In this note the mod-2-degree of smooth maps is used to prove a soft version of the Jordan curve theorem, extended to $\mathbb{R}^k$.

Let $M \subset \mathbb{R}^k$ be a compact $(k-1)$-dimensional smooth manifold, where $k > 1$. For each $p \notin M$ we define $g^p \colon M \to S^{k-1}$ by

$$g^p(x) = \frac{x-p}{\|x-p\|} \in S^{k-1}$$

for $x \in M$. Thus $g^p(x)$ is the unit vector pointing in direction towards x from p. The map g^p is smooth, since it is the restriction to M of $G^p \colon x \mapsto \frac{x-p}{\|x-p\|}$, which is defined and smooth $\mathbb{R}^k \setminus \{p\} \to \mathbb{R}^k$. Put $w(p) = \deg_2 g^p$, and

$$D = \{p \mid w(p) \text{ odd }\}, \qquad E = \{p \mid w(p) \text{ even }\}.$$

These sets are called the *inside* and the *outside* of M.

Theorem 1. *The complement $\mathbb{R}^k \setminus M$ is the disjoint union of D and E, and these are open sets. The set D is bounded, E is unbounded, and $\bar{D} = D \cup M$ is a smooth manifold with boundary M. Moreover, if M is connected, then D and E are connected.*

In the Jordan curve theorem $k = 2$, and M is the image of a simple closed smooth curve. The theorem asserts exactly that the complement of M can be separated in two components, an 'inside' and an 'outside'. In fact, the theorem is valid for continuous curves, but the proof of that is outside the scope of differential geometry.

Proof. It is clear that $\mathbb{R}^k \setminus M = D \cup E$ is a disjoint union.

First we explain the definition of D and E. Let $y \in S^{k-1}$, then $(g^p)^{-1}(y)$ is the set of points $x \in M$ which lie in direction y from p, that is, which belong to the ray $p + \mathbb{R}_+ y$ out of p. Thus by definition p belongs to D when (for a regular value y) this ray crosses M an odd number of times, and it belongs to E when the ray crosses M an even number of times. The choice of the direction y does not matter, as long as it is a regular value.

We can now prove that E is unbounded and D is bounded. Let R be a sufficiently large number such that M is contained in the ball of radius R, centered at 0. For p outside this ball, let $y = p/\|p\|$, then the ray $p + \mathbb{R}_+ y$ does not intersect M at all, hence y is regular, $w(p) = 0$, and p belongs to E. Thus E contains all points outside the ball, hence it is unbounded, and D is contained in the ball, hence it is bounded.

The following lemma explains which values $y \in S^{k-1}$ are regular for g^p.

Lemma 1. *Let $y \in S^{k-1}$. Then y is a critical value for g^p if and only if $y \in T_x M$ for some $x \in (g^p)^{-1}(y)$.*

Proof. That y is a regular value means that $d(g^p)_x \colon T_x M \to T_y S^{k-1}$ is surjective for all $x \in (g^p)^{-1}(y)$. Notice that the tangent space $T_y S^{k-1}$ is the $k-1$-dimensional subspace $y^\perp \subset \mathbb{R}^k$ of vectors orthogonal to y, and that $T_x M$ is also

$k-1$-dimensional. Hence by linear algebra $d(g^p)_x$ is surjective if and only if it is injective.

The differential $d(g^p)_x$ is the restriction to T_xM of the differential $d(G^p)_x\colon \mathbb{R}^k \to \mathbb{R}^k$ of the smooth extension. The latter differential can be computed explicitly, it is a non-zero constant times the projection onto $y^\perp$ along y. It follows that $d(g^p)_x(v) = 0$ if and only if v is proportional to y, so the kernel of $d(g^p)_x$ is the intersection of T_xM with the one-dimensional subspace $\mathbb{R}y$. It follows that $d(g^p)_x$ is injective if and only if $y \notin T_xM$. $\square$

Lemma 2. *Let $p, q \in \mathbb{R}^k \setminus M$. If p and q can be joined by a smooth curve in $\mathbb{R}^k \setminus M$, then $w(p) = w(q)$.*

Proof. Let $\gamma\colon [0,1] \to \mathbb{R}^k \setminus M$ be such a curve with $\gamma(0) = p$, $\gamma(1) = q$. then

$$F(x,t) = \frac{x - \gamma(t)}{\|x - \gamma(t)\|}$$

is a smooth homotopy from g^p to g^q. $\square$

It follows from this lemma that D and E are open sets, because if for example $p \in D$ and ϵ is sufficiently small so that the ball around p of radius ϵ does not meet M, then all points in this ball can be joined to p and hence also belong to D.

We will prove that the closure of D is $\bar{D} = D \cup M$, and that this is a smooth manifold with boundary $\partial\bar{D} = M$. Since D and E are open, all we have to prove is that every point of M belongs to the closure of D and is covered by a chart of the form required for a boundary point of a manifold with boundary.

Let $x \in M$. It follows from Theorem 1 in Note 5 (with $M = \mathbb{R}^k$ and N equal to the present M) that there exists $\epsilon > 0$ and a diffeomorphism σ of $U =]-\epsilon, \epsilon[^k$ onto an open neighborhood W of x in $\mathbb{R}^k$, such that the restriction of σ to the set $]-\epsilon, \epsilon[^{k-1} = \{u \in U \mid u_m = 0\}$ is a chart on M with image $W \cap M$.

Let $W^+ = \sigma(\{u \in U \mid u_m > 0\})$ and $W^- = \sigma(\{u \in U \mid u_m < 0\})$. It follows from Lemma 2 that $p \mapsto w(p)$ is constant on each of these two sets. We will show that w takes different values on W^+ and W^-. Choose a ball around x inside W, and choose p, q from this ball such that $p \in W^+$ and $q \in W^-$, and such that the direction y from p to q is a regular value for g^p (this is possible, according to Sard's theorem). The line from p to q intersects M at least once, and by choosing p and q sufficiently close to an intersection point we can arrange that there is exactly one such intersection between p and q. It follows that the set $(g^p)^{-1}(y)$ has exactly one element more than $(g^q)^{-1}(y)$, and hence $w(p) \neq w(q)$. We have obtained the desired conclusion, that w takes different values on W^+ and W^-. Thus one of these sets belongs to D and the other to E. Changing the sign on the last coordinate in σ, if necessary, we can arrange that $W^+ \subset D$ and $W^- \subset E$. It follows that σ satisfies the conditions required of a chart on a manifold with x in the boundary. Thus $D \cup M$ is a manifold with boundary M.

Assume finally that M is connected. We will prove that then D and E are connected as well. Assume $D = D_1 \cup D_2$ is a disjoint union, where D_1 and D_2 are open. For each $x \in M$ we choose a chart σ as above with image W such that

$W^+ \subset D$ and $W^- \subset E$. Being the image of a connected set in $\mathbb{R}^m$, the set W^+ is connected, and hence either $W^+ \subset D_1$ or $W^+ \subset D_2$. It follows that x, together with a full neighborhood in M, belongs to the closure of either D_1 or D_2, but not both. Therefore $M = (\bar{D}_1 \cap M) \cup (\bar{D}_2 \cap M)$ is a disjoint splitting of M in open sets. Since M is connected, one of these sets is empty, but this implies that either D_1 or D_2 has empty boundary. Hence D_1 or D_2 is empty, and D is connected. The proof that E is connected is analogous. $\square$