In this note the differential is defined for smooth maps between smooth manifolds. Let M, N be smooth manifolds of dimension m and n, respectively, and let $p \in M$. Let $f: M \to N$ be smooth. The *differential* of f at p wiill be a linear map from T_pM to $T_{f(p)}N$. It is a generalization of the derivative $f'(t)$ of a map $f: \mathbb{R} \to \mathbb{R}$.

If M and N are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, then by definition the differential df_p is the linear map $\mathbb{R}^m \to \mathbb{R}^n$ which has the Jacobian $Jf(p) = (\frac{\partial f_i}{\partial x_j}(p))_{i,j}$ as its matrix with respect to the standard basis vectors. It satisfies

$$f(p+h) - f(p) = df_p(h) + o(h),$$

which means that

$$\frac{|f(p+h) - f(p) - df_p(h)|}{|h|} \to 0$$

as $h \to 0$. The geometric interpretation is that the linear map df_p is an approximation to f at p.

The important *chain rule* is valid for the differential of a composed map $f \circ g$, where $g: \mathbb{R}^l \to \mathbb{R}^m$ and $f: \mathbb{R}^m \to \mathbb{R}^n$ are smooth. It reads

$$d(f \circ g)_p = df_{g(p)} \circ dg_p,$$

and it is an immediate consequence of the corresponding chain rule for the Jacobians.

If M and N are smooth manifolds in $\mathbb{R}^k$ and $\mathbb{R}^l$, respectively, the differential $df_p: T_pM \to T_{f(p)}N$ is defined as the restriction to T_pM of the differential dF_p of a local smooth extension F of f at p (see Milnor p. 6). It follows from the chain rule that

$$df_p(\gamma'(t_0)) = (f \circ \gamma)'(t_0) \tag{1}$$

for any smooth curve γ on M with $\gamma(t_0) = p$.

Assume now that M and N are arbitrary abstract manifolds, and let $f: M \to N$ be smooth (see Note 2). The definition of df_p is inspired by (1). Recall from Note 3 that by definition the tangent space T_pM is the set of $\sim_p$ equivalence classes of smooth curves γ on M through p. The class of a curve γ is denoted $[\gamma]$.

Definition 1. The differential $df_p: T_pM \to T_{f(p)}N$ is defined by

$$df_p([\gamma]) = [f \circ \gamma]$$

for all smooth curves γ on M through p.

Notice that $f \circ \gamma$ is a smooth curve on N, and $[f \circ \gamma]$ is its equivalence class in $T_{f(p)}N$. In fact, in order for $df_p([\gamma])$ to be well defined, we need the lemma below. Notice also that if M and N happen to be smooth manifolds in $\mathbb{R}^k$ and $\mathbb{R}^l$, then it follows from (1) that the differential defined in Definition 1 is the same as the previous one.

Lemma 1. *Let γ_1 and γ_2 be smooth curves on M with $\gamma_1(t_1) = \gamma_2(t_2) = p$, then*

$$\gamma_1 \sim_p \gamma_2 \quad \Rightarrow \quad f \circ \gamma_1 \sim_{f(p)} f \circ \gamma_2.$$

Proof. Let τ be a chart on N around $f(p)$, and let σ be a chart on M around p. Then for each curve on M through p,

$$\begin{aligned}(\tau^{-1} \circ f \circ \gamma)'(t_0) &= (\tau^{-1} \circ f \circ \sigma \circ \sigma^{-1} \circ \gamma)'(t_0) \\ &= d(\tau^{-1} \circ f \circ \sigma)_u((\sigma^{-1} \circ \gamma)'(t_0)) \end{aligned} \tag{2}$$

by the chain rule (where u is determined by $\sigma(u) = p$). Applying this to γ_1 and γ_2, the implication of the lemma follows from the definition of the equivalence relations $\sim_p$ and $\sim_{f(p)}$. □

Recall from Note 3 that if a chart σ around p has been chosen, then the map $[\gamma] \mapsto (\sigma^{-1} \circ \gamma)'(t_0)$ is a linear isomorphism $T_pM \to \mathbb{R}^m$. Likewise, if a chart τ around $f(p)$ has been chosen, then the map $[\beta] \mapsto (\tau^{-1} \circ \beta)'(t_0)$ is a linear isomorphism $T_{f(p)}N \to \mathbb{R}^n$. It follows from equation (2) above that the following diagram commutes

$$\begin{array}{ccc} T_pM & \xrightarrow{df_p} & T_{f(p)}N \\ \downarrow & & \downarrow \\ \mathbb{R}^m & \xrightarrow{d(\tau^{-1} \circ f \circ \sigma)_u} & \mathbb{R}^n. \end{array}$$

Since the map in the bottom of this diagram is linear, we conclude that $df_p \colon T_pM \to T_{f(p)}N$ is a linear map.

In Note 3 a different picture of the tangent space T_pM was introduced, which invokes the space $\mathfrak{D}(p)$ of derivations L of $C^\infty(p)$. It was shown that if γ is a smooth curve on M through p, then the map $L_\gamma \colon C^\infty(p) \to \mathbb{R}$ defined by

$$L_\gamma(\varphi) = (\varphi \circ \gamma)'(t_0)$$

is a derivation at p. Furthermore, it was shown that $[\gamma] \mapsto L_\gamma$ is an isomorphism of vector spaces $T_pM \to \mathfrak{D}(p)$. We will now introduce a map $\mathfrak{D}(p) \to \mathfrak{D}(f(p))$, which corresponds to the differential in this picture.

Lemma 2. *Let $f \colon M \to N$ be smooth, and let $L \in \mathfrak{D}(p)$. The map $C^\infty(f(p)) \to \mathbb{R}$ defined by*

$$\varphi \mapsto L(\varphi \circ f)$$

is a derivation at $f(p)$.

Proof. Notice that $\varphi \in C^\infty(f(p))$ implies $\varphi \circ f \in C^\infty(p)$. Let $\varphi, \psi \in C^\infty(f(p))$. Then

$$L((\varphi\psi) \circ f) = L((\varphi \circ f)(\psi \circ f)) = L(\varphi \circ f)\psi(f(p)) + \varphi(f(p))L(\psi \circ f),$$

since L is a derivation. □

Let $\mathfrak{d}f_p(L) \in \mathfrak{D}(f(p))$ denote the derivation defined in the lemma, that is

$$\mathfrak{d}f_p(L)(\varphi) = L(\varphi \circ f)$$

for all $\varphi \in C^\infty(f(p))$. The following lemma shows that $\mathfrak{d}f_p \colon \mathfrak{D}(p) \to \mathfrak{D}(f(p))$ is the map we are seeking.

Lemma 3. *The diagram*

$$\begin{array}{ccc} T_pM & \xrightarrow{df_p} & T_{f(p)}N \\ \downarrow & & \downarrow \\ \mathfrak{D}(p) & \xrightarrow{\mathfrak{d}f_p} & \mathfrak{D}(f(p)), \end{array}$$

with vertical arrows given by $[\gamma] \mapsto L_\gamma$*, is commutative.*

Proof. We have

$$\mathfrak{d}f_p(L_\gamma)(\varphi) = L_\gamma(\varphi \circ f) = (\varphi \circ f \circ \gamma)'(t_0) = L_{f\circ\gamma}(\varphi).$$

Since $[f \circ \gamma] = df_p([\gamma])$, the lemma follows. □

Because of this lemma the map $\mathfrak{d}f_p \colon \mathfrak{D}(p) \to \mathfrak{D}(f(p))$ is also called the differential of f at p, and it is denoted by the same symbol df_p as the differential $T_pM \to T_{f(p)}N$.

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