Chapter 4

Hilbert Spaces

4.1 Inner product spaces

In the following we will discuss both complex and real vector spaces. With $\mathbb{L}$ denoting either $\mathbb{R}$ or $\mathbb{C}$ we recall that a vector space over $\mathbb{L}$ is a set E equipped with addition, i.e., a map $(x, y) \to x + y$ from $E \times E$ to E and multiplication by scalars in $\mathbb{L}$, i.e., a map $(\lambda, x) \to \lambda x$ from $\mathbb{L} \times E$ to E satisfying the well known vector space axioms.

Examples of real vector spaces are known from linear algebra they include $\mathbb{R}^k$ and $\mathcal{F}(M, \mathbb{R})$ of real functions defined on a set M with pointwise addition and multiplication by real numbers. Likewise we have the complex vector spaces $\mathbb{C}^k$ and $\mathcal{F}(M, \mathbb{C})$ of complex valued functions on a set M with pointwise addition and multiplication by complex numbers. If M is a subset of $\mathbb{R}^k$ then the set $C(M, \mathbb{C})$ of continuous functions is a subspace of $\mathcal{F}(M, \mathbb{C})$, since addition of continuous functions results in a continuous function and multiplication of a continuous function by a number is also a continuous function. We will encounter many other interesting examples of subspaces of $\mathcal{F}(M, \mathbb{C})$.

We turn to the definition of inner product spaces.

Definition 4.1 (Inner product spaces). Let E be a vector space over $\mathbb{L}$ ($= \mathbb{R}$ or $\mathbb{C}$). An *inner product* (also called (*scalar product*)) on E is a map $(\cdot \, , \, \cdot) : E \times E \to \mathbb{L}$, satisfying the following conditions (where $\underline{0}$ denotes the null-vector in E):

  i) $\forall x \in E \setminus \{\underline{0}\} : (x, x) > 0$,

 ii) $\forall x, y \in E : (x, y) = \overline{(y, x)}$,

iii) $\forall x, y, z \in E : (x + y, z) = (x, z) + (y, z)$,

 iv) $\forall \lambda \in \mathbb{L} \; \forall x, y \in E : (\lambda x, y) = \lambda(x, y)$.

If $(\cdot \, , \, \cdot)$ is an inner product on E we call the pair $(E, (\cdot \, , \, \cdot))$ an *inner product space*.

Note that $(x, x) > 0$ in i) means that (x, x) is a real positive number. In the case $\mathbb{L} = \mathbb{R}$, complex conjugation in ii) is of course superfluous.

The last two conditions above express that the map $x \to (x, y)$ from E into $\mathbb{L}$ is linear for each fixed $y \in E$. If we combine this with ii) we see that

$$
\begin{aligned}
(x, y + z) &= (x, y) + (x, z) , & (4.1) \\
(x, \lambda y) &= \overline{\lambda}(x, y) , & (4.2)
\end{aligned}
$$

for all $x, y, z \in E$ and $\lambda \in \mathbb{L}$. We say that the inner product is *conjugate linear* in the second variable. In the case $\mathbb{L} = \mathbb{C}$ a map from $E \times E \to \mathbb{C}$, which is linear in the first variable and conjugate linear in the second variable is often said to be a *sesqui-linear form* on E. Thus we have seen that an inner product on a complex vector space E is a sesqui-linear form on E. Conversely, any sesqui-linear form, satisfying the property i), which is referred to as saying that the form is *positive definite*, is an inner product. To see this we only have to show that ii) is satisfied. Note first that by the linearity in the first variable we have

$$(\underline{0}, x) = 0 \quad \text{for all} \quad x \in E .$$

In particular, we obtain $(\underline{0}, \underline{0}) = 0$, and thus from i) we find

$$\forall x \in E : (x, x) = 0 \Leftrightarrow x = \underline{0} . \tag{4.3}$$

Together with i)this shows that $(x, x) \in \mathbb{R}$ for all $x \in E$. Applying

$$(x + y, x + y) = (x, x) + (y, y) + (x, y) + (y, x)$$

we obtain that $(x, y) + (y, x) \in \mathbb{R}$, i.e., $\text{Im}(x, y) = -\text{Im}(y, x)$ for all $x, y \in E$. Replacing x by ix and using the sesqui-linearity we find $i(x, y) - i(y, x) \in \mathbb{R}$, i.e., $\text{Re}(x, y) = \text{Re}(y, x)$, which is what we wanted to show.

Given an inner product we define the *norm* $\|x\|$ of $x \in E$ by

$$\|x\| = \sqrt{(x, x)} .$$

From i) above it follows that $\|x\| > 0$ for $x \neq 0$, and from iv) and (4.2) it follows that

$$\|\lambda x\|^2 = (\lambda x, \lambda x) = \lambda \overline{\lambda}(x, x) = |\lambda|^2 \|x\|^2 .$$

i.e.,

$$\|\lambda x\| = |\lambda| \|x\| . \tag{4.4}$$

Below, as a consequence of the Cauchy-Schwarz' inequality in Theorem 4.10 we will show the *triangle inequality*

$$\|x + y\| \leq \|x\| + \|y\| , \quad x, y \in E . \tag{4.5}$$

Positivity and the two previous relations are the characteristic properties of the norm. The norm defines a natural generalization of the Euclidean distance in $\mathbb{R}^3$ (see (1.1) in Chapter 1) by

$$d(x, y) = \|x - y\| . \tag{4.6}$$

Of special importance in what follows is that the norm defines a notion of convergence of sequences in E:

Definition 4.2 (Convergence of sequences)**.** Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in an inner product space E, i.e., a map $n \to x_n$ from $\mathbb{N}$ into E. We say that the sequence converges to $x \in E$, or has the limit x, and write $x_n \to x$ for $n \to \infty$ or $\lim_{n \to \infty} x_n = x$, if

$$\|x_n - x\| \to 0 \quad \text{for } n \to \infty.$$

Finally, we shall need the important notion of a Cauchy sequence.

Definition 4.3 (Cauchy sequence)**.** We say that the sequence $(x_n)_{n \in \mathbb{N}}$ in E is a *Cauchy sequence*, if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \varepsilon \quad \text{for all } n, m > N,$$

(which is sometimes written as $\|x_n - x_m\| \to 0$ as $n, m \to \infty$).

Proposition 4.4. *Every convergent sequence in an inner product space E is a Cauchy sequence*

Proof. Exercise 4.1 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

If the converse holds we say that the space E is complete or that it is a *Hilbert space*.

Definition 4.5 (Hilbert Space)**.** *If every Cauchy sequence in and inner product space E is convergent we say that E is a complete inner product space or that it is a Hilbert space.*

It is a fundamental property of the real numbers (equivalent to the supremum property, that every non-empty set of real numbers with an upper bound has a supremum, i.e., a least upper bound) that $\mathbb{R}$ is complete wrt. the usual inner product, i.e., the usual product. It is a simple consequence of this that $\mathbb{R}^k$ as well as $\mathbb{C}^k$ are complete for all k wrt. the usual inner products defined in the next example (see Exercise 4.2).

Example 4.6.   1. The usual inner product on $\mathbb{C}^k$ (corresponding to the usual inner product on $\mathbb{R}^k$) is defined by

$$((x_1, \ldots, x_k), (y_1, \ldots, y_k)) = x_1 \overline{y}_1 + \ldots x_k \overline{y}_k.$$

It is left as an exercise for he reader to show that the requirements i) – iv) are satisfied.

2. On the vector space $C([a, b])$ of continuous complex functions on the interval $[a, b]$ we define an inner product by

$$(f, g) = \int_a^b f(x) \overline{g(x)} dx \qquad\qquad\qquad (4.7)$$

for $f, g \in C([a, b])$. In general, for any *positive* continuous function $\rho : [a, b] \to \mathbb{R}_+ =]0, +\infty[$ we get an inner product $(\cdot\,,\,\cdot)_\rho$ by

$$(f, g)_\rho = \int_a^b f(x) \overline{g(x)} \rho(x) dx. \qquad\qquad\qquad (4.8)$$

We remind the reader that the integral of a continuous complex function is obtained by integrating the real an imaginary part, i.e., if $f = \text{Re} f + i \text{Im} f$ then

$$\int_a^b f(x)dx = \int_a^b \text{Re} f(x)dx + i \int_a^b \text{Im} f(x)dx \ . \tag{4.9}$$

Then $f \to \int_a^b f(x)dx$ defines a linear map from $C([a,b])$ into $\mathbb{C}$, which immediately implies that $(\cdot \ , \ \cdot)_\rho$ satisfies iii) and iv) in Definition 4.1. Property ii) follows from $\int_a^b \overline{f(x)}dx = \overline{\int_a^b f(x)dx}$, and i) is a consequence of the fact that if f is continuous and $f \geq 0$ then $\int_a^b f(x)dx = 0$ if and only if $f = 0$.

Note that if $\underline{\rho}$ and $\overline{\rho}$ denote the max and the min of the positive convex function ρ on $[a,b]$ such that $0 < \underline{\rho} \leq \rho \leq \overline{\rho} < +\infty$, then

$$\underline{\rho}\|f\| \leq \|f\|_\rho \leq \overline{\rho}\|f\|$$

for $f \in C([a,b])$, where $\| \cdot \|$ is the norm given by the inner product (4.7)and $\| \cdot \|_\rho$ is the norm given by (4.8).

3. Let $\ell_0(\mathbb{N})$ denote the set of complex sequences $(x_n)_{n\in\mathbb{N}}$ that vanish from a certain point, i.e., for which only a finite number of elements are different from 0. The set $\ell_0(\mathbb{N})$ is then a subspace of the vector space of all complex sequences, which with our previous notation is the space $\mathcal{F}(\mathbb{N},\mathbb{C})$. On $\ell_0(\mathbb{N})$ we define an inner product by

$$((x_n),(y_n)) = \sum_{n=1}^{\infty} x_n \overline{y}_n \ ,$$

where the sum on the right has only finitely many terms different from 0 (and thus obviously is convergent). That this defines an inner product on $\ell_0(\mathbb{N})$ is seen as in the case of $\mathbb{C}^k$.

4.2   Orthogonality

Let now $(E,(\cdot \ , \ \cdot))$ be an inner product space. We say that two vectors $x, y \in E$ are *orthogonal* and write $x \perp y$ if $(x,y) = 0$. We generalize this notion and say that a vector $x \in E$ is orthogonal to a subset $A \subseteq E$, and write $x \perp A$, if x is orthogonal to all vectors in A.

The *orthogonal complement* $A^\perp$ of A is defined to be the set of all vectors orthogonal to A, i.e.,

$$A^\perp = \{x \in E \mid (x,y) = 0 \ \text{ for all } \ y \in A\} \ . \tag{4.10}$$

We note that by iii) and iv) in Definition 4.1 $A^\perp$ is a subspace of E for any subset $A \subseteq E$. For the same reason

$$A^\perp = (\text{span}A)^\perp \ . \tag{4.11}$$

Here span$A$ is the subspace of E spanned by all elements in A, i.e., the space consisting of all linear combinations of vectors from A.

A family $(x_i)_{i \in I}$ of vectors from E, where I is any index set is said to be an *orthogonal family* if $(x_i, x_j) = 0$, whenever $i \neq j$, i.e., if the vectors in the set are pairwise orthogonal. If moreover $\|x_i\| = 1$ for all $i \in I$ we say that the family is *orthonormal*.

A family $(x_i)_{i \in I}$ of vectors from E is said to be linearly independent if any finite subfamily of $(x_i)_{i \in I}$ is linearly independent.

Lemma 4.7. *Let $(x_i)_{i \in I}$ be an orthogonal family in E, such that $x_i \neq 0$ for all $i \in I$. Then $(x_i)_{i \in I}$ is a linearly independent family.*

Proof. Let $(x_{i_1}, \ldots, x_{i_n})$ be a finite subfamily of $(x_i)_{i \in I}$ and assume that the scalars $\lambda_1, \ldots, \lambda_n$ satisfy $\lambda_1 x_{i_1} + \ldots + \lambda_n x_{i_n} = 0$. If we for all $j \in \{1, \ldots, n\}$ calculate the inner product with x_{i_j} on both sides of this equation we arrive at

$$\lambda_1 (x_{i_1}, x_{i_j}) + \ldots + \lambda_n (x_{i_n}, x_{i_j}) = 0 \ .$$

By the assumption that $(x_{i_k}, x_{i_j}) = 0$ for $k \neq j$ we arrive at $\lambda_j (x_{i_j}, x_{i_j}) = 0$. Hence $\lambda_j = 0$ because $(x_{i_j}, x_{i_j}) \neq 0$, since $x_{i_j} \neq 0$. $\qquad \square$

We also have the following generalization of the Pythagorean Theorem.

Theorem 4.8 (Pythagoras)**.** *Let $(x_1, \ldots, x_n)$ be a finite orthogonal family. Then*

$$\| \sum_{i=1}^n x_i \|^2 = \sum_{i=1}^n \|x_i\|^2 \ .$$

Proof. We have

$$\| \sum_{i=1}^n x_i \|^2 = \left(\sum_{i=1}^n x_i \ , \ \sum_{j=1}^n x_j \right) = \sum_{i,j=1}^n (x_i, x_j)$$

$$= \sum_{i=1}^n (x_i, x_i) = \sum_{i=1}^n \|x_i\|^2 \ ,$$

where the third equality used that only the diagonal terms corresponding to $i = j$ contribute to the sum by the orthogonality assumption. $\qquad \square$

The next theorem is also well known from elementary linear algebra.

Theorem 4.9 (Orthogonal projections and Bessel's inequality)**.** *Let $(e_1, \ldots, e_n)$ be a finite orthonormal family in E. For all vectors $x \in E$ there is a unique vector $u \in \mathrm{span}\{e_1, \ldots, e_n\}$ such that*

$$x - u \in \{e_1, \ldots, e_n\}^\perp.$$

It is given by

$$u = \sum_{i=1}^n (x, e_i) e_i \ . \tag{4.12}$$

and may be characterized as the unique vector in $\mathrm{span}\{e_1, \ldots, e_n\}$ with the shortest distance to x wrt. the norm $\| \cdot \|$. The vector u given in (4.12) is called the orthogonal projection of x on the subspace $\mathrm{span}\{e_1, \ldots, e_n\}$.

Moreover, we have **Bessel's inequality**

$$\sum_{i=1}^{n} |(x, e_i)|^2 \le \|x\|^2 \tag{4.13}$$

for all $x \in E$.

Proof. All vectors $u \in \operatorname{span}\{e_1, \ldots, e_n\}$ can be written in the form $u = \lambda_1 e_1 + \ldots + \lambda_n e_n$, where $\lambda_1, \ldots, \lambda_n \in \mathbb{L}$. Calculating the inner product with e_i on both sides of this equation we arrive at $(u, e_i) = \lambda_i (e_i, e_i) = \lambda_i$ since $(e_i, e_i) = 1$. Hence

$$u = \sum_{i=1}^{n} (u, e_i) e_i \tag{4.14}$$

for $u \in \operatorname{span}\{e_1, \ldots, e_n\}$. But $x - u \in \{e_1, \ldots, e_n\}^{\perp}$ is equivalent to $(x - u, e_i) = 0$ for all $i = 1, \ldots, n$. Thus $(x, e_i) = (u, e_i)$ for $i = 1, \ldots, n$. Together with (4.14) this gives the first part of the theorem.

If $v \in \operatorname{span}\{e_1, \ldots, e_n\}$ then $x - v = (x - u) + (u - v)$ where $(x - u) \perp (u - v)$. Thus

$$\|x - v\|^2 = \|x - u\|^2 + \|u - v\|^2 \ge \|x - u\|^2 \, ,$$

with equality if and only if $u = v$.

Bessel's inequality follow from Theorem 4.8 if we note that $x = u + (x - u)$ where $u \perp (x - u)$, since then

$$\|x\|^2 = \|u\|^2 + \|x - u\|^2 \ge \|u\|^2 = \sum_{i=1}^{n} |(x, e_i)|^2 \, .$$

In the last step we have used that $\|(x, e_i) e_i\|^2 = |(x, e_i)|^2$. $\qquad\qquad \square$

From Bessel's inequality we shall now derive the *Cauchy-Schwarz' inequality*

Theorem 4.10 (Cauchy-Schwarz' inequality)**.** *If E is an inner product space then for all $x, y \in E$ we have*

$$|(x, y)| \le \|x\| \, \|y\|. \tag{4.15}$$

Proof. In fact, if $y = \underline{0}$ then both sides of the inequality are zero and if $y \ne \underline{0}$ then $\|\frac{1}{\|y\|} y\| = 1$ and it follows from (4.13) with $n = 1$ and $e_1 = \frac{1}{\|y\|} y$ that

$$\left| \left(x, \frac{1}{\|y\|} y \right) \right| \le \|x\| \, ,$$

from which (4.15) follows after multiplication by $\|y\|$ on both sides. $\qquad\qquad \square$

From the Cauchy-Schwarz' inequality we derive the triangle inequality (4.5)

$$
\begin{aligned}
\|x + y\|^2 &= (x + y, x + y) = \|x\|^2 + (x, y) + (y, x) + \|y\|^2 \\
&= \|x\|^2 + 2\operatorname{Re}(x, y) + \|y\|^2 \\
&\le \|x\|^2 + 2|(x, y)| + \|y\|^2 \\
&\le \|x\|^2 + 2\|x\| \, \|y\| + \|y\|^2 \\
&= (\|x\| + \|y\|)^2 \, .
\end{aligned}
$$

4.3 Continuity of the inner product

In the following we will repeatedly make use of the fact that *the inner product* $(\cdot\,,\,\cdot)$: $E \times E \to \mathbb{L}$ *is continuous.* This is seen as follows. Let $x_0, y_0 \in E$ be given and choose $x, y \in E$ such that $\|x - x_0\| \le \delta$ and $\|y - y_0\| \le \delta$ where $\delta > 0$ is given. Then

$$
\begin{aligned}
|(x, y) - (x_0, y_0)| &= |(x, y - y_0) + (x - x_0, y_0)| \\
&\le |(x, y - y_0)| + |(x - x_0, y_0)| \\
&\le \|x\| \, \|y - y_0\| + \|x - x_0\| \, \|y_0\| \\
&\le \delta(\|x\| + \|y_0\|) \\
&\le \delta(\|x_0\| + \delta + \|y_0\|) \, .
\end{aligned}
\tag{4.16}
$$

Here we have used the Cauchy-Schwarz' inequality and that $\|x\| = \|(x - x_0) + x_0\| \le \|x - x_0\| + \|x_0\| \le \|x_0\| + \delta$.

Since the last expression in (4.16) tends to 0 as $\delta \to 0$ we conclude that for all $\varepsilon > 0$ there is a $\delta > 0$ such that $|(x, y) - (x_0, y_0)| < \varepsilon$ if $\|x - x_0\| \le \delta$ and $\|y - y_0\| \le \delta$. This is exactly the continuity of the inner product.

The continuity may equivalently be stated as saying that for all sequences (x_n) and (y_n) in E we have that

$$
(x_n, y_n) \to (x_0, y_0) \quad \text{as} \quad n \to \infty \, ,
\tag{4.17}
$$

if $x_n \to x_0$ and $y_n \to y_0$ as $n \to \infty$.

Definition 4.11 (Convergence of series). A series $\sum\limits_{n=1}^{\infty} x_n$ with terms in a normed vector space E is said to be convergent with sum $x = \sum\limits_{n=1}^{\infty} x_n$ in E if the sequence of partial sums $(s_k)_{k \in \mathbb{N}}$ defined by

$$
s_k = \sum_{n=1}^{k} x_n
\tag{4.18}
$$

converges to x as $k \to \infty$.

In Exercise 4.19 an alternative definition of Hilbert space is given using the concept of absolute convergence of a series.

From (4.17) and iii) in Definition 4.1 it follows that

$$
\left(\sum_{n=1}^{\infty} x_n, y \right) = \lim_{k \to \infty} \left(\sum_{n=1}^{k} x_n, y \right) = \lim_{k \to \infty} \sum_{n=1}^{k} (x_n, y) = \sum_{n=1}^{\infty} (x_n, y)
\tag{4.19}
$$

and likewise that

$$
\left(y, \sum_{n=1}^{\infty} x_n \right) = \sum_{n=1}^{\infty} (y, x_n)
\tag{4.20}
$$

for any convergent series $\sum\limits_{n=1}^{\infty} x_n$ in E and all $y \in E$.

An other consequence of the continuity of the inner product is that $A^\perp = (\overline{A})^\perp$ for any subset $A \subseteq E$. Here $\overline{A}$ denotes the closure of the set A, i.e., the set of all limit points of

sequences from A. In fact, if $x \in A^\perp$ and $y \in \overline{A}$ there is a sequence (y_n) in A such that $y = \lim y_n$. Thus we conclude that $(x, y) = \lim(x, y_n) = 0$, i.e., $x \perp y$. Since $y \in \overline{A}$ was arbitrary we see that $A^\perp \subseteq (\overline{A})^\perp$. The opposite inclusion follows immediately from $A \subseteq \overline{A}$. Together with (4.11) this implies that

$$A^\perp = (\mathrm{span} A)^\perp = (\overline{\mathrm{span} A})^\perp \ . \tag{4.21}$$

We call $\overline{\mathrm{span} A}$ the *closed subspace spanned* by A.

Likewise it it seen in Exercise 4.5 that $A^\perp$ *is a closed subspace of E* (i.e., $\overline{A^\perp} = A^\perp$ *for any subset $A \subseteq E$*).

4.4   Hilbert spaces

It is well known from linear algebra that any finite dimensional inner product space E has orthonormal bases. If we let $(e_1, \ldots, e_n)$ denote such a basis. If $x \in E$ we denote by $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{L}^n$ the coordinates of x wrt. this basis, i.e.,

$$x = x_1 e_1 + \ldots + x_n e_n \ ,$$

According to Pythagoras Theorem 4.8 we have

$$\|x\| = (|x_1|^2 + \ldots + |x_n|^2)^{1/2} \ .$$

It follows that the map $(x_1, \ldots, x_n) \mapsto x_1 e_1 + \ldots + x_n e_n$ is a linear isometry from $\mathbb{L}^n$ onto E and the inverse map is $x \mapsto ((x, e_1), \ldots, (x, e_n))$. Since the vector space $\mathbb{L}^n$ is known to be complete wrt. the usual norm we conclude from that E is also complete, i.e., a Hilbert space.

For general Hilbert spaces we will see that the requirement of completeness will ensure that several properties of finite dimensional spaces will generalize to infinite dimensional spaces.

We will first consider important examples of Hilbert spaces.

Example 4.12.   1. We have seen above that any finite dimensional inner product space is a Hilbert space. This is in particular true for $\mathbb{R}^k$ and $\mathbb{C}^k$ (see Exercise 4.2).

2. The space $\ell_0(\mathbb{N})$ (see Example 4.6) is not complete. In fact, let $x_n \in \ell_0(\mathbb{N})$ be given by

$$x_n = (1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots, \tfrac{1}{n}, 0, 0, \ldots) .$$

Then for $m \leq n$ we see that

$$\|x_n - x_m\|^2 = \sum_{k=m+1}^{n} \frac{1}{k^2}$$

and hence the sequence $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\ell_0(\mathbb{N})$ since the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent. The sequence $(x_n)_{n \in \mathbb{N}}$ is however obviously not convergent in $\ell_0(\mathbb{N})$. Thus $\ell_0(\mathbb{N})$ cannot be a Hilbert space.

If we instead consider the larger subspace $\ell^2(\mathbb{N})$ of $\mathcal{F}(\mathbb{N}, \mathbb{C})$ consisting of sequences of complex numbers $(a_n)_{n \in \mathbb{N}}$ that are square summable, i.e,

$$\sum_{n=1}^{\infty} |a_n|^2 < +\infty \,.$$

That $\ell^2(\mathbb{N})$ is a subspace of $\mathcal{F}(\mathbb{N}, \mathbb{C})$ follows from the well-known inequality $|a+b|^2 \leq 2(|a|^2 + |b|^2)$ for complex number a, b (this inequality is actually the Cauchy-Schwarz' inequality for the vectors $(1,1)$ and (a,b)). For two sequences (a_n) and (b_n) in $\ell^2(\mathbb{N})$ we conclude that

$$\sum_{n=1}^{\infty} |a_n + b_n|^2 \leq 2 \sum_{n=1}^{\infty} |a_n|^2 + 2 \sum_{n=1}^{\infty} |b_n|^2 < +\infty \,.$$

Thus $(a_n) + (b_n) \in \ell^2(\mathbb{N})$. Since we also have $\lambda(a_n) = (\lambda a_n) \in \ell^2(\mathbb{N})$ if $\lambda \in \mathbb{C}$ and $(a_n) \in \ell^2(\mathbb{N})$ we have shown that $\ell^2(\mathbb{N})$ is a subspace of $\mathcal{F}(\mathbb{N}, \mathbb{C})$.

Using the inequality $|ab| \leq \frac{1}{2}(|a|^2 + |b|^2)$ for $a, b \in \mathbb{C}$ we see that

$$((a_n),(b_n)) = \sum_{n=1}^{\infty} a_n \bar{b}_n$$

defines a map $(\cdot \, , \, \cdot) : \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \to \mathbb{C}$. In fact, the series above is absolutely convergent (see the definition of absolute convergence in Exercise 4.19). It is now clear that the above map defines an inner product on $\ell^2(\mathbb{N})$.

We will now show that $\ell^2(\mathbb{N})$ with this inner product is a Hilbert space. Thus assume that $(x^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\ell^2(\mathbb{N})$. We write $x^n = (a_1^n, a_2^n, \dots)$.

For every $k \in \mathbb{N}$ we have that $|a_k^n - a_k^m| \leq \|x^n - x^m\|$. Thus the sequence $(a_k^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for each $k \in \mathbb{N}$. Since $\mathbb{C}$ is complete we conclude that this sequence converges, i.e., there is a_k such that $a_k = \lim_{n \to \infty} a_k^n$ for all $k \in \mathbb{N}$. Let $x = (a_1, a_2, \dots)$. We will now show that $x \in \ell^2(\mathbb{N})$ and that $x^n \to x$ as $n \to \infty$.

Given $\varepsilon > 0$ we can find an $N \in \mathbb{N}$, such that

$$\sum_{k=1}^{K} |a_k^n - a_k^m|^2 \leq \sum_{k=1}^{\infty} |a_k^n - a_k^m|^2 = \|x^n - x^m\|^2 \leq \varepsilon^2$$

for all $n, m \geq N$ and all $K \in \mathbb{N}$. For $m \to \infty$ we see from this that $\sum_{k=1}^{K} |a_k^n - a_k|^2 \leq \varepsilon^2$ for $n \geq N$ and all $K \in \mathbb{N}$. For $K \to \infty$ we get

$$\|x^n - x\|^2 = \sum_{k=1}^{\infty} |a_k^n - a_k|^2 \leq \varepsilon^2$$

for $n \geq N$. This shows that $x^N - x \in \ell^2(\mathbb{N})$ and hence that $x = x^N - (x^N - x) \in \ell^2(\mathbb{N})$ and also that $x^n \to x$ as $n \to \infty$ as was claimed.

3. The space $C([a, b])$ with inner product given by (4.8) is not complete as we shall now discuss. The norm on this space is given by

$$\|f\|_\rho^2 = \int_a^b |f(x)|^2 \rho(x) dx \ .$$

Let f_n denote the function on $[0, 2]$ which is equal to 0 on $[0, 1]$, grows linearly from 0 to 1 on $[1, 1 + \frac{1}{n}]$ and is equal to 1 on $[1 + \frac{1}{n}, 2]$ (draw the graph!),

It is easy to see that (f_n) is a Cauchy sequence in $C([0, 2])$ w.r.t. the norm $\| \cdot \|_\rho$, but that it is not convergent in $C([0, 2])$. As in the discussion of ℓ_0 above we can extend $C([a, b])$ to a Hilbert space $L^2([a, b])$ consisting of square integrable functions on $[a, b]$, i.e., functions $f : [a, b] \to \mathbb{C}$, such that

$$\int_a^b |f(x)|^2 \, dx < \infty \ .$$

We here refer to a generalization of the Riemann integral, called the Lebesgue integral. We also generalize what is meant by functions or rather what is meant by two functions being equal. The details of this is beyond the scope of these notes. We shall only need the following three facts:

i) Two functions f and g in $L^2([a, b])$ are considered to be equal if

$$\int_a^b |f(x) - g(x)|^2 \, dx = 0 \quad \text{i.e., if} \quad \|f - g\| = 0 \, ,$$

and we say that f is equal to g *almost everywhere* on $[a, b]$, or that $f(x) = g(x)$ for *almost all* $x \in [a, b]$. This is, in particular, true for any two functions that differ at only finitely many points of the interval. This strictly speaking means that $L^2([a, b])$ is not a space of functions, but of classes of functions that agree almost everywhere. The proof that $L^2([a, b])$ is a vector space with inner product given by (4.7) can now be completed as for $\ell^2(\mathbb{N})$ above. That $L^2([a, b])$ is a Hilbert space is one of the most fundamental results in measure theory known as the Riesz-Fischer Theorem.

ii) $L^2([a, b])$ is a minimal extension of $C([a, b])$ in the sense that the closure $\overline{(C([a, b]))}$ in $L^2([a, b])$ is equal to all of $L^2([a, b])$. In other words to every $f \in L^2([a, b])$ there is sequence $(f_n)_{n \in \mathbb{N}}$ in $C([a, b])$, such that $\|f_n - f\| \to 0$ as $n \to \infty$.

iii) In contrast to the Riemann integral the Lebesgue integral is defined on an arbitrary interval $I \subseteq \mathbb{R}$ and for continuous positive functions it agrees with the improper Riemann integral. We may thus define the Hilbert space $L^2(I)$ for all intervals I, in particular, we may consider $L^2(\mathbb{R})$. For this space we have instead of (ii) above that the closure of the subspace $C_0(\mathbb{R})$, consisting of all continuous functions vanishing outside a bounded interval, is all of $L^2(\mathbb{R})$. In Section 5.5 we will need that the same holds for the subspace $C_0^\infty(\mathbb{R})$ consisting of C^∞-functions vanishing outside a bounded interval.

For more details on measure theory and Lebesgue integration the interested reader may consult the book M.Reed and B. Simon: Methods of modern mathematical physics, Vol I, Academic press 1972.

In the following we will let H denote a Hilbert space with inner product $(\cdot\,,\,\cdot)$. Any subspace X of H is an inner product space with the inner product defined as the restriction of $(\cdot\,,\,\cdot)$ to $X \times X$. Then *$X \subseteq H$ is a Hilbert space if and only if X is a closed subspace of H*, i.e., if $X = \overline{X}$ (see Exercise 4.6). In particular, all finite-dimensional subspaces of H are closed.

We will need the following extension of Pythagoras' Theorem.

Theorem 4.13 (Infinite Pythagorean Theorem). *Let $(x_i)_{i \in \mathbb{N}}$ be an orthogonal family in a Hilbert space H. Then $\sum\limits_{i=1}^{\infty} x_i$ is convergent in H if and only if*

$$\sum_{i=1}^{\infty} \|x_i\|^2 < +\infty \,,$$

and

$$\| \sum_{i=1}^{\infty} x_i \|^2 = \sum_{i=1}^{\infty} \|x_i\|^2 \,. \tag{4.22}$$

Proof. Since H is a Hilbert space $\sum\limits_{i=1}^{\infty} x_i$ is convergent in H if and only if the sequence of partial sums (s_n) is a Cauchy sequence. This means that for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$\|s_n - s_m\|^2 = \| \sum_{i=m+1}^{n} x_i \|^2 = \sum_{i=m+1}^{n} \|x_i\|^2 \leq \varepsilon^2 \tag{4.23}$$

for all $n > m \geq N$ where we have used Theorem 4.8. Since $\mathbb{R}$ is also complete we have on the other hand that $\sum_{i=1}^{\infty} \|x_i\|^2$ is convergent if and only if the sequence (r_n) given by

$$r_n = \sum_{i=1}^{n} \|x_i\|^2$$

is a Cauchy sequence in $\mathbb{R}$. This means that for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$|r_n - r_m| = \sum_{i=m+1}^{n} \|x_i\|^2 < \varepsilon^2 \tag{4.24}$$

for all $n > m \geq N$. The first claim in the theorem follows by comparing (4.23) and (4.24). Finally, we see from the continuity of $x \mapsto \|x\|$ and Theorem 4.8 that

$$\| \sum_{i=1}^{\infty} x_i \|^2 = \| \lim_{n \to \infty} \sum_{i=1}^{n} x_i \|^2 = \lim_{n \to \infty} \| \sum_{i=1}^{n} x_i \|^2 = \lim_{n \to \infty} \sum_{i=1}^{n} \|x_i\|^2 = \sum_{i=1}^{\infty} \|x_i\|^2 \,,$$

which implies (4.22). $\qquad\square$

We want to extend the notion of orthonormal basis to the infinite dimensional case. We may characterize an orthonormal basis in a finite dimensional inner product space H as an orthonormal family that spans H. In the infinite dimensional case we will only require that an orthonormal basis spans a *dense subspace*, i.e., a set whose closure is the whole space. Since as we just saw any finite dimensional subspace in an inner product space is closed the following notion of orthonormal basis space is, indeed, an extention of the finite dimensional notion.

Definition 4.14 (Orthonormal basis)**.** An orthonormal basis for A Hilbert space H is an orthonormal family $(e_i)_{i \in I}$ in H such that $\overline{span\{e_i \mid i \in I\}} = H$.

We emphasize that this notion of orthonormal basis is different from the purely algebraic notion of a basis, where it is required that the basis spans the whole set.

Since $H^\perp = \{\underline{0}\}$ it follows that from (4.21)that

$$\{e_i \mid i \in I\}^\perp = \{\underline{0}\} \ .$$

for any orthonormal basis $(e_i)_{i \in I}$ for H. This means any orthonormal basis is a maximal orthonormal family in H, i.e., that there is no vector $e \in H$ with $\|e\| = 1$, such that e together with $(e_i)_{i \in I}$ is an orthonormal family. The converse that any maximal orthonormal family in H is an orthonormal basis will be shown in Theorem 4.16 below.

Note that the definition of orthonormal basis implies that any orthonormal family $(e_i)_{i \in I}$ is an orthonormal basis for $\overline{span\{e_i \mid i \in I\}}$, i.e., for the closure of the subspace spanned by the family.

Any Hilbert space has an orthonormal basis. This claim however relies on the what is known as *the axiom of choice*, which we will not discuss here. In the following we shall restrict attention to *separable Hilbert spaces*, i.e., Hilbert spaces that have an orthonormal basis which is either finite or countable, i.e., can be indexed by $\mathbb{N}$.

We know from elementary linear algebra that in finite dimensional vector spaces all bases have the same number of elements equal to the dimension of the space. In Exercise 4.21 we show that in an infinite dimensional separable Hilbert space all orthonormal bases are infinite and have countably many elements.

Example 4.15. In the space $\ell^2(\mathbb{N})$ discussed in Example 4.12 we can get an orthonormal basis $(e_i)_{i \in \mathbb{N}}$ by letting e_i be the sequence with all elements equal to 0 except element number i which we choose to be 1, i.e.,

$$(e_i)_j = \begin{cases} 1 & \text{for} \quad j = i \\ 0 & \text{for} \quad j \neq i \ . \end{cases}$$

That this is an orthonormal basis follows by observing that it is obviously an orthonormal family and that $span\{e_i \mid i \in \mathbb{N}\} = \ell_0(\mathbb{N})$ whose closure is $\ell^2(\mathbb{N})$ (see Exercise 4.15) We call $(e_i)_{i \in \mathbb{N}}$ the *natural orthonormal basis* for $\ell^2(\mathbb{N})$.

In Section 4.6 we shall determine an orthonormal basis for the space $L^2([-L, L])$. It will in this case turn out to be more convenient to use all integers $\mathbb{Z}$ as an index ser for the basis. Of course if one can use $\mathbb{Z}$ one could also have used $\mathbb{N}$ (why?). In Exercise 4.18 an alternative construction of an orthonormal basis for $L^2([0, 1])$ is given

4.5 Orthonormal expansions

Our goal in this section is to generalize the expansion of vectors in orthonormal bases known in finite dimensional Hilbert spaces to the case of infinite dimensional separable Hilbert spaces. *Thus we will in this section assume that H is an infinite dimensional separable Hilbert space.* All results in this section generalize to non-separable spaces but we shall not discuss this here.

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Let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal family in H (assumed to be infinite dimensional) and consider a vector $x \in H$. From Bessel's inequality (4.13) we have that $\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2$ for all $n \in \mathbb{N}$. This implies the *generalized Bessel inequality*

$$\sum_{i=1}^{\infty} |(x, e_i)|^2 \leq \|x\|^2 . \tag{4.25}$$

By Theorem 4.13 together with (4.25) we therefore conclude that the series $\sum_{i=1}^{\infty} (x, e_i) e_i$ is convergent in H. If we write

$$u = \sum_{i=1}^{\infty} (x, e_i) e_i \tag{4.26}$$

it is clear that $u \in \overline{span\{e_i \mid i \in \mathbb{N}\}}$. From Definition 4.11 for convergence of a series the vector u could in principle depend on the order in which we perform the sum in (), i.e., on the order we chose for the basis $e_1, e_2, e_3, \ldots$. That this is not the case follows from the next theorem.

Theorem 4.16 (Projections in infinite dimensional spaces). *Let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal family in the separable Hilbert space H. To any vector $x \in H$ there is a unique vector $u \in \overline{span\{e_i \mid i \in \mathbb{N}\}}$ such that*

$$x - u \in \{e_i \mid i \in \mathbb{N}\}^{\perp} ,$$

and u is given by (4.5). In particular, u does not depend on the order of the summation and we also write

$$u = \sum_{i \in \mathbb{N}} (x, e_i) e_i .$$

Among all vectors in $\overline{span\{e_i \mid i \in \mathbb{N}\}}$ the vector u has the shortest distance to x.

Proof. The vector u given by (4.5) satisfies that for each $j \in \mathbb{N}$

$$(u, e_j) = \left(\sum_{i=1}^{\infty} (x, e_i) e_i, e_j \right) = \sum_{i=1}^{\infty} (x, e_i)(e_i, e_j) = (x, e_j).$$

Thus, $(x - u, e_j) = 0$ for all $j \in \mathbb{N}$ and hence $x - u \in \{e_j \mid j \in \mathbb{N}\}^{\perp}$. This proves the existence of u we show the uniqueness below.

If $v \in \overline{span\{e_i \mid i \in \mathbb{N}\}}$ then $u - v \in \overline{span\{e_i \mid i \in \mathbb{N}\}}$ and thus Pythagoras Theorem gives

$$\|x - v\|^2 = \|x - u\|^2 + \|u - v\|^2 \geq \|x - u\|^2.$$

This shows the last claim of the theorem. If v moreover, assumes that $x - v \in \{e_i \mid i \in \mathbb{N}\}^{\perp}$ then $u - v = (x - v) - (x - u) \in \{e_i \mid i \in \mathbb{N}\}^{\perp} = (\overline{span\{e_i \mid i \in \mathbb{N}\}})^{\perp}$. In particular, $(u - v, u - v) = 0$, i.e., $u = v$ showing that u is unique. $\qquad\square$

We now have the following important result on orthonormal bases.

Theorem 4.17 (Properties of Orthonormal Basis). *For an orthonormal family $(e_i)_{i\in\mathbb{N}}$ in a separable Hilbert space H the following statements are equivalent*

(i) $(e_i)_{i\in\mathbb{N}}$ *is an orthonormal basis in H.*

(ii) $\{e_i \mid i \in \mathbb{N}\}^\perp = \{\underline{0}\}$.

(iii) **The orthonormal expansion**

$$x = \sum_{i\in\mathbb{N}} (x, e_i) e_i$$

holds for all $x \in H$.

(iv) **Parseval's identity**

$$\|x\|^2 = \sum_{i=1}^{\infty} |(x, e_i)|^2 \tag{4.27}$$

holds for all $x \in H$.

Proof. We have previously seen that (i) $\Rightarrow$ (ii). The implication (ii) $\Rightarrow$ (iii) follows from Theorem 4.16 since with the notation used there we have $x = u$. That (iii) $\Rightarrow$ (iv) follows immediately from (4.22).

If we finally assume that (iv) holds we find for all $x \in H$

$$\|x - \sum_{i=1}^{n} (x, e_i) e_i\|^2 = \|x\|^2 - \sum_{i=1}^{n} |(x, e_i)|^2 \to 0 \quad \text{for } n \to \infty .$$

Thus $span\{e_i \mid i \in I\}$ is dense in H and hence (iv) $\Rightarrow$ (i). $\qquad\square$

As a further consequence of Theorem 4.16 we obtain the *projection theorem*.

Theorem 4.18 (Projection Theorem). *Let X be a closed subspace of a separable Hilbert space H. Then for each $x \in H$ there exist unique vectors $u \in X$ and $v \in X^\perp$ such that*

$$x = u + v . \tag{4.28}$$

Proof. Since X is a closed subspace of the separable Hilbert space H then X is a separable Hilbert space (see Exercise 4.17). We may thus choose and orthonormal basis $(e_i)_{i\in I}$ for X, where I is finite or equal to $\mathbb{N}$. Hence $X = \overline{span\{e_i \mid i \in I\}}$. The claim is now an immediate consequence of Theorems 4.9 and 4.16. $\qquad\square$

As we have seen u in (4.28) may also be characterized as the vector in X with the shortest distance to x.

In general if two subspaces V and W of a vector space E satisfy that any vector $x \in E$ can be uniquely written as $x = v + w$, where $v \in V$ and $w \in W$ we say that V and W are complementary and write $E = V \oplus W$. We also express this as saying that E is the *direct sum* of V and W.

We may thus rephrase Theorem 4.18 as

$$H = X \oplus X^\perp$$

for any closed subspace X of H. The vector u in (4.28) is called the *orthogonale projection* of x onto X and as we have seen it can be calculated as

$$u = \sum_{i \in I} (x, e_i) e_i \, ,$$

where $(e_i)_{i \in I}$ is an orthonormal basis for X.

Since $X^\perp$ is also a closed subspace of H we have $H = X^{\perp\perp} \oplus X^\perp$. If we use that any vector $x \in X^{\perp\perp}$ may be written uniquely as $x = u + v$, with $u \in X$ and $v \in X^\perp$ and the obvious inclusion $X \subseteq X^{\perp\perp}$ (why?) we see that, in fact, $u = x$ and $v = 0$. Thus

$$X = X^{\perp\perp}$$

for every closed subspace X of H. It then follows that v in (4.28) is the orthogonal projection of x onto $X^\perp$.

Proposition 4.19. *Let X be a closed subspace in a separable Hilbert space H. Assume we have orthonormal bases $(e_i)_{i \in I}$ for X and $(e_j)_{j \in J}$ for $X^\perp$. Then the combined family $(e_i)_{i \in I \cup J}$ is an orthonormal basis for H*

Proof. Exercise 4.22 □

4.6 Fourier series

The celebrated theory of Fourier series which has its origin in J. Fourier's 1807 analysis of the heat equation may be conveniently formulated in terms of orthonormal expansions. We shall now discuss this very briefly.

The relevant Hilbert space is $H = L^2([-L, L])$ with inner product

$$(f, g) = \frac{1}{2L} \int_{-L}^{L} f(\theta) \overline{g(\theta)} \, d\theta \, .$$

We have included the factor $\frac{1}{2L}$ for convenience. The key result about Fourier series is stated in the next theorem. We will give a proof based on the Stone-Weierstrass Theorem (see, e.g., W. Rudin: Functional analysis, Chapter 5).

Theorem 4.20 (Fourier basis). *Let the functions $e_n \in C([-L, L])$ be given by*

$$e_n(\theta) = e^{in\frac{\pi}{L}\theta} \, , \quad \theta \in [-L, L] \, , \, n \in \mathbb{Z} \, .$$

Then $(e_n)_{n \in \mathbb{Z}}$ is an orthonormal basis in $H = L^2([-L, L])$.

Proof. That $(e_n)_{n \in \mathbb{Z}}$ is an orthonormal family is seen from the calculation

$$
\begin{aligned}
(e_n, e_m) &= \frac{1}{2L} \int_{-L}^{L} e_n(x) \overline{e_m(x)} dx \\
&= \frac{1}{2L} \int_{-L}^{L} e^{i(n-m)\frac{\pi}{L}\theta} \, d\theta \\
&= \begin{cases} \frac{1}{2L} \left[\frac{L}{i(n-m)\pi} e^{i(n-m)\frac{\pi}{L}\theta} \right]_{-L}^{L} = 0 & \text{for} \quad n \neq m \\ \frac{1}{2L} [x]_{-L}^{L} = 1 & \text{for} \quad n = m \, , \end{cases}
\end{aligned}
$$

Here we have used that e_n is a periodic function of $\theta \in \mathbb{R}$ with period $2L$.

It remains to prove that $\overline{\text{span}\{e_n \mid n \in \mathbb{Z}\}} = H$. To conclude this we note that it is enough to show that $C([-L, L]) \subseteq \overline{\text{span}\{e_n \mid n \in \mathbb{Z}\}}$ since $\overline{C([-L, L])} = H$ as already remarked. It is thus enough to show that for all functions $f \in C(-L, L)$ and all $\varepsilon > 0$ there is an $f_1 \in \text{span}\{e_n \mid n \in \mathbb{Z}\}$ such that

$$\|f - f_1\| < \varepsilon. \tag{4.29}$$

To see this we first choose a function (why is this possible?) $f_2 \in C([-L, L])$ such that $f_2(-L) = f_2(L) = 0$ and

$$\|f - f_2\| < \frac{\varepsilon}{2}. \tag{4.30}$$

Then f_2 may be extended to a continuous periodic function on $\mathbb{R}$ with period $2L$. It the follows from the Stone-Weierstrass' Theorem that there is a function $f_1 \in \text{span}\{e_n \mid n \in \mathbb{Z}\}$, such that

$$|f_2(\theta) - f_1(\theta)| < \frac{\varepsilon}{\sqrt{4L}}, \quad \theta \in [-L, L].$$

Hence

$$\|f_2 - f_1\| < \frac{\varepsilon}{2}. \tag{4.31}$$

From (4.30), (4.31), and the triangle inequality we arrive at (4.29). $\qquad \square$

If for $f \in L^2([-L.L])$ we define *the Fourier coefficients* $c_n(f)$ by

$$c_n(f) = \frac{1}{2L} \int_{-L}^{L} f(\theta) e^{-in\frac{\pi}{L}\theta} \, d\theta, \tag{4.32}$$

the above theorem and Theorem 4.17 imply that

$$f(\theta) = \sum_{n \in \mathbb{Z}} c_n(f) e^{in\frac{\pi}{L}\theta}. \tag{4.33}$$

This series is called the *Fourier series* for f. We emphasize that it converges in sense of the Hilbert space $L^2([-L, L])$, i.e., with respect to the L^2-norm $\|\cdot\|$. More precisely, this means that

$$\| \sum_{n=-N}^{N} c_n(f) e_n - f \| \to 0 \text{ for } N \to \infty.$$

It is important to realize that this is not the same as uniform or pointwise convergence. For a discussion on uniform and pointwise convergence of Fourier series we refer the reader to more advanced texts on the subject.

Exercises

Opgave 4.1. Proof Proposition 4.4. *Hint:* Use the triangle inequality (4.5).

Opgave 4.2. (a) Show that a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^k$ (or $\mathbb{C}^k$) is convergent or a Cauchy sequence wrt. the standard norm if and only if each coordinate sequence

$$(x_n^i)_{n\in\mathbb{N}}, \ i = 1, \ldots, k,$$

is convergent or a Cauchy sequence respectively (in $\mathbb{R}$ or $\mathbb{C}$). We have used the notation $x_n = (x_n^1, \ldots, x_n^k)$.

(b) Use (a) to conclude that $\mathbb{R}^k$ and $\mathbb{C}^k$ are complete, i.e., Hilbert spaces, for all $k \geq 2$. The completeness of $\mathbb{R}$ is assumed known.

Opgave 4.3. Consider the finite dimensional complex Hilbert space $H = \mathbb{C}^k$, with the inner product from Example 4.6 (1).
   Show that $((1, 0, \ldots, 0), (0, 1, 0 \ldots, 0), \ldots, (0, \ldots, 0, 1))$ is an orthonormal basis for H.

Opgave 4.4. Show that an inner product on a complex Hilbert space Hilbert satisfies the polarization identity

$$(x, y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2) .$$

Opgave 4.5. Let H be a Hilbert space. Show that for any subset $A \subseteq H$ the subset $A^\perp$ is a *closed subspace* of H.

Opgave 4.6. Let H be a Hilbert space. Show that a subspace X of H is a Hilbert space if and only if X is closed. Show also that the closure of a subspace of H is itself a subspace. (here you should use the continuity of addition and of scalar multiplication).

Opgave 4.7. Show that $(\sin n\theta)_{n\in\mathbb{N}}$ is an orthogonal family in $C([0, \pi])$ with the inner product given in (4.7).

Opgave 4.8. Determine $a_1, a_2, a_3 \in \mathbb{C}$ such that

$$\int_0^\pi |\cos\theta - \sum_{n=1}^3 a_n \sin n\theta|^2 d\theta$$

is as small as possible.

Opgave 4.9. Let the polynomials $p_0(x), p_1(x), \ldots$ be given such that $p_n(x)$ is a polynomial of degree n in the variable x with the coefficient of x^n being 1 and $(p_0(x), p_1(x), \ldots)$ being an orthonormal family in $L^2([0, 1])$. Find $p_0(x), p_1(x)$ and $p_2(x)$.

Opgave 4.10. Show that $(\sin(n - \frac{1}{2})\theta)_{n\in\mathbb{N}}$ is an orthogonal family in $C([0, \pi])$ with inner product given by (4.7).

Opgave 4.11. Let H be an infinite dimensional separable Hilbert space and let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal family in H.

(1) Show that the series $\sum_{n=1}^\infty \frac{1}{n}e_n$ is convergent in H and determine for which $\alpha \in \mathbb{R}$ the series $\sum_{n=1}^\infty n^\alpha e_n$ is convergent in H.

(2) Determine the orthogonal projection of the vectors $e_1 \pm 2e_2$ on the subspace spanned by the vector $\sum_{n=1}^{\infty} n^{-1} e_n$. You may use that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Opgave 4.12. Show that $\lim_{n \to \infty} \int_0^{\pi} \log \theta \sin n\theta d\theta = 0$.
    Hint: This can be seen as a corollary to Bessel's inequality (4.25) used on the the orthogonal family in Exercise 4.7.

Opgave 4.13. Let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal basis for the Hilbert space H. Show that the following generalization of Parseval's equation holds for all $x, y \in H$:

$$(x, y) = \sum_{i=1}^{\infty} (x, e_i)\overline{(y, e_i)} .$$

Opgave 4.14. Consider the inner product space $\ell_0(\mathbb{N})$ defined in Example 4.6 (3). Let

$$X = \left\{ (x_n)_{n \in \mathbb{N}} \in \ell_0(\mathbb{N}) \mid \sum_{n=1}^{\infty} x_n \frac{1}{n} = 0 \right\} .$$

Show that X is a closed subspace of $\ell_0(\mathbb{N})$ and that $X \oplus X^{\perp} \neq \ell_0(\mathbb{N})$.

Opgave 4.15. Argue that the closure of $\ell_0(\mathbb{N})$ is equal to $\ell^2(\mathbb{N})$, i.e., that any sequence x in $\ell^2(\mathbb{N})$ is the limit of a convergent sequence $(x^n)_{n \in \mathbb{N}}$ in $\ell_0(\mathbb{N})$.
    Hint: Let x^n be the sequence which is equal to x up to the n-th term and zero afterwards.

Opgave 4.16. Let H be a complex Hilbert space and let $n \in \mathbb{N}$, $a \in \mathbb{C}$ be such that $a^n = 1$ and $a^2 \neq 1$. Show the *generalized polarization identity*

$$(x, y) = \frac{1}{n} \sum_{\nu=0}^{n-1} a^{\nu} \|x + a^{\nu} y\|^2 .$$

Opgave 4.17. (a) Show that the Hilbert space H is separable if and only if it has a countable dense subset (a countable set whose closure is the whole space). Hint to the if-part: Construct an orthonormal basis using the Gram-Schmidt procedure.

(b) Let H be a separable Hilbert space. Show that any subset M of H has an at most countable dense subset (an at most countable subset of M whose closure is M) *Hint:* Assume that M is not finite. Let $\{x_i \mid i \in \mathbb{N}\}$ be a countable dense subset of H. For $i, n \in \mathbb{N}$ let $x_{i,n}$ denote any arbitrarily chosen point in $\{x \in M \mid \|x - x_i\| \leq \frac{1}{n}\}$ assuming that this set is non-empty. Otherwise let $x_{i,n}$ be an arbitrarily chosen point in M. Now show that $\{x_{i,n} \mid i, n \in \mathbb{N}\}$ is a countable dense subset of M.

(c) Show that a closed subspace X of a separable Hilbert space is itself a separable Hilbert space. Hint: Use the characterization of separable in question (a) and the result of (b).

Opgave 4.18. (Difficult) Consider the Hilbert space $H = L^2([0,1])$ with the inner product $(f, g) = \int f(\theta)\overline{g(\theta)}d\theta$. Let $F_0 : \mathbb{R} \to \mathbb{R}$ be given by $F_0(t) = 1$ if $0 \leq t < 1$ and $F(t) = -1$ if $1 \leq t < 2$. Define f_n, $n = 0, 1, 2, \ldots$ by

$$F_n(t) = 2^{n-1} F_0(2^n t).$$

Let $f_{n,k} \in H$, $n = 0, 1, 2, \ldots$, $k = 0, 1, \ldots, 2^{n-1} - 1$ be given by

$$f_{n,k}(\theta) = f_n\left(\theta - k2^{n-1}\right), \quad \theta \in [0.1]$$

(a) Show that the family $(f_{n,k}|n = 0, 1, 2, \ldots, \ k = 0, 1, \ldots, 2^{n-1} - 1)$ is a countable orthonormal family.

(b) Show that the subspace

$$\text{span}\{f_{n,k} \mid n = 0, 1, 2, \ldots, N, \ k = 0, 1, \ldots, 2^{n-1} - 1\}$$

consists of all functions that are constant on the intervals $[k2^{-N}, (k+1)2^{-N})$, $k = 0, 1, \ldots, 2^{-N} - 1$.

(c) Show that[1]

$$C^1([a, b]) \subseteq \overline{\text{span}\{f_{n,k} \mid n = 0, 1, 2, \ldots, \ k = 0, 1, \ldots, 2^{n-1} - 1\}}.$$

Hint: Use the Mean Value Theorem.

(d) Conclude from the information $L^2([0, 1]) = \overline{C^1([a, b])}$ that $(f_{n,k}|n = 0, 1, 2, \ldots, \ k = 0, 1, \ldots, 2^{n-1} - 1)$ is an orthonormal basis for $L^2([0, 1])$.

Opgave 4.19. Show that an inner product space E is a Hilbert space if and only if any series $\sum_{n=1}^{\infty} x_n$ with terms in E satisfying $\sum_{n=1}^{\infty} \|x_n\| < \infty$ is convergent.
On a Hilbert space a series $\sum_{n=1}^{\infty} x_n$ satisfying $\sum_{n=1}^{\infty} \|x_n\| < \infty$ is said to be absolutely convergent.

Opgave 4.20. Consider the Hilbert space $H = L^2([-1, 1])$ with the inner product

$$(f, g) = \int_{-1}^{1} f(t)\overline{g(t)}dt$$

and the elements $f_1, f_2, g \in H$ given by $f_1(t) = \sqrt{3/2}\,t$, $f_2(t) = \sqrt{5/2}\,t^2$, and $g(t) = 1$. Show that f_1, f_2 are orthonormal and find the closest vector to g in $\text{span}\{f_1, f_2\}$.

Opgave 4.21. Show that if H is an infinite dimensional separable Hilbert space then all orthonormal bases are infinite and have countably many elements. *Hint*: From the characterization of separability in Exercise 4.17 you know that H has a countable dense set D. Given an orthonormal basis show that the balls centered at basis vectors with radius less than 1 are disjoint. Argue that these balls have non-empty intersection with D and then that the orthonormal basis is is at most countable.

Opgave 4.22. Prove Proposition 4.19.

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[1] It is also possible to show that $C([a, b]) \subseteq \overline{\text{span}\{f_{n,k} \mid n = 0, 1, 2, \ldots, \ k = 0, 1, \ldots, 2^{n-1} - 1\}}$ but this is most easily done by fist showing that continuous functions on $[a, b]$ are uniformly continuous. Uniform continuity means that for any $\varepsilon > 0$ there is $\delta > 0$, such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$, for all $x, y \in [0, 1]$.