Let V be a real vector space. In Note 11 the tensor spaces $T^k(V)$ were defined, together with the tensor product

$$(S,T) \mapsto S \otimes T, \quad T^k(V) \times T^l(V) \to T^{k+l}(V).$$

There is an important construction of vector spaces which resemble tensor powers of V, but for which there is a more refined structure. These are the so-called exterior powers of V, which play an important role in differential geometry because the theory of differential forms is built on them. They are also of importance in algebraic topology and many other fields.

A multilinear map

$$\varphi: V^k = V \times \cdots \times V \to U$$

is called *alternating* if for all $v_1, \dots, v_k \in V$ the value $\varphi(v_1, \dots, v_k)$ changes sign whenever two of the vectors $v_1, \dots, v_k$ are interchanged, that is

$$\varphi(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -\varphi(v_1, \dots, v_j, \dots, v_i, \dots, v_k). \tag{1}$$

Since every permutation of the numbers $1, \dots, k$ can be decomposed into transpositions, it follows that

$$\varphi(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sgn} \sigma \, \varphi(v_1, \dots, v_k) \tag{2}$$

for all permutations $\sigma \in S_k$ of the numbers $1, \dots, k$.

Examples. 1. Let $V = \mathbb{R}^3$. The vector product $(v_1, v_2) \mapsto v_1 \times v_2 \in V$ is alternating from $V \times V$ to V.

2. Let $V = \mathbb{R}^n$. The $n \times n$ determinant is multilinear and alternating in its columns, hence it can be viewed as an alternating map $(\mathbb{R}^n)^n \to \mathbb{R}$.

Lemma 1. *Let $\varphi: V^k \to U$ be multilinear. The following conditions are equivalent:*
(a) φ is alternating,
(b) $\varphi(v_1, \dots, v_k) = 0$ whenever two of the vectors $v_1, \dots, v_k$ coincide,
(c) $\varphi(v_1, \dots, v_k) = 0$ whenever the vectors $v_1, \dots, v_k$ are linearly dependent.

Proof. (a)⇒(b) If $v_i = v_j$ the interchange of v_i and v_j does not change the value of $\varphi(v_1, \dots, v_k)$, so (1) implies $\varphi(v_1, \dots, v_k) = 0$.

(b)⇒(a) Consider for example the interchange of v_1 and v_2. By linearity

$$\begin{aligned} 0 &= \varphi(v_1 + v_2, v_1 + v_2, \dots) \\ &= \varphi(v_1, v_1, \dots) + \varphi(v_1, v_2, \dots) + \varphi(v_2, v_1, \dots) + \varphi(v_2, v_2, \dots) \\ &= \varphi(v_1, v_2, \dots) + \varphi(v_2, v_1, \dots). \end{aligned}$$

It follows that $\varphi(v_2, v_1, \dots) = -\varphi(v_1, v_2, \dots)$.

(b)⇒(c) If the vectors $v_1, \dots, v_k$ are linearly dependent then one of them can be written as a linear combination of the others. It follows that $\varphi(v_1, \dots, v_k)$ is a linear combination of terms in each of which some v_i appears twice.

(c)⇒(b) Obvious. □

In particular, if $k > \dim V$ then every set of k vectors is linearly dependent, and hence $\varphi = 0$ is the only alternating map $V^k \to U$.

Definition 1. An *alternating k-form* is an alternating k-tensor $V^k \to \mathbb{R}$. The space of these is denoted $A^k(V)$, it is a linear subspace of $T^k(V)$.

We define $A^1(V) = V^*$ and $A^0(V) = \mathbb{R}$.

Example. Let $\eta, \zeta \in V^*$. The 2-tensor $\eta \otimes \zeta - \zeta \otimes \eta$ is alternating.

The following lemma exhibits a standard procedure to construct alternating forms.

Lemma 2. *For each $T \in T^k(V)$ the element $\mathrm{Alt}(T) \in T^k(V)$ defined by*

$$\mathrm{Alt}(T)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \, T(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \tag{3}$$

is alternating. Moreover, if T is already alternating, then $\mathrm{Alt}(T) = T$.

Proof. Let $\tau \in S_k$ be the transposition corresponding to an interchange of two vectors among $v_1, \dots, v_k$. We have

$$\mathrm{Alt}(T)(v_{\tau(1)}, \dots, v_{\tau(k)}) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \, T(v_{\tau\circ\sigma(1)}, \dots, v_{\tau\circ\sigma(k)}).$$

Since $\sigma \mapsto \tau \circ \sigma$ is a bijection of S_k we can substitute σ for $\tau \circ \sigma$. Using $\operatorname{sgn}(\tau \circ \sigma) = -\operatorname{sgn}(\sigma)$, we obtain the desired equality with $-\mathrm{Alt}(T)(v_1, \dots, v_k)$.

If T is already alternating, all the summands of (3) are equal to $T(v_1, \dots, v_k)$. Since $|S_k| = k!$ we conclude that $\mathrm{Alt}(T) = T$. □

Let $e_1, \dots, e_n$ be a basis for V, and $\xi_1, \dots, \xi_n$ the dual basis for V^*. We saw in Note 11 that the elements $\xi_{i_1} \otimes \dots \otimes \xi_{i_k}$ form a basis for $T^k(V)$. We will now exhibit a similar basis for $A^k(V)$. We have seen already that $A^k(V) = 0$ if $k > n$.

Theorem 1. *Assume $k \leq n$. For each subset $I \subset \{1, \dots, n\}$ with k elements, let $1 \leq i_1 < \dots < i_k \leq n$ be its elements, and let*

$$\xi_I = \mathrm{Alt}(\xi_{i_1} \otimes \dots \otimes \xi_{i_k}) \in A^k(V). \tag{4}$$

These elements ξ_I form a basis for $A^k(V)$. In particular, $\dim A^k(V) = \frac{n!}{k!(n-k)!}$.

Proof. It follows from the last statement in Lemma 2 that $\mathrm{Alt}\colon T^k(V) \to A^k(V)$ is surjective. Applying Alt to the basis elements $\xi_{i_1} \otimes \dots \otimes \xi_{i_k}$ for $T^k(V)$, we therefore obtain a spanning set for $A^k(V)$. Notice that

$$\mathrm{Alt}(\xi_{i_1} \otimes \dots \otimes \xi_{i_k})(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \; \xi_{i_1}(v_{\sigma(1)}) \cdots \xi_{i_k}(v_{\sigma(k)}),$$

which is $1/k!$ times the determinant of the $k \times k$ matrix $(\xi_{i_p}(v_q))_{p,q}$. It follows that $\mathrm{Alt}(\xi_{i_1} \otimes \cdots \otimes \xi_{i_k}) = 0$ if there are repetitions among the $i_1, \ldots, i_k$. Moreover, we can rearrange these numbers in increasing order at the cost only of a possible change of the sign. Therefore $A^k(V)$ is spanned by the elements ξ_I in (4).

Consider a linear combination $T = \sum_I a_I \xi_I$ with some coefficients a_I. Applying the k-tensor ξ_I to an element $(e_{j_1}, \ldots, e_{j_k}) \in V^k$, where $1 \leq j_1 < \cdots < j_k \leq n$, we obtain 0 except when $(j_1, \ldots, j_k) = I$, in which case we obtain 1. It follows that $T(e_{j_1}, \ldots, e_{j_k}) = a_J$ for $J = (j_1, \ldots, j_k)$. Therefore, if $T = 0$ we conclude $a_J = 0$ for all the coefficients. Thus the elements ξ_I are independent. □

In analogy with the tensor product $(S, T) \mapsto S \otimes T$, from $T^k(V) \times T^l(V)$ to $T^{k+l}(V)$, there is a construction of a product $A^k(V) \times A^l(V) \to A^{k+l}$. Since tensor products of alternating tensors are not alternating, the construction is more delicate.

Definition 2. Let $S \in A^k(V)$ and $T \in A^l(V)$. The *wedge product* $S \wedge T \in A^{k+l}(V)$ is defined by

$$S \wedge T = \mathrm{Alt}(S \otimes T).$$

Example Let $\eta_1, \eta_2 \in A^1(V) = V^*$. Then by definition $\eta_1 \wedge \eta_2 = \frac{1}{2}(\eta_1 \otimes \eta_2 - \eta_2 \otimes \eta_1)$.

Since the operator Alt is linear, the wedge product depends linearly on the factors S and T. It is more cumbersome to verify the associative rule for $\wedge$. In order to do this we need the following lemma.

Lemma 3. *Let $S \in T^k(V)$ and $T \in T^l(V)$. Then*

$$\mathrm{Alt}(\mathrm{Alt}(S) \otimes T) = \mathrm{Alt}(S \otimes \mathrm{Alt}(T)) = \mathrm{Alt}(S \otimes T).$$

Proof. We will only verify

$$\mathrm{Alt}(\mathrm{Alt}(S) \otimes T) = \mathrm{Alt}(S \otimes T).$$

The proof for the other expression is similar.

Let $G = S_{k+l}$ and let $H \subset G$ denote the subgroup of permutations leaving each of the last elements $k+1, \ldots, k+l$ fixed. Then H is naturally isomorphic to S_k. Now

$$\begin{aligned}
&\mathrm{Alt}(\mathrm{Alt}(S) \otimes T)(v_1, \ldots, v_{k+l}) \\
&= \tfrac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}\,\sigma \ \mathrm{Alt}(S)(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) T(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}) \\
&= \tfrac{1}{(k+l)!k!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}\,\sigma \sum_{\tau \in S_k} \mathrm{sgn}\,\tau \ S(v_{\sigma(\tau(1))}, \ldots, v_{\sigma(\tau(k))}) T(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}) \\
&= \tfrac{1}{(k+l)!k!} \sum_{\tau \in H} \sum_{\sigma \in G} \mathrm{sgn}(\sigma \circ \tau) \ S(v_{\sigma(\tau(1))}, \ldots, v_{\sigma(\tau(k))}) T(v_{\sigma(\tau(k+1))}, \ldots, v_{\sigma(\tau(k+l))}).
\end{aligned}$$

Since $\sigma \mapsto \sigma \circ \tau$ is a bijection of G we can substitute σ for $\sigma \circ \tau$, and we obtain the desired expression, since there are $k!$ elements in H. □

Lemma 4. *Let $R \in A^k(V)$, $S \in A^l(V)$ and $S \in A^m(V)$. Then*

$$(R \wedge S) \wedge T = R \wedge (S \wedge T) = \mathrm{Alt}(R \otimes S \otimes T).$$

Let $S \in T^k(V)$ and $T \in T^l(V)$.

Proof. It follows from the preceding lemma that

$$(R \wedge S) \wedge T = \mathrm{Alt}(\mathrm{Alt}(R \otimes S) \otimes T) = \mathrm{Alt}(R \otimes S \otimes T)$$

and

$$R \wedge (S \wedge T) = \mathrm{Alt}(R \otimes \mathrm{Alt}(S \otimes T)) = \mathrm{Alt}(R \otimes S \otimes T). \quad \square$$

Since the wedge product is associative, we can write any product $T_1 \wedge \cdots \wedge T_r$ of tensors $T_i \in A^{k_i}(V)$ without specifying brackets. In fact, it follows by induction from Lemma 4 that

$$T_1 \wedge \cdots \wedge T_r = \mathrm{Alt}(T_1 \otimes \cdots \otimes T_r)$$

regardless of how brackets are inserted in the wedge product.

In particular we see that the basis elements ξ_I in Theorem 1 are given by

$$\xi_I = \xi_{i_1} \wedge \cdots \wedge \xi_{i_k}$$

where $I = (i_1, \dots, i_k)$ is an increasing sequence from $1, \dots, n$, and the basis elements ξ_i from V^* are viewed as 1-forms. This will be our notation for ξ_I from now on.

Lemma 5. *Let $\eta, \zeta \in V^*$, then*

$$\zeta \wedge \eta = -\eta \wedge \zeta. \tag{5}$$

More generally, if $S \in T^k(V)$ and $T \in T^l(V)$ then

$$T \wedge S = (-1)^{kl}\, S \wedge T \tag{6}$$

Proof. The identity (5) follows immediately from the fact that $\eta \wedge \zeta = \frac{1}{2}(\eta \otimes \zeta - \zeta \otimes \eta)$.

Since $A^k(V)$ is spanned by elements of the type $S = \eta_1 \wedge \cdots \wedge \eta_k$, and $A^l(V)$ by elements of the type $T = \zeta_1 \wedge \cdots \wedge \zeta_l$, where $\eta_i, \zeta_j \in V^*$, it suffices to prove (6) for these forms. In order to rewrite $T \wedge S$ as $S \wedge T$ we must let each of the k elements η_i pass the l elements ζ_j. The total number of sign changes is therefore kl. $\square$