_{i+j=n} q_i c_j t^n = 0.$$

Collecting all terms with t^n we see that this is formally valid if

$$(n+2)(n+1)c_{n+2} = -\sum_{i+j=n} p_i(j+1)c_{j+1} - \sum_{i+j=n} q_i c_j$$

for all $n \geq 0$. This is the recursion formula that the method provides, and it allows us to determine c_{n+2} for all $n \geq 0$ recursively from c_0 and c_1, regardsless of which values these two parameters have.

Let $0 < r < R$ and choose s with $r < s < R$. Then $\sum_{k=0}^{\infty} |p_k| s^k$ converges. In particular the terms $|p_k| s^k$ are bounded above as functions of k. We see similarly that $|q_k| s^k$ is bounded. Hence we can find $C > 0$ such that $|p_k| \leq C s^{-k}$ and $|q_k| \leq C s^{-k}$ for all $k \geq 0$.

We will now prove that the sequence $|c_n| r^n$ is bounded. We shall do this by induction, and to prepare for the inductive step we prove the following lemma.

4.3 Lemma. *There exists a number N (depending on r, s, C from above) such that*

$$|c_m| r^m \leq \max_{0 \leq k < m} |c_k| r^k.$$

for all integers $m \geq N$.

Proof. Let $m \geq 2$ and $M = \max_{0 \leq k < m} |c_k| r^k$. Since $m \geq 2$ we can write $m = n+2$ where $n \geq 0$ and apply the recursion formula for c_{n+2}. It follows that

$$\begin{aligned}
(n+2)(n+1)|c_{n+2}| &\leq \sum_{i+j=n} C s^{-i} (j+1) M r^{-j-1} + \sum_{i+j=n} C s^{-i} M r^{-j} \\
&= CM \sum_{i+j=n} (j+1+r) s^{-i} r^{-j-1} \\
&\leq CM(n+1+r) r^{-1} \sum_{i+j=n} s^{-i} r^{-j}.
\end{aligned}$$

Now $n + 1 + r \leq (n+1)(1+r)$ and

$$\sum_{i+j=n} s^{-i} r^{-j} = r^{-n} \sum_{i=0}^{n} (r/s)^i \leq r^{-n} \sum_{i=0}^{\infty} (r/s)^i.$$

Since $r < s$ we have $\sum_{i=0}^{\infty} (r/s)^i = \frac{s}{s-r} < \infty$ and hence the bound above yields

$$|c_{n+2}| \leq \frac{CM(1+r)s}{(n+2)(s-r)} r^{-n-1}.$$

Hence if

$$m = n + 2 \geq N := \frac{C(1+r)sr}{s-r}$$

it follows that

$$|c_m| \leq M r^{-m}.$$

□

The boundedness of $|c_n| r^n$ easily follows: Let $M = \max_{0 \leq k < N} |c_k| r^k$ where N is as in the lemma. By induction it follows that $M = \max_{0 \leq k < m} |c_k| r^k$ for all $m \geq N$.

From the bound $|c_n| \leq Mr^{-n}$ it follows that the series $c_n t^n$ converges for $|t| < r$. Hence the radius of convergence is at least r, and as r was an arbitrary number below R, it is at least R.

Finally, we saw above that the recursive relation can be solved for all values of c_0 and c_1. These are exactly the values $x(0)$ and $x'(0)$ of the sum function, and hence every set of initial values $x(0)$ and $x'(0)$ is accounted for with the power series. It follows that all the solutions on I are accounted for. This completes the proof of Theorem 4.1.

In order to derive Theorem 4.2 from Theorem 4.1 we only need to prove that the rational functions $P(t) = B(t)/A(t)$ and $Q(t) = C(t)/A(t)$ are sums of power series on the interval $I =]-R, R[$ mentioned in Theorem 4.2. Since B and C are polynomials it suffices to prove that $1/A(t)$ is sum of a power series on I (this follows from (5)). By the fundamental theorem of algebra $A(t)$ is a constant times the product of the functions $\alpha - t$ where $\alpha \in \mathbb{C}$ are the roots of A. In particular $|\alpha| \geq R$ by assumption. Using (5) we easily reduce to the statement that $(\alpha - t)^{-1}$ is the sum of a power series on I. This is true, since

$$(\alpha - t)^{-1} = \sum_{n=0}^{\infty} \alpha^{-n-1} t^n$$

which has radius of convergence $|\alpha|$. □

5 Legendre's equation

This equation is important in many applications. It has the form

$$(1 - t^2)x'' - 2tx' + l(l+1)x = 0$$

with l a complex constant (often an integer). The substitution of (2) into the equation leads to

$$\sum_{n=2}^{\infty} n(n-1)c_n t^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n t^n - \sum_{n=1}^{\infty} 2nc_n t^n + l(l+1) \sum_{n=0}^{\infty} c_n t^n = 0.$$

Here we have to shift the first sum so that it has the same power of t as the others. It becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} t^n$$

We also observe that the other sums might as well start at $n = 0$, since their coefficients are zero for the added values of n. Collecting terms we obtain

$$\sum_{n=0}^{\infty} \left((n+2)(n+1)c_{n+2} - n(n-1)c_n - 2nc_n + l(l+1)c_n\right) t^n = 0$$

and by the identity principle we conclude

$$c_{n+2} = \frac{n(n-1) + 2n - l(l+1)}{(n+2)(n+1)} c_n = \frac{n(n+1) - l(l+1)}{(n+2)(n+1)} c_n$$

for all $n \geq 0$. This recurrence formula allows us to determine the even numbered coefficients from c_0 and the odd numbered from c_1. This leads to two series, one in even powers of t obtained with $c_0 = 1$ and $c_1 = 0$, and the other in odd powers, obtained with $c_0 = 0$ and $c_1 = 1$. The roots of $A(t) = 1 - t^2$ are ± 1, and hence Theorem 4.2 tells us that the two series converge on $]-1, 1[$, and that every solution in this interval is a linear combination of these to functions.

It is interesting to consider the case where l is a non-negative integer. In this case it follows from the recursion formula that $c_{n+2} = 0$ for $n = l$, and then all the later coefficients of the same parity vanish as well. If l is even it follows that the power series solution with even powers terminates at $n = l$, and if l is odd then the solution with odd powers terminates at $n = l$. In both cases the equation thus allows a solution which is polynomial of degree l. This polynomial (appropriately normalized) is called the *Legendre polynomial* of degree l.

6 Regular singular points

In Theorem 4.2 we considered the equation

$$A(t)x'' + B(t)x' + C(t)x = 0.$$

under the assumption that $A(0) \neq 0$. We will now consider a situation where this is not the case. We assume that $A(t)$, $B(t)$ and $C(t)$ are polynomials, and that there is no non-trivial common polynomial factor to all three of them. Then a point $t = t_0$ is said to be *singular* if $A(t_0) = 0$, and *ordinary* otherwise. For the Legendre equation, for example, the singular points are $t_0 = \pm 1$.

Let us assume that $t = 0$ is singular. Then $A(0) = 0$. We will deal with two cases of this. The simplest case occurs when the multiplicity of $t = 0$ as a zero of the polynomial $A(t)$ is one. The second case occurs when the multiplicity is two and in addition $B(0) = 0$. The type of the solutions turns out to be similar in the two cases, and it is customary to refer to them together by saying that $t = 0$ is a *regular singular* point. Note that the first case, where $A(0) = 0$ with multiplicity one, can be transformed into a special case of the second case by multiplication of the entire equation with t. The common feature of the two cases is thus that the equation has or can be brought to the form

$$t^2 a(t)x''(t) + tb(t)x'(t) + c(t)x(t) = 0 \tag{8}$$

with polynomials $a(t)$, $b(t)$ and $c(t)$ such that $a(0) \neq 0$. This means that we can treat the two cases simultaneously through the equation (8).

In general (8) cannot be solved by a power series (2). This can be seen already in the case where a, b and c are constants,

$$a(t) = a_0, \quad b(t) = b_0, \quad c(t) = c_0.$$

In this case we can solve with the Ansatz $x(t) = t^\lambda$. Substitution into (8) gives

$$t^2 a_0 \lambda(\lambda - 1)t^{\lambda-2} + t b_0 \lambda t^{\lambda-1} + c_0 t^\lambda = 0$$

or equivalently

$$a_0\lambda(\lambda - 1) + b_0\lambda + c_0 = 0.$$

Thus t^λ solves (8) if (and only if) λ solves this quadratic equation. If a solution λ is not a non-negative integer, there is no way to develop t^λ in a power series as (2).

The proper solution method for the general equation (8) with a regular singularity at $t = 0$ was found by Frobenius, and it consists of substituting

$$x(t) = t^\lambda \sum_{n=0}^{\infty} c_n t^n = \sum_{n=0}^{\infty} c_n t^{\lambda+n}, \tag{9}$$

rather than the plain power series (2), into (8). Frobenius' method works in general, but here we will just indicate it briefly through an important example.

7 Bessel's equation

This is the equation

$$t^2 x''(t) + t x'(t) + (t^2 - p^2)x(t) = 0 \tag{10}$$

where $p \geq 0$ is constant. The substitution of (9) in (10) leads to

$$\sum_{n=0}^{\infty}(\lambda+n)(\lambda+n-1)c_n t^{\lambda+n} + \sum_{n=0}^{\infty}(\lambda+n)c_n t^{\lambda+n} + \sum_{n=0}^{\infty} c_n t^{\lambda+n+2} - p^2 \sum_{n=0}^{\infty} c_n t^{\lambda+n} = 0,$$

which we divide by t^λ and obtain

$$\sum_{n=0}^{\infty}(\lambda+n)(\lambda+n-1)c_n t^{n} + \sum_{n=0}^{\infty}(\lambda+n)c_n t^{n} + \sum_{n=0}^{\infty} c_n t^{n+2} - p^2 \sum_{n=0}^{\infty} c_n t^{n} = 0.$$

The first, second and fourth sum combined gives

$$\sum_{n=0}^{\infty} \left((\lambda+n)^2 - p^2\right) c_n t^n.$$

In the third sum we have to shift the summation in order to have the same power of t as in the other terms. It becomes

$$\sum_{n=2}^{\infty} c_{n-2} t^n.$$

We conclude

$$(\lambda^2 - p^2)c_0 + ((\lambda+1)^2 - p^2)c_1 t + \sum_{n=2}^{\infty} \left(((\lambda+n)^2 - p^2)c_n + c_{n-2}\right) t^n = 0$$

for all $t > 0$. By the identity principle it follows that

$$(\lambda^2 - p^2)c_0 = 0, \tag{11}$$

$$\left((\lambda+1)^2 - p^2\right) c_1 = 0 \tag{12}$$

$$\left((\lambda+n)^2 - p^2\right) c_n + c_{n-2} = 0, \quad (n \geq 2). \tag{13}$$

Here (11) is valid when $\lambda = \pm p$. We choose $\lambda = p$. Then (12) forces $c_1 = 0$. Finally (13) leads to the formula

$$c_n = \frac{-c_{n-2}}{(p+n)^2 - p^2} = \frac{-c_{n-2}}{n(2p+n)}, \quad n \geq 2,$$

from which all c_n can be determined recursively. In particular all the odd numbered terms vanish. We thus reach a formal solution of the form (9), but with only even powers of t. It is easily seen with the ratio test that the series $\sum_{m=0}^{\infty} c_{2m} t^{2m}$ converges for all t, and as before it then follows that the sum is actually a solution to Bessel's equation for $t > 0$. The resulting function, suitably normalized, is called the *Bessel function* of order p. With the first terms it reads

$$J_p(t) = t^p \left(1 - \frac{t^2}{2(2p+2)} + \frac{t^4}{2(2p+2)4(2p+4)} - \dots\right)$$

up to normalization. A second solution, linearly independent from J_p, can be derived in the same fashion by choosing $\lambda = -p$ instead of $\lambda = p$, but only if p is not an integer.

Bessel functions are among the most important functions of mathematics because they have many applications.