Derivable Functors, 2004–06

Very first draft

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Preface.

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Categories

1. Categories.

(1.1). Here are a few concepts from the jungle of categories. Explain what they mean.

Identity; morphism, isomorphism, endomorphism, automorphism; monomorphism, epimorphism; subobject, quotient object; initial object, final object, zero object; equalizer or kernel, coequalizer or cokernel; sum, product; fibered product or pullback diagram, amalgamated sum or pushout diagram; opposite category $\mathfrak{C}^{\mathrm{op}}$ of a category $\mathfrak{C}$; functor and contravariant functor; transformation of functors;

(1.2) Examples. Fundamental are the categories **Sets** of sets and **Ab** of abelian groups.

Basic examples from set theory and combinatorics: the category $\mathbf{Sets}_0$ of pointed (based) sets, the category **FiniteSets** of finite sets; the category **POS** of partially ordered sets (with strictly increasing maps as morphisms); the category **Cat** of small categories (with functors as morphisms).

Basic algebraic examples: the category **Gr** of groups; the category **Rings** of (unital) commutative rings; the category ($_k\mathbf{Alg}$ of k-algebras (of some given fixed type, for instance associative and unital (also, for $k = \mathbb{Z}$ called noncommutative rings), or Lie, or Jordan, or ...); the category $_R\mathbf{Mod}$ of R-modules.

Basic geometrical examples: the category **Top** of topological spaces, the category $\mathbf{Top}_0$) of pointed topological spaces, the category **Mfld** of manifolds (of some given fixed type, for instance topological manifolds, pl-manifolds, algebraic manifolds (over a given field), C^∞-manifolds (real or complex), analytic manifolds, ...); the category **Schemes** of schemes (possibly over a fixed base scheme).

Some very small examples: the category $\emptyset$ with no objects; the category $\mathbf{0}$ (or $*$) with one object and the identity as the only morphism; the category $\mathbf{0} \rightrightarrows \mathbf{1}$ with two objects, say 0 and 1, and two morphisms $0 \to 1$ (and two identities).

Every set M defines a *discrete category:* its objects are the elements of M and the only morphisms are the identities. An additional category derived from M has the same objects, and exactly one morphism $i \to j$ for any pair of elements (i, j) in M.

Every pre-ordered set $(M, \preccurlyeq)$ defines a category, denoted M: its objects are the elements of M, and for elements $i, j \in M$ there is a single morphism $i \to j$ if $i \preccurlyeq j$, and no morphism otherwise. In fact, a pre-ordered sets may be indentified with a categories in which, for any pair of objects i, j, there is at most one morphism $i \to j$.

(1.3) Example. For every nonnegative integer n, denote by $[n]$ the finite set,

$$[n] = \{0, 1, \ldots, n\}.$$

Three important categories, **s**, **ss**, and **sss**, have as objects the set of nonnegative integers $0, 1, 2, \ldots$; the sets of morphisms,

$$\mathrm{Hom}_{\mathbf{s}}(p, q), \quad \mathrm{Hom}_{\mathbf{ss}}(p, q), \quad \mathrm{Hom}_{\mathbf{sss}}(p, q),$$

are the sets of all maps $[p] \to [q]$ that are, respectively, arbitrary, weakly increasing, or strictly increasing.

(1.4) Diagram categories. A *quiver* D is an oriented multigraph. It consists of a set V of *vertices*, a set E of *edges* or *arrows*, and two maps $b, e : E \to V$. For $u, v \in V$ and $\alpha \in E$, we write $\alpha : u \to v$ for the statement $b(\alpha) = u$ and $e(\alpha) = v$.

If D is a quiver, then a *D-diagram* in the category $\mathfrak{C}$ is function F associating with every vertex v of D an object $F(v)$ of $\mathfrak{C}$ and with every arrow of D a morphism $F(\alpha)$ of $\mathfrak{C}$ such that if $\alpha : u \to v$ then $F(\alpha) : F(u) \to F(v)$. There is an obvious category $\mathfrak{C}^D$ of D-diagrams of $\mathfrak{C}$.

The *path category* $\mathcal{P}(D)$ of a quiver D has as objects the set of vertices of D and as morphisms the set of all strings,

$$(u, \alpha_1, \ldots, \alpha_n, v),$$

where u, v are vertices of D and the α_i are n arrows of D and either $n = 0$ and $u = v$, or $n > 0$ and $u = e(\alpha_1), b(\alpha_i) = e(\alpha_{i+1})$ for $1 \leqslant i < n$, $\; b(\alpha_n) = v$. Composition of composable morphisms is essentially concatenation.

2. Exact categories.

Fix a categori $\mathfrak{A}$.

(2.1) Definition. An object of $\mathfrak{A}$ is called a *zero object*, and usually denoted 0, if it is both a final and initial object of $\mathfrak{A}$. If $\mathfrak{A}$ has a zero object, then for any pair A, B of objects, the *zero morphism*,

$$0 = 0_{AB} \colon A \to B,$$

is the composition $A \to 0 \to B$. It behaves like a zero in the sense that $f0 = 0$ and $0g = 0$ for morphisms $f \colon B \to B'$ and $g \colon A' \to A$.

(2.2) Definition. Assume that $\mathfrak{A}$ has a zero object. Let $f \colon A \to B$ be a morphism. By definition, a *kernel* for f, denoted Ker f, is an equalizer for the pair $f, 0 \colon A \to B$,

$$\operatorname{Ker}(f) := \operatorname{Ker}(f, 0),$$

and a *Cokernel* for f, denoted Cok f, is a coequalizer for the pair $f, 0$.

In other words, a morphism $K \to A$ is a kernel of f, if and only if $K \to A \to B$ is the zero morphism and, for every morphism $h \colon A' \to A$ such that $A' \to A \to B$ is the zero morphism, there exists a unique morphism $h' \colon A' \to K$ such that h is the composition $A' \to K \to A$. The morphism h' is said to be *induced* by h. Clearly, there is a dual description of the cokernel, and a similar notation of *induced* maps. The setup is indicated in the following diagrams:

$$\begin{tikzcd} & A' \arrow[dl, dashed] \arrow[d, "h"] \arrow[dr, "0"] & \\ K \arrow[r] & A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[dr, "0"'] & B \arrow[r] \arrow[d] & C \arrow[dl, dashed] \\ & B & \end{tikzcd}$$

Note that the kernel as a morphism $K \to A$ is monic, unique up to canonical isomorphism; hence, equivalently, we may think of *the kernel* of $f \colon A \to B$ as a subobject K of A with the canonical injection $K \hookrightarrow A$. Dually, we may think of *the cokernel* as a quotient C of B with the canonical projection $B \twoheadrightarrow C$.

Note that if K is the kernel of $f \colon A \to B$, and $u \colon B \to B'$ is a monomorphism, then K is also the kernel of $uf \colon A \to B'$. Dually, if C is the cokernel of $f \colon A \to B$ and $v \colon A' \to A$ is epic, then C is also the cokernel of $fv \colon A' \to B$.

(2.3) Definition. Assume that $\mathfrak{A}$ has a zero object, and kernels and cokernels. Let $f \colon A \to B$ be a morphism in $\mathfrak{A}$. Then *the image* of f, denoted Im f, is the kernel of $B \to \operatorname{Cok} f$, and *the coimage* of f, denoted Coim f, is the cokernel of $\operatorname{Ker} f \to A$.

There is an induced morphism, called the *canonical morphism*, making the following diagram commmutative:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & \operatorname{Coim} f \arrow[d, "\tilde{f}"] \\ B & \operatorname{Im} f . \arrow[l] \end{tikzcd}$$

Inded, use the definition of Coim f as a cokernel to obtain the map $\hat{f}$ of the first of the following diagrams; the composition Coim $f \to B \to$ Cok f is the zero morphism, and the definition of Im f as a kernel yields $\tilde{f}$.

$$
\begin{tikzcd}
\operatorname{Ker} f \arrow[r] \arrow[dr, "0"'] & A \arrow[r] \arrow[d, "f"'] & \operatorname{Coim} f \arrow[dl, dashed, "\hat{f}"] \\
\operatorname{Cok} f & B \arrow[l] & \operatorname{Im} f, \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{Ker} f \arrow[r] & A \arrow[r] \arrow[d, "f"'] \arrow[dl, "0"'] & \operatorname{Coim} f \arrow[dl, "\hat{f}"] \arrow[d, dashed, "\tilde{f}"] \\
\operatorname{Cok} f & B \arrow[l] & \operatorname{Im} f. \arrow[l]
\end{tikzcd}
$$

Note. It is easy to see that all three morphisms $A \to B$, $A \to$ Coim, and $A \to$ Im f have Ker f as kernel. Dually, all three morphisms $A \to B$, Coim $f \to B$, and Im $f \to B$ have Cok f as cokernel.

(2.4) Definition. The category $\mathfrak{A}$ is an *exact category* if it has a zero object and kernels and cokernels, and if for every morphism $f\colon A \to B$ the canonical morphism $\tilde{f}\colon$ Coim $f \to$ Im f is an isomorphism.

(2.5) Proposition. *Assume that $\mathfrak{A}$ is an exact category, and let $f\colon A \to B$ be a morphism in $\mathfrak{A}$. Then:*

(1) *f is monic, iff* Ker $f = 0$, *iff $A \to$* Im f *is an isomorhism.*
(2) *f is epic, iff* Cok $f = 0$, *iff* Im $f \to B$ *is an isomorhism.*
(3) *f is isomorphic, iff f is monic and epic, iff* Ker $f =$ Cok $f = 0$.

Moreover, for a given factorization of f:

$$
\begin{tikzcd}
 & C \arrow[dr, "v"] & \\
A \arrow[ur, "u"] \arrow[rr, "f"'] & & B.
\end{tikzcd}
$$

(4) *If u is monic and v is epic, then $A \xrightarrow{u} C$ is a kernel of v, iff $C \xrightarrow{v} B$ is a cokernel of u.*
(5) *If u is epic and v is monic, then $C =$* Im $f =$ Coim f.

Proof. (1) If $A \to$ Im f is an isomorphism, then it is in particular monic; hence, so is the composition $A \to$ Im $f \to B$, that is, f is monic. Clearly, if f is monic, then Ker $f = 0$. Finally, if Ker $f = 0$ then $A \to$ Coim f is an isomorphism and hence, by the exactness property $A \to$ Im f is an isomorphism.

The proofs of the remaining assertions are similar. ▯

Note. The property (4) produces a bijective correspondance between the subobjects A of C and the quotient objects B of C. The quotient object corresponding to the subobject A of C is usually denoted C/A.

(2.6) Definition. Assume that $\mathfrak{A}$ is an exact category. A sequence of morphisms of $\mathfrak{A}$,

$$\cdots \longrightarrow X^{n-1} \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1} \longrightarrow \cdots,$$

is a *complex* or a *zero sequence*, if $f^n f^{n-1} = 0$ or, equivalently, $\operatorname{Im} f^{n-1} \subseteq \operatorname{Ker} f^n$, for all n. For a zero sequence, the n'th *cohomology* H^n is the cokernel of the morphism $X^{n-1} \to \operatorname{Ker} f^n$, or, equivalently, the quotient quotient object:

$$H^n := \operatorname{Ker} f^n / \operatorname{Im} f^{n-1};$$

it may also be described as the kernel of the morphism $\operatorname{Cok} f^{n-1} \to X^{n+1}$. The sequence is called an *exact sequence* if it is a zero sequence and $H^n = 0$ for all n.

The word 'complex' is reserved for an infinite sequence as indicated in the notation, but otherwise the definitions apply with obvious modifications to finite sequences. A diagram is called an *exact diagram* if every sequence formed by consequtive, composable morphism on a stragth line in the diagram is exact.

Note the following cases:
The sequence $0 \to A \to A'$ is exact, iff $A' \to A$ is monic.
The sequence $0 \to A' \to A \xrightarrow{u} B''$ is exact, iff $A' \to A$ is a kernel of u.
The sequence $A' \xrightarrow{u} A \xrightarrow{f} B \xrightarrow{v} B'$ is exact, iff it is a zero sequence and the induced map $\operatorname{Cok} u \to \operatorname{Ker} v$ is an isomorphism.

(2.7) Note. Clearly, for any commutative diagram in $\mathfrak{A}$,

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r] & B', \end{tikzcd}$$

there is a unique (induced) morphism $\operatorname{Ker} f \to \operatorname{Ker} f'$ making the following diagram commutative:

$$\begin{tikzcd} \operatorname{Ker} f \arrow[r] \arrow[d] & \operatorname{Ker} f' \arrow[d] \\ A \arrow[r] & A'. \end{tikzcd}$$

Similarly, the given diagram induces morphims between the cokernels, the images and the coimages.

(2.8) The 3-lemma. *Assume that $\mathfrak{A}$ is an exact category. Consider in $\mathfrak{A}$ an exact, commutative diagram,*

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "f'"'] & A \arrow[r] \arrow[d, "f"'] & A'' \arrow[d, "f''"'] \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B''. \end{tikzcd} \qquad (*)$$

Then: (1) *The induced sequence of kernels* $0 \to \operatorname{Ker} f' \to \operatorname{Ker} f \to \operatorname{Ker} f''$ *is exact.*
(2) *If f'' is monic, the induced map* $\operatorname{Cok} f' \to \operatorname{Cok} f$ *is monic.*

Proof. (1) It suffices to note that $\operatorname{Ker} f' \to \operatorname{Ker} f$ is a kernel of $\operatorname{Ker} f \to \operatorname{Ker} f''$.

(2) Assume that f'' is monic. Clearly, to prove (2), we may replace in the diagram the object A'' with $\mathrm{Im}(A \to A'')$. So we may assume that $A \to A''$ is epic. In an obvious choice of notation, let $0 \to C' \to C \to C''$ be the induced sequence of images of the f's. It fits into a commutative diagram,

$$\begin{tikzcd} 0 \arrow[r] & C' \arrow[r] \arrow[d, "i'"'] & C \arrow[r] \arrow[d, "i"'] & C'' \arrow[d, "i''"'] \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B''. \end{tikzcd} \tag{$\bar{*}$}$$

We will prove that the top row of ($\bar{*}$) is exact. Note first that with $K' := \mathrm{Ker}\, f'$, $K := \mathrm{Ker}\, f$, and $K'' := \mathrm{Ker}\, f''$, the sequence $C' \to C \to C'' \to 0$ is the sequence of cokernels induced from the following commutative diagram,

$$\begin{tikzcd} K' \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & K'' \arrow[r] \arrow[d] & 0 \\ A' \arrow[r] & A \arrow[r] & A'' \arrow[r] & 0. \end{tikzcd} \tag{\#}$$

The rows of (#) are exact. Indeed, in the top row we have $K'' = 0$, and so exactness follows from (1); the bottom row is exact since we assumed that $A \to A''$ is epic. So, by the dual assertion (1)*, the sequence $C' \to C \to C'' \to 0$ is exact. Moreover, in the commutative diagram ($\bar{*}$), the injection $C' \to B'$ and the morphism $B' \to B$ are monic. Hence $C' \to C$ is monic. Therefore, the top row of ($\bar{*}$) is exact.

So the commutative diagram ($\bar{*}$) is exact. The vertical morphisms of ($\bar{*}$) are monic, and their cokernels are the same as those of the original diagram ($*$), So we may assume in the original diagram ($*$) that all the vertical morphisms are monic. Under this assumption it is easy to verify that the morphism $f' : A' \to B'$ is a kernel of the composition $h := (B' \to B \to D)$, where $D := \mathrm{Cok}\, f$. So $\mathrm{Cok}\, f'$ is the coimage, and hence equal to the image, of h. As an image, it injects into D. So the morphism $\mathrm{Cok}\, f' = \mathrm{Im}\, h \to D = \mathrm{Cok}\, f$ is a monomorphism. Hence (2) has been proved. ∎

(2.9) The 4-Lemma. *Assume that $\mathfrak{A}$ is an exact category, and consider an exact commutative diagram,*

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d, "f'"'] & A \arrow[r] \arrow[d, "f"'] & A'' \arrow[d, "f''"'] \\ B_0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \end{tikzcd}$$

If $A_0 \to B_0$ is epic, then $\mathrm{Ker}\, f' \to \mathrm{Ker}\, f \to \mathrm{Ker}\, f''$ *is exact.*

Proof. Split the diagram into two commutative diagrams:

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d, "f'"'] & \bar{A} \arrow[d, "\bar{f}"'] \\ B_0 \arrow[r] & B' \arrow[r] & \bar{B}, \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & \bar{A} \arrow[r] \arrow[d, "\bar{f}"'] & A \arrow[r] \arrow[d, "f"'] & A'' \arrow[d, "f''"'] \\ 0 \arrow[r] & \bar{B} \arrow[r] & B \arrow[r] & B'', \end{tikzcd}$$

where $\bar{f}\colon \bar{A} \to \bar{B}$ is the induced morphism of images.

The two diagrams are exact. Therefore, by Assertion (2.8)(1), the sequence $0 \to \operatorname{Ker} \bar{f} \to \operatorname{Ker} f \to \operatorname{Ker} f''$ is exact, and by the assertion dual to (2.8)(2), the morphism $\operatorname{Ker} f' \to \operatorname{Ker} \bar{f}$ is epic. The assertion of the Proposition is a consequence. ∎

(2.10) The Ker-Coker sequence of a composition. *Assume that $\mathfrak{A}$ is exact. Then, for two composable morphisms $f\colon A \to B$ and $g\colon B \to C$, there is an exact sequence,*

$$0 \to \operatorname{Ker} f \to \operatorname{Ker} gf \to \operatorname{Ker} g \to \operatorname{Cok} f \to \operatorname{Cok} gf \to \operatorname{Cok} g \to 0.$$

Proof. Apply the 4-lemma (2.9) three times to parts of the following diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \operatorname{Ker} f \arrow[r] \arrow[d] & A \arrow[r, "f"] \arrow[d, "gf"'] & B \arrow[r] \arrow[d, "g"] & \operatorname{Cok} f \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & C \arrow[r, equal] & C \arrow[r] & 0,
\end{tikzcd}$$

to obtain the exact sequence,

$$0 \to \operatorname{Ker} f \to \operatorname{Ker} gf \to \operatorname{Ker} g \to \operatorname{Cok} f.$$

Conclude by duality. ∎

(2.11) Corollary. The second Noether Isormorphism Theorem. *Assume that $\mathfrak{A}$ is exact. Then for subobjects $C_0 \subseteq C \subseteq A$ of an object A, there is a canonical isomorphism,*

$$C/C_0 \xrightarrow{\sim} \operatorname{Ker}(A/C_0 \to A/C).$$

(2.12) The Snake Lemma. *Assume that $\mathfrak{A}$ is an exact category. Then an exact commutative diagram in $\mathfrak{A}$,*

$$\begin{tikzcd}
 & & & & 0 \arrow[d] \\
A_0 \arrow[r] \arrow[d, "f_0"'] & A' \arrow[r] \arrow[d, "f'"'] & A \arrow[r] \arrow[d, "f"'] & A'' \arrow[r] \arrow[d, "f''"'] & A^0 \arrow[d, "f^0"'] \\
B_0 \arrow[r] \arrow[d] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & B^0 \\
0, & & & &
\end{tikzcd}$$

induces an exact sequence,

$$\operatorname{Ker} f' \longrightarrow \operatorname{Ker} f \longrightarrow \operatorname{Ker} f'' \xrightarrow{\delta} \operatorname{Cok} f' \longrightarrow \operatorname{Cok} f \longrightarrow \operatorname{Cok} f''.$$

More precisely, set $K := \operatorname{Ker}(A \to B'')$ and $L := \operatorname{Cok}(A' \to B)$. Then the induced morphism $K \to \operatorname{Ker} f''$ is epic, the induced morphism $\operatorname{Cok} f' \to L$ is monic, and δ is the uniqe morphism making the following diagram commutative:

$$
\begin{tikzcd}
& K \arrow[r] \arrow[d] & \operatorname{Ker} f'' \arrow[dddll, dashed, "\delta"] \\
& A \arrow[d] & \\
& B \arrow[d] & \\
\operatorname{Cok} f' \arrow[r] & L, &
\end{tikzcd}
$$

Proof. Let $C \to B''$ be the kernel of $B'' \to B^0$. Then the morphism $A \to B''$ lifts uniquely to a morphism $A \to C$ having kernel K, and fitting into a commutative, exact diagram,

$$
\begin{tikzcd}
A_0 \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & A^0 \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & C \arrow[r] & B'' \arrow[r] & B^0,
\end{tikzcd}
$$

Apply The 4-Lemma (2.9) twice to obtain the first of the following exact sequences:

$$A' \longrightarrow K \longrightarrow \operatorname{Ker} f'' \longrightarrow 0, \qquad 0 \longrightarrow \operatorname{Cok} f' \longrightarrow L \longrightarrow B''. \tag{*}$$

The second exact sequence is obtained by the dual argument. Since the compositions $A' \to K \to L$ and $K \to L \to B''$ are zero morphisms, the existence and uniqueness of δ are obtained from the exactness in (*). By the 4-Lemma (2.9) and duality, to prove that the long sequence of kernels and cokernels is exact, we need only to prove that the following sequence is exact:

$$\operatorname{Ker} f \longrightarrow \operatorname{Ker} f'' \xrightarrow{\delta} \operatorname{Cok} f';$$

as the morhism $\operatorname{Cok} f' \to L$ is a monomorphism, it suffices to prove that the following sequence is exact:

$$\operatorname{Ker} f \longrightarrow \operatorname{Ker} f'' \longrightarrow L \tag{2.12.2}$$

is exact. Apply the 4-Lemma (2.9) to the following diagram:

$$
\begin{tikzcd}
A' \arrow[r] \arrow[d, equal] & K \arrow[r] \arrow[d] & \operatorname{Ker} f'' \arrow[r] \arrow[d] & 0 \arrow[d] \\
A' \arrow[r] & B \arrow[r] & L \arrow[r] & 0.
\end{tikzcd}
$$

It follows that the morphism $\operatorname{Ker}(K \to B) \to \operatorname{Ker}(\operatorname{Ker} f'' \to L)$ is an epimorphism. This epimorphism factors through the morphism $\operatorname{Ker} f \to \operatorname{Ker}(\operatorname{Ker} f'' \to L)$. Hence the latter morphism is an epimorphism as well. Therefore, (2.12.2) is exact. ∎

(2.13) Definition. In the setup of the Snake Lemma, the exact sequence of kernels and cokernels will be called the *snake sequence*, and the morphism δ will be called the *snake morphism* or the *connecting morphism*.

(2.14) The 5-Lemma. *Under the hypotheses of the Snake Lemma* (2.13), *if the morphisms f' and f'' are isomorphisms, then so is f. In particular, the middle morphism f is an isomorphism if the other f vertical morphisms are isomorphisms.*

Proof. Exactness of the snake sequence yields $\operatorname{Ker} f = \operatorname{Cok} f = 0$. ∎

(2.15) The Push-out Lemma. *Assume that $\mathfrak{A}$ is exact. Consider at push-out diagram,*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A'' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\ B & \xrightarrow{\beta} & B''. \end{array}$$

(1) *The induced morphism* $\operatorname{Cok}\alpha \to \operatorname{Cok}\beta$ *is an isomorphism.*
(2) *If f or α is an epimorhism, then* $\operatorname{Ker}\alpha \to \operatorname{Ker}\beta$ *is an epimorphism.*

Proof. (1) Let $\alpha''\colon A \to A^0$ be the cokernel of $\alpha\colon A \to A''$. Then, by the assumed push-out properties, applied to α'' and the zero morphism $B \to A^0$, there is a unique morphism $\beta''\colon B'' \to A^0$ such that $\beta'' f'' = \alpha''$ and $\beta''\beta = 0$. Now check that $\beta''\colon B'' \to A^0$ is a cokernel of $\beta\colon B \to B''$.

(2) Assume that α is epic. We have to prove that both induced morphisms $\operatorname{Ker}\alpha \to \operatorname{Ker}\beta$ and $\operatorname{Ker} f \to \operatorname{Ker} f''$ are epic. First, it follows from (1) that β is epic. Therefore, completing the given diagram with the kernels of α and β, we obtain an exact, commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \xrightarrow{\alpha} & A'' & \longrightarrow & 0 \\ & & {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \xrightarrow{\beta} & B'' & \longrightarrow & 0. \end{array}$$

By the Snake Lemma there is an induced exact sequence,

$$\operatorname{Ker} f \longrightarrow \operatorname{Ker} f'' \xrightarrow{\delta} \operatorname{Cok} f' \to \operatorname{Cok} f \to \operatorname{Cok} f''.$$

If suffices to prove that δ is the zero morphism. Indeed, if $\delta = 0$, then it follows that the first morphism $\operatorname{Ker} f \to \operatorname{Ker} f''$ in the sequence is epic; moreover, as the last morphism in the sequence is an isomorphism by (1), it follows that $\operatorname{Cok} f' = 0$, that is, the morphism $f'\colon \operatorname{Ker}\alpha \to \operatorname{Ker}\beta$ is epic.

Now, by the Snak Lemma, δ is induced by the composition,

$$\operatorname{Ker}(A \to B'') \to A \to B \to \operatorname{Cok}(A' \to B).$$

It follows from the assumed push-out properties that $\beta\colon B \to B''$ is a cokernel of $A' \to B$. So δ is induced by the zero morphism, and hence equal to zero. ∎

(2.16) Proposition. *Let $F: \mathfrak{A} \to \mathfrak{B}$ be a functor between exact categories.*

(1) *If F is faithfull then it reflects zero object, zero morphisms, and exact sequences.*
(2) *The functor F preserves exact sequences, if and only if for every short exact sequence in $\mathfrak{A}$,*
$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0,$$
the following sequence is exact in $\mathfrak{B}$,
$$0 \longrightarrow FA' \longrightarrow FA \longrightarrow FA'' \longrightarrow 0,$$

Proof. (1) The zero object 0 of $\mathfrak{A}$ is the only object for which $\operatorname{End}(0)$ consists of single morphism. Therefore, if $FA = 0$, then $A = 0$.

Assume for $\alpha: A \to A'$ that $F(\alpha) = 0$. The zero morphism $0_{AA'}$ is the composition $0_{AA'} = \alpha 0_{AA}$. Therefore, $F(0_{AA'}) = 0_{FAFA'} F(0_{AA}) = 0$. In particular, $F(\alpha) = F(0_{AA'})$. Therefore, $\alpha = 0_{AA'}$.

Assume for sequences
$$A' \xrightarrow{\alpha'} A \xrightarrow{\alpha} A'' \quad \text{and} \quad FA' \xrightarrow{F(\alpha')} FA \xrightarrow{F(\alpha)} FA''$$
that the last sequence is exact. First, since $F(\alpha\alpha') = F(\alpha)F(\alpha') = 0$, it follows that $\alpha\alpha' = 0$; hence the first sequence is a zero sequence. Consider the two diagrams,
$$\begin{tikzcd} & \operatorname{Ker}\alpha \arrow[d] \arrow[dr, "0"] & \\ A' \arrow[r, "\alpha'"] \arrow[dr, "0"'] & A \arrow[r, "\alpha"'] \arrow[d] & A'', \\ & \operatorname{Cok}\alpha' & \end{tikzcd} \qquad \begin{tikzcd} & F(\operatorname{Ker}\alpha) \arrow[d] \arrow[dr, "0"] & \\ FA' \arrow[r, "F(\alpha')"] \arrow[dr, "0"'] & FA \arrow[r, "F(\alpha)"'] \arrow[d] & FA'', \\ & F(\operatorname{Cok}\alpha'). & \end{tikzcd}$$

[Why is $F(0): F(\operatorname{Ker}\alpha) \to F(A'')$ equal to the zero morphism in $\mathfrak{B}$???]

It follows that $F(\operatorname{Ker}\alpha) \to F(\operatorname{Cok}\alpha'')$ is the zero morphism. Hence $\operatorname{Ker}\alpha \to \operatorname{Cok}\alpha'$ is the zero morphism. Consequently, $A' \to A \to A''$ is exact.

(2) The proof of the second assertion is immediate. ▯

(2.17) Exercises.

1. Let $\mathfrak{A}$ be a category with a zero object 0, and let $\mathfrak{B} = \mathfrak{A}^{\mathbb{Z}}$ be the category of all $\mathbb{Z}$-indexed families of objects $X = (X^i)$ from $\mathfrak{A}$. For any object $A \in \mathfrak{A}$ and any integer n, let $A(-n)$ denote the family with $A(-n)^n = A$ and $A(-n)^i = 0$ for $i \neq n$. Prove for any family $X \in \mathfrak{B}$ that the two natural morphisms are isomorphisms:
$$\bigoplus_{n \in \mathbb{Z}} X^n(-n) = X = \prod_{n \in \mathbb{Z}} X^n(-n),$$
[Hint: don't assume in advance the existence of the product and the coproduct.]

3. Additive categories.

Let $\mathfrak{A}$ be a category.

(3.1) Definition. The categori $\mathfrak{A}$ is said to *have a semi-additive Hom-structure* if there is given, for any pair of objects X, Y in $\mathfrak{A}$, a structure on $\mathrm{Hom}_{\mathfrak{A}}(X, Y)$ as a commutative (additive) monoid such that composition of morphisms,

$$\mathrm{Hom}_{\mathfrak{A}}(Y, Z) \times \mathrm{Hom}_{\mathfrak{A}}(X, Y) \to \mathrm{Hom}_{\mathfrak{A}}(X, Z),$$

is *bi-additive*, that is, a homomorphism of monoids in each variable. The monoid composition in $\mathrm{Hom}_{\mathfrak{A}}(X, Y)$, called *addition*, is denoted $(f, g) \mapsto f + g$, and the neutral element in $\mathrm{Hom}_{\mathfrak{A}}(X, Y)$, called the *zero morphism*, is denoted 0_{XY} or simply 0. Note that it is part of the definition of bi-additivity, and not a consequence of the definition, that any composition with a zero morphisms yields a zero morphism.

(3.2) Definition. Assume that the category $\mathfrak{A}$ has finite products. Then we say that *finite products are finite sums*, if the following two conditions hold:

(add 1) The final object of $\mathfrak{A}$ is also initial (and hence a zero object, denoted 0).
(add 2) For every pair of objects X, Y of $\mathfrak{A}$, the diagram,

$$X \xrightarrow{\binom{1}{0}} X \times Y \xleftarrow{\binom{0}{1}} Y,$$

is a direct sum of X and Y.

More symmetrically, the two conditions hold if $\mathfrak{A}$ has finite products, finite sums, a zero-object, and if for all objects X, Y the morphism,

$$X \vee Y \xrightarrow{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} X \times Y, \tag{3.2.1}$$

from the sum to the product, is an isomorphism.

If the conditions hold, it is common to denote by $X \oplus Y$ both the sum and the product under the canonical isomorphim. It comes with four morphisms:

$$\begin{array}{ccccc} X_1 & \xrightarrow{\mathrm{in}_1} & X_1 \oplus X_2 & \xleftarrow{\mathrm{in}_2} & X_2 \\ & & \| & & \\ X_1 & \xleftarrow{\mathrm{pr}_1} & X_1 \oplus X_2 & \xrightarrow{\mathrm{pr}_2} & X_2 \end{array} \qquad \text{such that } \mathrm{pr}_i\, \mathrm{in}_j = \begin{cases} 0, & \text{if } i \neq j, \\ 1. & \text{if } i = j. \end{cases}$$

We use the well known matrix-notation to describe morphisms between direct sums: A morphism *from* a direct sum is given by a row, a morphim *into* a direct sum is given by a column; accordingly, a morhism from a direct sum to a direct sum is given by a column of rows (or a row of columns), that is, by a matrix.

(3.3) Proposition. *The following two conditions on the category $\mathfrak{A}$ are equivalent:*
(i) *$\mathfrak{A}$ has finite products, and finite products are finite sums in the sense of* (3.2).
(ii) *$\mathfrak{A}$ has finite products and a semi-additive Hom-structure.*
Assume that the conditions are satisfied. Then the Hom-structure is unique: The sum of two morphisms $f, g\colon X \to Y$ is determined by the equations,

$$f + g = \nabla_Y \begin{pmatrix} f \\ g \end{pmatrix} = (f, g)\Delta_X = \nabla_Y \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \Delta_X. \tag{3.3.1}$$

Moreover, the maps of the diagram in (3.2) *satisfies the equation* $\mathrm{pr}_1 \mathrm{in}_1 + \mathrm{pr}_2 \mathrm{in}_2 = 1$. *Finally, a diagram,*

$$A_1 \xleftarrow{\pi_1} P \xrightarrow{\pi_2} A_2,$$

is a product of A_1 and A_2, if and only if there is a diagram,

$$A_1 \xrightarrow{\iota_1} P \xleftarrow{\iota_2} A_2,$$

such that

$$\pi_i \iota_j = \begin{cases} 0, & \text{if } i \neq j, \\ 1. & \text{if } i = j, \end{cases} \qquad \text{and } \iota_1\pi_1 + \iota_2\pi_2 = 1.$$

Proof. Assume first that there is given a semi-additive Hom-structure on $\mathfrak{A}$. Then the last assertion of the Proposition holds. Indeed, it is well-known that the assertion holds in the category (SemiAb) of commutative semigroups with zero element. By applying the special case to the semigroups $\mathrm{Hom}_{\mathfrak{A}}(X, A_1)$, $\mathrm{Hom}_{\mathfrak{A}}(X, A_2)$, and $\mathrm{Hom}_{\mathfrak{A}}(X, P)$, for an arbitrary object of $\mathfrak{A}$, it follows in particular that the induced map of semigroups,

$$\mathrm{Hom}_{\mathfrak{A}}(X, P) \to \mathrm{Hom}_{\mathfrak{A}}(X, A_1) \times \mathrm{Hom}_{\mathfrak{A}}(X, A_2),$$

is a bijection. Whence P is the product of A_1 and A_2.

Let us note in addition that the existence of a semi-additive Hom-structure on $\mathfrak{A}$ implies that a final object 0 is also initial. Indeed, assume that 0 is a final object. There is only one element i $\mathrm{End}(0)$. So the identity of 0 is equal to the zero element of $\mathrm{End}(0)$. Hence it follows from the bi-aditivity of composition that the zero element in $\mathrm{Hom}(0, Y)$ is the unique element in $\mathrm{Hom}(0, Y)$. It follows that the object 0 is also an initial object, and the composition $X \to 0 \to Y$ is the zero element in $\mathrm{Hom}(X, Y)$.

Now assume Condition (ii) on $\mathfrak{A}$. Then, first, there is a final object 0 of $\mathfrak{A}$, and as we have just noted, this object is also initial. Hence Condition (3.3)(add 1) holds. Consider Condition (3.3)(add 2), by definition of the morphism in_i, it follows that $\mathrm{pr}_i \mathrm{in}_i = 1$ and that $\mathrm{pr}_i \mathrm{in}_j = 0$ when $i \neq j$. Moreover, we have the equation $\mathrm{in}_1 \mathrm{pr}_2 + \mathrm{in}_2 \mathrm{pr}_2 = 1$ in $\mathrm{End}(A_1 \times A_2)$. Indeed, to verify the equation, we have to prove that the two endomorphisms are equalized by pr_1 and by pr_2, and this follows from the first equations and bi-additivity:

$$\mathrm{pr}_1(\mathrm{in}_1 \mathrm{pr}_1 + \mathrm{in}_2 \mathrm{pr}_2) = \mathrm{pr}_1 \mathrm{in}_1 \mathrm{pr}_1 + \mathrm{pr}_1 \mathrm{in}_2 \mathrm{pr}_2) = 1 \,\mathrm{pr}_1 + 0 \,\mathrm{pr}_2 = \mathrm{pr}_1,$$

with a similar computation for pr_2. Therefore, by the assertion dual to the last assertion of the proposition, the diagram is a direct sum of A_1 and A_2. Thus Condition (3.2)(add 2) holds.

In particular, we have seen that Condition (i) holds. Moreover, for two morphisms $f, g\colon X \to Y$ it follows with $A_1 = A_2 = X$ that $\mathrm{in}_1 \mathrm{pr}_1 + \mathrm{in}_2 \mathrm{pr}_2 = 1$. Moreover $(f, g)\,\mathrm{in}_1 = f$ etc, and $\mathrm{pr}_j\, \Delta_X = 1$. Hence,

$$(f, g)\Delta_X = (f, g)(\mathrm{in}_1 \mathrm{pr}_1 + \mathrm{in}_2 \mathrm{pr}_2)\Delta_X = (f, g)\,\mathrm{in}_1 \mathrm{pr}_1 \Delta_X + (f, g)\,\mathrm{in}_2 \mathrm{pr}_2 \Delta_X = f + g.$$

The remaining equations of (3.3.1) are proved similarly.

Conversely, assume Condition (i). It is easy to see that the composition on the Hom-sets defined by the equation above is a semi-additive Hom-structure on $\mathfrak{A}$. ▯

(3.4) Definition. The category $\mathfrak{A}$ is said to be *semi-additive* if the equivalent conditions of (3.3) hold. It is called an *additive category* if, under the semi-additive Hom-structure, each Hom-set is a commutative group, that is, every morphism $f\colon X \to Y$ has an additive inverse $-f$, such that $f + (-f) = 0$. It suffices that the identity 1_X, for any object X, has an additive inverse.

If $\mathfrak{A}$ and $\mathfrak{B}$ are categories with a semi-additive Hom-structure, a functor $F\colon \mathfrak{A} \to \mathfrak{B}$ is said to be a *Hom-additive functor*, if the maps,

$$\mathrm{Hom}_{\mathfrak{A}}(X, Y) \to \mathrm{Hom}_{\mathfrak{B}}(FX, FY),$$

induced by the functor, are homomorphisms of monoids. It is easy to see that if the categories $\mathfrak{A}$ and $\mathfrak{B}$ are semi-additive categories, then a functor $F\colon \mathfrak{A} \to \mathfrak{B}$ is Hom-additive, if and only if it is right or left additive (Recall that the functor F is *right additive* if it commutes with finite direct sums, and *left additive* if it commutes with finite products, and *additive* if it is both right and left additive.)

(3.5) Proposition. *A semi-additive category $\mathfrak{A}$ is additive, if and only if for every object X of $\mathfrak{A}$, the following morphism is an isomorphim:*

$$X \oplus X \xrightarrow{\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)} X \oplus X.$$

In particular, an exact, semi-additive category is additive.

Proof. A left inverse is necessarily of the form $\begin{pmatrix}1 & f\\ 0 & 1\end{pmatrix}$, where $f + 1 = 0$.

It is easy to the see that the morphism has kernel and cokernel equal to zero. Hence, if $\mathfrak{A}$ is exact, the morphism is an isomorphism. ▯

(3.6) Proposition. *Let $F\colon \mathfrak{A} \to \mathfrak{B}$ be a functor between semi-additive categories $\mathfrak{A}$ and $\mathfrak{B}$. Assume that F has a left adjoint functor $G\colon \mathfrak{B} \to \mathfrak{A}$. Then F and G are additive functors, and the adjunction bijection is an isomorphism of commutative monoids,*

$$u\colon \mathrm{Hom}_{\mathfrak{A}}(GX, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{B}}(X, FY).$$

Proof. The functor F is left exact, because it has a left adjoint. In particular, F is left additive. Hence, as observed in (3.4), F is Hom-additive and right additive. Similarly, the functor G is additive.

It follows from the description of addition in Proposition (3.3) that the bijection is a homomorphism of monoids. ∎

4. Abelian categories.

Fix a categori $\mathfrak{A}$.

(4.1) Definition. The category $\mathfrak{A}$ is said to be *abelian* if it is exact and has finite products and finite coproducts.

Proposition. *An abelian category is additive.*

Proof. Assume that $\mathfrak{A}$ is abelian. Let A, B be objects of $\mathfrak{A}$. Consider the commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\mathrm{in}_1"] \arrow[d, equal] & A \vee B \arrow[r, "{(0,1)}"] \arrow[d, "\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)"] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"'] & A \times B \arrow[r, "\mathrm{pr}_2"'] & B \arrow[r] & 0.
\end{tikzcd}$$

Its rows are easily seen to be exact. Hence, by the Five-lemma, $\mathfrak{A}$ is semi-additive. By Proposition (3.5), $\mathfrak{A}$ is additive. ∎

(4.2) Definition. Let $\mathfrak{Q}$ be a subclass of objects in an abelian category $\mathfrak{A}$.

The class $\mathfrak{Q}$ is said to *thick*, if it contains the zero object and the following condition holds: Given any exact sequence in $\mathfrak{A}$,

$$0 \to Q' \to Q \to Q'' \to 0, \tag{4.2.1}$$

then $Q \in \mathfrak{Q}$ if and only if $Q', Q'' \in \mathfrak{Q}$.

The class $\mathfrak{Q}$ is said to be (right) *dense* in $\mathfrak{A}$ if for every object $A \in \mathfrak{A}$ there exists a monomorphism $A \hookrightarrow Q$ into an object $Q \in \mathfrak{Q}$.

The class $\mathfrak{Q}$ is called a $^+$class, if for any exact sequence (4.2.1), if $Q', Q \in \mathfrak{Q}$, then $Q'' \in \mathfrak{Q}$.

The class $\mathfrak{Q}$ is said to be of (*right*) *dimension* $\leqslant n$ if it is additive and the following condition holds:

for any exact sequence in $\mathfrak{A}$,

$$Q_0 \to Q_1 \to Q_2 \to Q_{n-1} \to Q \to 0,$$

if $Q_1, \dots, Q_n \in \mathfrak{Q}$, then $Q \in \mathfrak{Q}$.

The category $\mathfrak{A}$ is said to be of dimension $\leqslant n$ if the class of injective objects is right dense and of dimension $\leqslant n$.

(4.3) Exercises.

1. Let $A \xrightarrow{\partial} A' \xrightarrow{\partial'} A''$ be a a zero-sequence. It is said to be *split* if there are morphisms $s\colon A' \to A$ and $s'\colon A'' \to A'$ such that $\partial s + s'\partial' = 1_{A'}$. Set $B := \operatorname{Im}\partial \subseteq A$ and $B' := \operatorname{Im}\partial' \subseteq A''$, and denote by $\bar{\partial}\colon A \to B$ and $\bar{\partial}'\colon A' \to B'$ the induced epimorphisms. Show that if the sequence is split, then there is a natural isomorphism $A \to B \oplus B'$ making the following diagram commutative:

$$\begin{tikzcd} A \arrow[r, "\partial"] \arrow[d, equal] & A' \arrow[r, "\partial'"] \arrow[d, "\wr"] & A'' \arrow[d, equal] \\ A \arrow[r, "\binom{\bar{\partial}}{0}"'] & B \oplus B' \arrow[r, "(0\ \bar{\partial}')"'] & B'; \end{tikzcd}$$

Conclude in particular, that the sequence is exact.

5. Triangulated categories.

(5.1) Setup. Fix an additive category $\mathfrak{K}$. When an additive automorphism $\Sigma\colon \mathfrak{K} \to \mathfrak{K}$ is given, we will often write $X(k) := \Sigma^k X$ for the powers of Σ, and for a morphism $u\colon X \to Y$, we write simply $u\colon X(k) \to Y(k)$ for the morphism $u(k) = \Sigma^k(u)$. A *triangle* (with respect to the given automorphism Σ) is a 6-tuple $(X, Y, Z; u, v, w)$ of three objects X, Y, Z and three morphisms $u\colon X \to Y$, $v\colon Y \to Z$, and $w\colon Y \to X(1)$. We will indicate by the notation $f : U \rightsquigarrow V$ that f is a morphism $f : U \to V(1)$. In this notation, a triangle may be pictured by diagrams,

$$X \xrightarrow{u} Y \xrightarrow{v} Z \overset{w}{\rightsquigarrow} X, \qquad\qquad \text{or} \qquad \begin{tikzcd} & Z \arrow[dl, rightsquigarrow, "w"'] & \\ X \arrow[rr, "u"] & & Y. \arrow[ul, "v"'] \end{tikzcd}$$
$$X \xrightarrow[u]{} Y \xrightarrow[v]{} Z \xrightarrow[w]{} X(1),$$

Note that a triangle induces an infinite sequence,

$$\cdots \to X(-1) \to Y(-1) \to Z(-1) \to X \to Y \to Z \to X(1) \to Y(1) \to Z(1) \to \cdots .$$

A *morphism of triangles* $(X, Y, Z; u, v, w) \to (X', Y', Z'; u', v', w')$ is a triple (x, y, z) of morphisms $x\colon X \to X'$, $y\colon Y \to Y'$, and $z\colon Z \to Z'$ such that the obvious squares commute:

$$\begin{tikzcd} & Y \arrow[rr, "y"] \arrow[dl, "v"'] & & Y' \arrow[dl, "v'"'] \\ Z \arrow[rr, "z"] \arrow[dr, rightsquigarrow, "w"] & & Z' \arrow[dr, rightsquigarrow, "w'"] & \\ & X \arrow[uu, "u"] \arrow[rr, "x"'] & & X'. \arrow[uu, "u'"'] \end{tikzcd}$$

(5.2) Definition. The additive category $\mathfrak{K}$ is said to be *triangulated* if there is given an additive automorphism Σ of $\mathfrak{K}$, called the *shift functor* (or the *translation functor* or the *suspension functor*), and a class of triangles (with respect to Σ), called the *exact triangles* (or the *distinguished triangles*), such that the following conditions hold:

(TR 1) (a) Any triangle isomorphic to an exact triangle is exact.

(b) Any morphism $u\colon X \to Y$ embeds in an exact triangle $(X, Y, Z; u, v, w)$.

(c) For any object X, the triangle $(X, X, 0; 1, 0, 0)$ is exact:

$$\begin{tikzcd} & 0 \arrow[dl, rightsquigarrow, "0"'] & \\ X \arrow[rr, equal] & & X. \arrow[ul, "0"'] \end{tikzcd}$$

(TR 2) (*Rotation Axiom*). A triangle $(X, Y, Z; u, v, w)$ is exact, if and only if the triangle $(Y, Z, X(1); v, w, -u)$ exact:

$$\begin{tikzcd} & Z \arrow[dl, rightsquigarrow, "w"'] & \\ X \arrow[rr, "u"'] & & Y, \arrow[ul, "v"'] \end{tikzcd} \qquad \begin{tikzcd} & Z \arrow[dl, "w"'] & \\ X(1) \arrow[rr, rightsquigarrow, "-u"'] & & Y. \arrow[ul, "v"'] \end{tikzcd}$$

(TR 3) (*Prism Axiom*). For any two triangles $(X, Y, Z; u, v, w)$ and $(X', Y', Z; u', v', w')$ and morphisms $x\colon X \to X'$ and $y\colon Y \to Y'$ such that $u'x = yu$, there exists a morphism $z\colon Z \to Z'$ such that (x, y, z) is a morphisms of triangles:

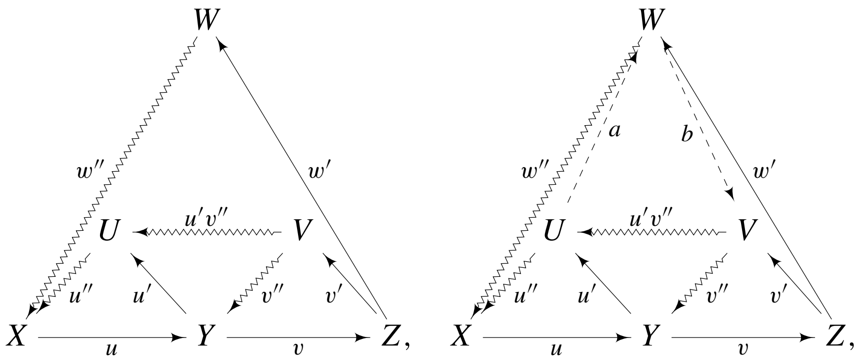
$$\begin{tikzcd}
Y \arrow[r, "y"] \arrow[d, "v"'] & Y' \arrow[d, "v'"] \\
Z \arrow[r, dashed, "z"] \arrow[d, squiggly, "w"'] & Z' \arrow[d, squiggly, "w'"] \\
X \arrow[r, "x"'] \arrow[uu, bend left, "u"] & X'. \arrow[uu, bend right, "u'"']
\end{tikzcd}$$

(TR 4) (*Octahedron Axiom*). Consider two composable morphisms $u\colon X \to Y$ and $v\colon Y \to Z$, and the composition $w = vu\colon X \to Z$. Assume that the morphisms are embedded in exact triangles $(X, Y, U; u, u', u'')$, $(Y, Z, V; v, v', v'')$, and $(X, Z, W; w, w', w'')$. Consider the composition $u'v''\colon V \to Y(1) \to U(1)$. Then there are two morphisms $a\colon U \to W$ and $b\colon W \to V$ such that

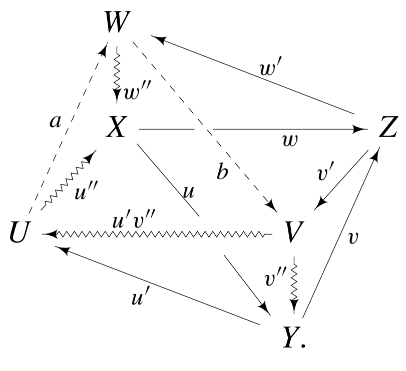
(i) the triangle $(U, W, V; a, b, u'v'')$ is exact, and

(ii) the following equalities hold: $w''a = u''\colon V \to X(1)$, $bw' = v'\colon Z \to V$, $au' = w'v\colon V \to W$, and $uw'' = v''b\colon W \to Y(1)$.

The morphisms in the axiom may be pictured as the edges in the following diagrams:



or as the edges of the following octahedron (with $w :== vu$ on a separate edge):



Of the eight faces of the octahedron, four are exact triangles and four are commutative triangles; of the three diagonal squares, two are commutative, and in the third square $UXZV$ the composition of any two consecutive morphisms is equal to zero.

A functor $T\colon \mathfrak{K} \to \mathfrak{K}'$ between triangulated categories is said to be *triangular* or *exact*, if it commutes with the shifts and transforms exact triangles to exact triangles. A functor $H\colon \mathfrak{K} \to \mathfrak{A}$ from a triangulated category to an ablian category is said to be *cohomological* or *exact*, if it transforms exact triangles to exact sequences, that is, if for any exact triangle $(X, Y, Z; u, v, w)$ in $\mathfrak{K}$ the sequence $H(X) \to H(Y) \to H(Z)$ is exact in $\mathfrak{A}$.

(5.3). In the rest of this section we assume that a triangulation in $\mathfrak{K}$ is given. Let $u\colon X \to Y$ be a morphism. By Axiom (5.2)(1)(b), the morphism v embeds into an exact triangle,

$$\begin{tikzcd} & Z \arrow[dl, squiggly, "w"'] & \\ X \arrow[rr, "u"] & & Y. \arrow[ul, "v"'] \end{tikzcd} \tag{5.3.1}$$

In analogy with the case of complexes we will often say that the triangle (5.3.1) is a *cone* for the morphism u, or sometimes even that the top vertex Z is a *cone* for u. If a second cone for u is given, then by the Prism Axiom, applied with $X = X'$ ($x = 1_X$) and $Y = Y'$ ($y = 1_Y$), there exists a morphism $z\colon Z \to Z'$ such that $(1_X, 1_Y, z)$ is a morphism of triangles $(X, Y, Z) \to (X, Y, Z')$. It follows from Corollary (5.7) below that $z\colon Z \to Z'$ is necessarily an isomorphism. So, a cone of u is determined up to isomorphism. But it should be emphasized that the isomorphism is not unique, and, strictly speaking, no triangle should be called *the* cone of u.

If two of the three morphism in a triangle are multiplied by -1, then the resulting triangle is isomorphic to the original triangle. Indeed, an isomorphism is determined by multipliction by -1 in one of the three vertices. In particular, if the original triangle is exact, then so is the resulting triangle. If the exact triangle (5.3.1) is rotated three times as described in Axiom (TR 2), then the result is the exact triangle with the three vertices shifted 1, and the three morphisms multiplied by -1. The resulting triangle is also exact if two of its morphisms are again multiplied by -1, leaving a sign change on only one of the original morphisms. So the following triangle is a cone for the shifted morphism $u\colon X(1) \to Y(1)$:

$$\begin{tikzcd} & Z(1) \arrow[dl, squiggly, "-w"'] & \\ X(1) \arrow[rr, "u"] & & Y(1). \arrow[ul, "v"'] \end{tikzcd} \tag{5.3.2}$$

(5.4) Lemma. *Let $(X, Y, Z; u, v, w)$ be an exact triangle. Then a morphism $f\colon Y \to A$ extends to a morphism $\tilde{f}\colon Z \to A$, if and only if $fu = 0$. Similarly, at morphism $g\colon B \to X$ lifts to a morphism $\tilde{g}\colon B \to Z(-1)$, if and only if $ug = 0$:*

$$\begin{tikzcd} & Z \arrow[dl, squiggly, "w"'] \arrow[drr, dashed, "\tilde{f}"] & & \\ X \arrow[r, "u"'] & Y \arrow[u, "v"'] \arrow[r, "f"'] & A. \end{tikzcd} \qquad \begin{tikzcd} & & Z(-1) \arrow[dl, "w"] & \\ B \arrow[urr, dashed, "\tilde{g}"] \arrow[r, "g"'] & X \arrow[rr, "u"'] & & Y. \arrow[ul, squiggly, "-v"'] \end{tikzcd}$$

Proof. To prove the only if part it suffices to prove that $vu = 0$. The vanishing follows by applying the prism axiom to the triangles $(X, X, 0; 1, 0, 0)$ and $(X, Y, Z; u, v, w)$, with $x := 1\colon X \to X$ and $y := u\colon X \to Y$.

Conversely, assume that $fu = 0$. The extension $\tilde{f}$ is obtained by applying The Prism Axiom to the triangles $(X, Y, Z; u, v, w)$ and $(0, A, A; 0, 1, 0)$ (the latter is exact by axioms 1(b) and 2), with $x := 0\colon X \to 0$ and $y := f\colon Y \to A$. □

(5.5) Comment. The cone $Z = \operatorname{Con} f$ of a morphism $f\colon X \to Y$ in a triangulated category may in many ways be seen as a (poor) substitute for the kernel/cokernel pair of a morphism in an abelian category. For instance, (5.4) shows that the cone Z has the "versal" property of a cokernel of u and that $Z(-1)$ has the versal property of a kernel. For a composition gf, part of the octahedron axiom asserts the exactnes of a triangle,

$$\begin{tikzcd} & \operatorname{Con} g \arrow[dl, squiggly] & \\ \operatorname{Con} f \arrow[rr] & & \operatorname{Con} gf. \arrow[ul] \end{tikzcd}$$

It should be seen as the analogue of the exact kernel–cokernel sequence of a composition in an abelian category.

(5.6) Proposition. *For any exact triangle $(X, Y, Z; u, v, w)$ and any object A, the following two long sequences are exact:*

$$\cdots \to \operatorname{Hom}(X(1), A) \to \operatorname{Hom}(Z, A) \to \operatorname{Hom}(Y, A) \to \operatorname{Hom}(X, A) \to \cdots$$
$$\cdots \to \operatorname{Hom}(A, X) \to \operatorname{Hom}(A, Y) \to \operatorname{Hom}(A, Z) \to \operatorname{Hom}(A, X(1)) \to \cdots .$$

Proof. That the first sequence is exact at $\operatorname{Hom}(Y, A)$ is the contents of the Lemma. It follows, by repeated application of the Rotation Axiom, that the first sequence is exact everywhere. By duality, or by an analogous proof, the second sequence is exact. □

(5.7) Corollary. *If, in a morphism (x, y, z) of exact triangles, two of the morphisms are isomorphisms, then so is the third.*

A morphism $u\colon X \to Y$ is an isomorphism if and only if its cone is zero.

Proof. The first assertion follows from exactness of (say) the second long exact sequence. Indeed, assume that (x, y, z) is a morphism from $(X, Y, Z; u, v, w)$ to $(X', Y', Z'; u', v', w')$ and that x and y are isomorphisms. The long exact sequences for the first and the second triangle are the rows in two-row diagram whose vertical map are induced by x, y and z. Consider in the diagram the map $z_*\colon \operatorname{Hom}(A, Z) \to \operatorname{Hom}(A, Z')$ induced by z. Its neighbors in the diagram, two to the left and two to the right, are isomorphisms, being induced by u or v. Therefore, by the Five-lemma, z_* is a bijection. Since A was an arbitrary object of $\mathfrak{K}$, it follows that z is an isomorphism.

Let Z be a cone of $u\colon X \to Y$ and let Z' be a cone of $u'\colon X' \to Y'$. Assume there is given an isomorphism x, y from $u\colon X \to Y$ to $u'\colon X' \to Y'$. By the prism axiom, x, y

extend to a morphism of triangles (x, y, z), and by the first part of the proof, $z\colon Z \to Z'$ is an isomorphism.

If u is an isomorphism, we may take $u' = 1_Y$ and $x = u$, $y = 1_Y$. By Axiom (1)(c), we may take $Z' = 0$. As z is an isomorphism it follows that $Z = 0$. Conversely, if $Z = 0$, then, in the exact sequence, every third group is zero. Hence the map $\mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y)$ is zero. Since A was arbitrary, it follows that $u\colon X \to Y$ is an isomorphism. □

(5.8) Example. *The cone of the zero morphism* $X \xrightarrow{0} Y$ *is equal to* $X(1) \oplus Y$.

The precise version is the following statement: The triangle $(X, Y, Z; 0, i, p)$, where $Z := X(1) \oplus Y$ and $i\colon Y \to Z$ and $j\colon Z \to X(1)$ are the natural morphisms, is exact.

To prove the statement, consider a cone $(X, Y, Z; 0, v, w)$ for the zero morphism. Then, in the second long exact sequence of (5.6), every third map is the zero map. Consequently, the following sequence is exact for every object A:

$$0 \to \mathrm{Hom}(A, Y) \to \mathrm{Hom}(A, Z) \to \mathrm{Hom}(A, X(1)) \to 0. \qquad (*)$$

By the surjectivity in (*), for $A := X(1)$, there is a morphism $\iota\colon X(1) \to Z$ with $w\iota = 1$. Then the following diagram is commutative:

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "i"] \arrow[d, equal] & X(1) \oplus Y \arrow[r, "p"] \arrow[d, "{(\iota, v)}"'] & X(1) \arrow[r] \arrow[d, equal] & 0 \\ & Y \arrow[r, "v"] & Z \arrow[r, "w"] & X(1). & \end{tikzcd}$$

Apply the functor $\mathrm{Hom}(A, -)$ to the diagram. In the result, the top row split exact, and the bottom row is the short exact sequence (*). So, by the 5-lemma the middle vertical map $\mathrm{Hom}(A, X(1) \oplus Y) \to \mathrm{Hom}(A, Z)$ is bijective. Therefore the morphism $X(1) \oplus Y \to Z$ is an isomorphism.

(5.9) Lemma. *Any pair of morphisms* $x\colon X \to X'$ *and* $u\colon X \to Y$ *with the same source can be completed with a pair* $y\colon Y \to Y'$ *and* $u'\colon X' \to Y'$ *to a commutative square* ($yu = u'x$) *such that the morphisms* u *and* u' *have the same cone and the morphisms* x *and* y *have the same cone. The dual conclusion holds in the dual setup, that is, when the pair* (y, u') *is given.*

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "u"'] & X' \arrow[d, dashed, "u'"] \\ Y \arrow[r, dashed, "y"] & Y', \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, dashed, "x"] \arrow[d, dashed, "u"'] & X' \arrow[d, "u'"] \\ Y \arrow[r, "y"] & Y'. \end{tikzcd}$$

Proof. Here are two constructions: To prove the first assertion, define Y' as the cone of $(u\,x)^{\mathrm{tr}}\colon X \to Y \oplus X'$, with the two morphisms $Y \oplus X' \to Y'$ and $Y \oplus X' \rightsquigarrow X$. Let $(y, -u')$ denote the two coordinates of the morphism $Y \oplus X' \to Y'$. The the first square is commutative, since the composition $(y\, -u')(u\,x)^{\mathrm{tr}}$ is the zero morphism. The composition

$(0\,1)(u\,x)^{\mathrm{tr}}$ is equal to X. Let X'' be the cone of x, and apply the octahedron axiom to obtain the following octahedral diagram:

$$\begin{tikzcd}
 & & X'' \arrow[ddll, squiggly] \arrow[dr, dashed, "b"] & & \\
 & Y' \arrow[ur, dashed, "a"] \arrow[dl, squiggly] & & Y(1) \arrow[ll, squiggly, "y"'] \arrow[dl, squiggly] & \\
X \arrow[rr, "\binom{u}{x}"'] & & Y \oplus X' \arrow[ul, "(y\ -u')"] \arrow[rr, "(0\,1)"'] & & X' \arrow[uull] \arrow[ul, "0"']
\end{tikzcd}$$

So X'' is the common cone of x and y.

The second assertion holds by duality arguments.

Here's a second construction: form a cone of $u\colon X \to Y$, and rotate it to obtain an exact triangle $(Z, X, Y; w, u, v)$. Consider the composition $xw\colon Z \to X \to X'$, and form a cone $(Z, X', Y'; xw, u', v')$. Finally, let V be a cone of x. Then there are morphism $y\colon Y \to Y'$ and $b\colon Y' \to V$ with the octahedral properties:

$$\begin{tikzcd}
 & & Y' \arrow[ddll, squiggly] \arrow[dr, dashed, "b"] & & \\
 & Y \arrow[ur, dashed, "y"] \arrow[dl, squiggly] & & V \arrow[ll, squiggly] \arrow[dl, squiggly] & \\
Z \arrow[rr, "w"'] & & X \arrow[ul, "u"] \arrow[rr, "x"'] & & X' \arrow[uull, "u'"'] \arrow[ul, "v'"']
\end{tikzcd}$$

In particular, $yu = u'x$, the morphisms u and u' have $Z(1)$ as their common cone, and the morphisms x and y have V as their common cone. □

(5.10) Definition. A class $\mathfrak{M} \subseteq \mathfrak{K}$ of objects is called a *triangular subclass* if it is nonempty and if, for any exact triangle in $\mathfrak{K}$, if two of the vertices belong to $\mathfrak{M}$, then so does the third. It is an implicit part of the condition that any object of $\mathfrak{K}$ isomorphic to an object of $\mathfrak{M}$ is itself in $\mathfrak{M}$.

The cone of $1\colon X \to X$ is the zero object. Hence, since $\mathfrak{M}$ is nonempty it follows that the zero object is in $\mathfrak{M}$. Moreover, if $X \in \mathfrak{M}$ it follows from the Rotation Axiom (TR 2) that $X(1)$ and $X(-1)$ are in $\mathfrak{M}$. It follows form Example (5.8) that the class is additive, that is, closed under finite direct sums. So, clearly, the full category determined by $\mathfrak{M}$ is a triangulated: the exact triangles are the exact triangles of $\mathfrak{K}$ with all three vertices in $\mathfrak{M}$.

If $\mathfrak{M}$ is a triangular subclass of $\mathfrak{K}$, a morphism $u\colon X \to Y$ is called an *$\mathfrak{M}$-morphism* if its cone belongs to $\mathfrak{M}$.

(5.11) Proposition. *If $\mathfrak{M} \subseteq \mathfrak{K}$ is a triangular subclass, then the system $S_{\mathfrak{M}}$ of all $\mathfrak{M}$-morphisms is a multiplicative denominator system in $\mathfrak{K}$.*

The assertion corresponds to the following statements about the system $S_{\mathfrak{M}}$:

(LOC 0) The system is multiplicative: every identity 1_X is an $\mathfrak{M}$-morphism and the composition $u'u$ of two $\mathfrak{M}$-morphisms $u\colon X \to X'$ and $u'\colon X' \to X''$ is an $\mathfrak{M}$-morphism.

The system $S_{\mathfrak{M}}$ has in fact an additonal property: if $u'u$ is an $\mathfrak{M}$-morphism, then u is an $\mathfrak{M}$-morphism if and only if u' is.

(LOC 1) The system has the following *left denominator property*: Any pair of morphisms $s\colon X \to X'$ and $f\colon X \to Y$ where s is an $\mathfrak{M}$-morphism may be completed to a commutative diagram,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[d, dashed, "s'"] \\ X' \arrow[r, dashed, "f'"] & Y', \end{tikzcd}$$

where s' is an $\mathfrak{M}$-morphism. And it has the corresponding *right denominator property*: conversely, if f' and s' are given with $s' \in S_{\mathfrak{M}}$, then they may be completed to the commutative diagram with $s \in S_{\mathfrak{M}}$.

(LOC 2) The system has the *left equalizer property*: If two morphisms $f, g\colon X \to Y$ are equalized by an $\mathfrak{M}$-morphism s, say $s\colon X' \to X$ with $fs = gs$, then they are coequalized by an $\mathfrak{M}$-morphism s':

$$\begin{tikzcd} X' \arrow[r, "s"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, dashed, "s'"] & Y'. \end{tikzcd}$$

And it has the corresponding *right equalizer property*: conversely, if two morphisms f, g are coequalised by an $\mathfrak{M}$-morphism, then they are equalised by an $\mathfrak{M}$-morphism.

The following two conditions are natural for denominator system S in a triangulated category:

(LOC 3) (1) A morphism $s\colon X \to X'$ belongs to S, if and only if its shift $s(1)\colon X(1) \to X'(1)$ belongs to S.

(2) If, in a morphism (x, y, z) of exact triangles, two of the morphisms belong to S, then so does the third.

Clearly, the system $S_{\mathfrak{M}}$ has the property (LOC 3)(1). The property (2) is much more delicate, and it does not hold for a general class $\mathfrak{M}$.

Proof of the denominator properties. (LOC 0): The cone of the identity 1_X is, by Axiom (1)(c), the zero object, and it belong to $\mathfrak{M}$. Hence $1_X \in S_{\mathfrak{M}}$.

The cones of the three morphisms u, u', and $u'' := u'u$ fit, by the octahedron axiom, into an exact triangle. Hence, if two of the three morphsims are in $S_{\mathfrak{M}}$, then so is the third.

(LOC 1): Consider the cone of $s\colon X \to X'$ and rotate it to obtain an exact triangle $(U, X, X'; u, s, w)$. Let v be the composition $v = fu\colon U \to Y$, and let Y' be its cone. Then we have the following diagram with two exact triangles:

$$
\begin{tikzcd}
& X \arrow[rr, "f"] \arrow[dd, "s"] & & Y \arrow[dd, "s'"] \\
U \arrow[ur, "u"] \arrow[rr, equal] & & U \arrow[ur, "fu"] & \\
& X' \arrow[ul, squiggly] \arrow[rr, dashed, "f'"'] & & Y'. \arrow[ul, squiggly]
\end{tikzcd}
$$

Use the prism axiom to complete 1_U and fu with a morphism f' to a morpyhism of triangles. Then, in particular, the square with the morphisms s, f, s', f' is commutative. Morever, the cone of s' is $U(1)$ which is in $\mathfrak{M}$. Hence $s' \in S_{\mathfrak{M}}$. So the required square has been obtained.

The right denominator property is proved similarly. Or it may be noticed that it follows by duality.

(LOC 2): To coequalize f and g it suffices to coequalize $f - g$ and 0; hence we may assume that $g = 0$. Assume that $X' \xrightarrow{s} X \xrightarrow{f} Y$ is the zero morphism. Let M be the cone of s. It follows from Lemma (5.4) that f factors over M as a product $f\colon X \xrightarrow{a} M \xrightarrow{b} Y$. Embedd b into an exact triangle $(M, Y, Y''; b, s', c)$:

$$
\begin{tikzcd}
& & Y' \arrow[dl, squiggly] & \\
X \arrow[r, "a"] & M \arrow[rr, "b"] & & Y. \arrow[ul, "s'"']
\end{tikzcd}
$$

Then $s' \in S_{\mathfrak{M}}$, because $M \in \mathfrak{M}$ and as $s'b = 0$, it follows that $s'f = s'ba = 0$.

(LOC 3): If X'' is the cone of $s\colon X \to X'$ then $X''(1)$ is the cone $s(1)$. Hence, $s \in S_{\mathfrak{M}}$ if and only if $s(1) \in S_{\mathfrak{M}}$.

The last assertion ought to be a consequence of octahedron axiom. When z is of the form discussed in (5.15) below, the proof is easy. ▯

(5.12) Note. The definition of the system $S_{\mathfrak{M}}$ of $\mathfrak{M}$-morphisms makes sense for an arbitrary subclass $\mathfrak{M}$ of $\mathfrak{K}$: A morphism $s\colon X \to X'$ belongs to $S_{\mathfrak{M}}$ if its cone belongs to $\mathfrak{M}$. Conversely, to any system S of morphisms of $\mathfrak{K}$ there is associated class $\mathfrak{Z}(S)$ of *S-acyclic objects*: An object Z belongs to $\mathfrak{Z}(S)$ if the zero morphism $0 \to Z$ belongs to S.

Obviously, if $\mathfrak{M}$ is given, then for any object M of $\mathfrak{K}$ we have that $M \in \mathfrak{M}$ if and only if the zero-morphism $0 \to M$ is an $\mathfrak{M}$-morphism.

If S contains all identities and satisfies (LOC 3), then for any morphism $s\colon X \to X'$ we have that $s \in S$ if and only if the cone of s is S-acyclic. Indeed, there is a morphism of exact triangles,

$$
\begin{tikzcd}
& X \arrow[dl] \arrow[rr, "s"] \arrow[dd, equal] & & X' \arrow[dl] \\
0 \arrow[rr] \arrow[dr, squiggly] & & X'' \arrow[dr, squiggly] & \\
& X \arrow[rr, equal] & & X, \arrow[uu, "s"']
\end{tikzcd}
$$

and the lower horizontal identity morphism 1_X is in S. Hence the zero-morphism $0 \to X''$ belongs to S if and only if s belongs to S.

It is easy to see under this correspondence that systems of morphisms of $\mathfrak{K}$ satisfying the four conditions (LOC 0)–(LOC 3) correspond to triangular subclasses of objects of $\mathfrak{K}$.

(5.13) Note. *The Octahedron Axiom implies the Prism Axiom*, assuming the axioms (TR 1) and (TR 2). Indeed, consider as in (TR 2) a commutative square, say two morphisms $u\colon X \to Y$ and $u'\colon X' \to Y'$ and a morphism (x, y) from u to u'. Embed the morphisms at each of the four sides in an exact triangle. In addition, embed the composition $p := u'x = yu$ in an exact triangle $(X, Y', W; p, q, r)$:

$$
\begin{tikzcd}
 & & Z \arrow[dl, squiggly, "w"'] \arrow[dd, dashed, "a"] & & \\
 & X \arrow[rr, "u"] \arrow[dd, "x"] & & Y \arrow[ul, "v"'] \arrow[dd, "y"] & \\
X'' \arrow[ur, squiggly, "x''"] \arrow[rr, dashed, "c"] & & W \arrow[ul, squiggly, "r"] \arrow[rr, dashed, "b"] \arrow[dd, dashed, "d"] & & Y'' \arrow[ul, squiggly, "y''"'] \\
 & X' \arrow[ul, "x'"] \arrow[rr, "u'"] & & Y' \arrow[ul, "q"'] \arrow[ur, "y'"'] \arrow[dl, "v'"] & \\
 & & Z' \arrow[ul, squiggly, "w'"] & &
\end{tikzcd}
\tag{5.13.1}
$$

Apply the octahedron axiom to the composition $p = yu$ to obtain the morphisms a and b. In particular, by the commutativity asserted in the axiom, $ra = w$, $bq = y'$, $av = qy$, and $ur = y''b$. Similarly, apply the axiom to $p = u'x$ to obtain the morphisms c and d with $rc = x''$, $dq = v'$, $xr = w'd$, and $cx' = qu'$.

Consider the compositions,

$$z = da\colon Z \to Z'.$$

It follows in particular that

$$w'z = w'da = xra = xw, \text{ and } zv = dav = dqy = v'y.$$

Thus (x, y, z) is a morphism of triangles, and the Prism Axiom has been proved).

(5.14) The cone of the cones. Continue with the setup of Section (5.13): Z and Z' are the cones of the horizontal morphisms in the square, and the morphism $z = da\colon Z \to Z'$ completes (x, y) to a morphism of triangles. Similarly, X'' and Y'' are the cones of the vertical morphisms, and the morphism $u'' := bc\colon X'' \to Y''$ completes (u, u') to a morphism of triangles. In addition, form the *cone of the horizontal cones*, that is, embedd $z\colon Z \to Z'$ into an exact triangle $(Z, Z', Z''; z, z', z'')$. The morphisms appear in the following diagram of exact triangles and commutative squares,

$$
\begin{tikzcd}
 & X \arrow[rr, "u"] \arrow[dd, "x"] & & Y \arrow[rr, "v"] \arrow[dd, "y"] & & Z \arrow[rr, "w"] \arrow[dd, "z"] & & X(1) \arrow[dd, "x"] \\
X'' \arrow[ur, squiggly, "x''"] \arrow[rr, "u''"] & & Y'' \arrow[ur, squiggly, "y''"] & & Z'' \arrow[ur, squiggly, "z''"] & & X''(1) \arrow[ur, squiggly, "-x''"] & \\
 & X' \arrow[rr, "u'"'] \arrow[ul, "x'"] & & Y' \arrow[rr, "v'"'] \arrow[ul, "y'"] & & Z' \arrow[rr, "w'"'] \arrow[ul, "z'"] & & X'(1).
\end{tikzcd}
$$

We claim that there are two morphism $v'' \colon Y'' \to Z''$ and $w'' \colon Z'' \rightsquigarrow X''$ such that, in the following diagram, (1) the horizontal tripples (v, v', v'') and (w, w', w'') are morphisms of triangles, and (2) the triangle $(X'', Y'', Z''; u'', v'', z'')$ is exact:

$$
\begin{tikzcd}
 & X \arrow[rr, "u"] \arrow[dd, "x"] & & Y \arrow[rr, "v"] \arrow[dd, "y"] & & Z \arrow[rr, "w"] \arrow[dd, "z"] & & X(1) \arrow[dd, "x"] \\
X'' \arrow[ur, squiggly, "x''"] \arrow[rr, "u''"] & & Y'' \arrow[ur, squiggly, "y''"] \arrow[rr, "v''"] & & Z'' \arrow[ur, squiggly, "z''"] \arrow[rr, squiggly, "w''"] & & X''(1) \arrow[ur, squiggly, "-x''"] & \\
 & X' \arrow[rr, "u'"'] \arrow[ul, "x'"] & & Y' \arrow[rr, "v'"'] \arrow[ul, "y'"] & & Z' \arrow[rr, "w'"'] \arrow[ul, "z'"] & & X'(1).
\end{tikzcd}
$$

Indeed, consider the composition $z = da$. The cone of d is determined by rotation from the exact triangle $(X'', W, Z'; c, d, x'w')$. Application of the octahedron axiom yields the two morphisms v'' and w'' as in the following diagram,

$$
\begin{tikzcd}
 & & Z'' \arrow[ddll, squiggly, "z''"'] \arrow[dr, dashed, "w''"] & & \\
 & Y'' \arrow[ur, dashed, "v''"] \arrow[dl, squiggly, "vy''"] & & X''(1) \arrow[ll, squiggly, "-u''"'] \arrow[dl, squiggly, "-c"] & \\
Z \arrow[rr, "a"'] & & W \arrow[rr, "d"'] \arrow[ul, "b"] & & Z'. \arrow[ul, "x'w'"] \arrow[uull, "z'"']
\end{tikzcd}
$$

with the commutation equations,

$$
z''v'' = vy'', \quad w''z' = x'w', \quad v''b = z'd, \quad -cw'' = az'', \tag{5.14.2}
$$

and such the the triangle $(Y'', Z'', X''(1); v'', w'', -u'')$ is exact. From the exactness we deduce by rotation that the triangle $(X'', Y'', Z''; u'', v'', w'')$ is exact. The two first commutation equations state that two is the asserted squares are commutative. Commutativity of the remaining two squares follows from the last two equations in (5.14.2) and commutation properties of the diagram (5.13.1).

(5.15) Note. The *Enriched Octahedron Axiom* is the following: In the Octahedron setup the morphisms $a\colon U \to W$ and $b\colon W \to V$ can be chosen so that in addition the following two triangles are exact:

$$
\begin{tikzcd}
 & X(1) \oplus V \arrow[dl, rightsquigarrow, "(u\ v'')"'] & \\
Y \arrow[rr, "au'=w'v"'] & & W, \arrow[ul, "\binom{-w''}{b}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Z \oplus U \arrow[dl, "(w'\ -a)"'] & \\
W \arrow[rr, rightsquigarrow, "uw''=v''b"'] & & Y. \arrow[ul, "\binom{v}{u'}"']
\end{tikzcd}
$$

(5.16) Exercises.

1. Do $u\colon X \to Y$ and $-u\colon X \to Y$ have the same cone? [Hint: In what sense is the answer yes and in what sense is it no?]

2. Is the shift functor in a triangulated category a triangular functor?

3. Prove that first property i (LOC 3), the shift invariance, is a consequence of the second.

4. Prove that an exact functor $T : \mathfrak{K} \to \mathfrak{K}'$ between triangulated categories is additive. Prove that an exact functor $H : \mathfrak{K} \to \mathfrak{A}$ from a triangulated category to an abelian category is additive.

6. Spectral sequences.

(6.1) Setup. Fix an abelian category $\mathfrak{A}$. At several places we will meet a diagram in $\mathfrak{A}$ of the form,

$$\begin{tikzcd} & & G \arrow[d] & \\ A' \arrow[r] & F' \arrow[r] \arrow[d] & F \arrow[r] & A \\ & G', & & \end{tikzcd}$$

where the row is exact. The diagram induces a morphism,

$$\mathrm{Ker}(G \to A) \xrightarrow{\delta} \mathrm{Cok}(A' \to G'),$$

defined as the composition,

$$\mathrm{Ker}(G \to A) \to \mathrm{Im}(F' \to F) = \mathrm{Coim}(F' \to F) \to \mathrm{Cok}(A' \to G').$$

(6.2) The Spectral Lemma. *An exact commutative diagram,*

$$\begin{tikzcd} A'_0 \arrow[dr] & F'_0 \arrow[d] \arrow[dr] & G \arrow[d] & \\ A' \arrow[r] & F' \arrow[r] \arrow[d] \arrow[dr] & F \arrow[r] \arrow[d] \arrow[dr] & A \\ & G' & F_0 & A_0, \end{tikzcd}$$

induces an exact sequence,

$$0 \to \mathrm{Ker}(G \to A_0) \to \mathrm{Ker}(G \to A) \xrightarrow{\delta} \mathrm{Cok}(A' \to G') \to \mathrm{Cok}(A'_0 \to G') \to 0.$$

Proof. Clearly, the kernel $\mathrm{Ker}(G \to A_0)$ is unchanged if A_0 is replaced by the image of $F_0 \to A_0$. So we may assume that $F_0 \to A_0$ is epic. Similarly, we may assume that $F \to A$ is epic, and that $A'_0 \to F'$ and $A' \to F'$ are monic. Then, in particular, the diagram induces an epic $A \to A_0$ and a monic $A' \to A'_0$. The morphism $G \to A$ factors through $A_0 \to A$, and $A' \to G'$ factors through $A'_0 \to A'$.

Clearly, the first part of the sequence, the inclusion of two kernels,

$$0 \to \mathrm{Ker}(G \to A_0) \xrightarrow{i} \mathrm{Ker}(G \to A),$$

is exact. Moreover, the two kernels contain the kernel of $G \to F$. So, the cokernel of i is unchanged, if G is replaced by its image in F. Hence we may assume that $G \to F$ is a monomorphism. Similarly, we may assume that $F' \to G'$ is an epimorphism.

Consider the following diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(G{\to}A_0) \arrow[r] & \mathrm{Ker}(G{\to}A) \arrow[r] \arrow[d, equal] & \mathrm{Ker}(A_0{\to}A) \arrow[d, equal] & & \\
& & \mathrm{Cok}(A'{\to}A'_0) \arrow[r] & \mathrm{Cok}(A'{\to}G') \arrow[r] & \mathrm{Cok}(A'_0{\to}G') \arrow[r] & 0.
\end{tikzcd}$$

The top row in the diagram is the first part of the exact kernel-cokernel sequence of the composition $G \to F \to A$. Hence the top row is exact. Similarly, the bottom row is exact. The first vertical isomorphism is induced from the kernel-cokernel sequence of the composition $F' \to F \to F_0$. The second vertical isomorphism is induced similarly. The diagram is commutative. In fact, both compositions in the diagram, from $\mathrm{Ker}(G{\to}A)$ to $\mathrm{Cok}(A'{\to}G')$, are equal to the morphism of (6.1).

As a consequence, the asserted sequence is exact. ▯

(6.3) Remark. Note that any exact diagram,

$$\begin{tikzcd}
F'_0 \arrow[d] & G \arrow[d] \\
F' \arrow[r, "f"] \arrow[d] & F \arrow[d] \\
G' & F_0,
\end{tikzcd}$$

may be completed to a diagram as in the Spectral Lemma by adding first the compositions $F'_0 \to F' \to F$ and $F' \to F \to F_0$, and next the kernels and cokernels of the appropriate morphisms.

⋮

⋮

(6.20) Triangular filtrations. Consider in a triangular category $\mathfrak{K}$ an object X *with an increasing filtration*. By definition, the filtration of X consists of a sequence of exact triangles,

$$\begin{tikzcd}
& {} & & G_{q-1} \arrow[dl, squiggly] & & G_q \arrow[dl, squiggly] & & {} \arrow[dl, squiggly] & \\
\cdots \arrow[rr] & & F_{q-2} \arrow[ul] \arrow[rr] & & F_{q-1} \arrow[ul] \arrow[rr] & & F_q \arrow[ul] \arrow[rr] & & \cdots,
\end{tikzcd}$$

and a sequence of morphisms $F_q \to X$ compatible with the morphisms $F_{q-1} \to F_q$.

The filtration may more precisely be called a filtration *to* X. It determines a filtration *from* X as follows: Complete the morphism $F_q \to X$ to an exact triangle, the first of the following two:

$$\begin{tikzcd}
& \overline{F}_{q+1} \arrow[dl, squiggly] & \\
F_q \arrow[rr] & & X, \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
& G_q \arrow[dl] & \\
\overline{F}_q \arrow[rr] & & \overline{F}_{q+1}. \arrow[ul, squiggly]
\end{tikzcd}$$

The second triangle, also exact, is obtained by applying the octahedral axiom to the composition $F_{q-1} \to F_q \to X$. In addition to the exact triangle, a number of commutative diagrams result. In particular, the sequence of morphisms $X \to \overline{F}_q$ is compatible with the morphisms $\overline{F}_q \to \overline{F}_{q+1}$. Note that the triangles are only unique up to a noncanonical isomorphism.

(6.21) The spectral sequence of an increasing triangular filtration. *In the setup of* (6.20), *for any exact functor $T : \mathfrak{K} \to \mathfrak{A}$, from $\mathfrak{K}$ to an abelian category $\mathfrak{A}$, there is an induced* 2*-spectral sequence,*

$$E_2^{p,q} = T^{p+q} G_q \implies T^n X, \quad \textit{with} \quad F_q T^n X := \operatorname{Im}(T^n F_q {\to} T^n X). \tag{6.21.1}$$

The spectral sequence is defined as follows: For convenience, set $F_{-\infty} := 0$ and $F_\infty := X$. Complete, for every s, t with $-\infty \leqslant s \leqslant q \leqslant \infty$, the composition $F_s \to F_q$ to an exact triangle,

$$\begin{tikzcd}
& G_{s,q} \arrow[dl, squiggly] & \\
F_s \arrow[rr] & & F_q \arrow[ul]
\end{tikzcd}$$

The triangles for $s = q$ and for $s = -\infty$ are trivial: $G_{q,q} = 0$ and $G_{-\infty,q} = F_q$. For $s = q - 1$ they are taken to be the triangles of the given filtration: $G_{q-1,q} = G_q$. Finally, for $q = \infty$, the triangles are those considered in (6.20): $G_{q,\infty} = \overline{F}_{q+1}$.

Some of the morphisms in the triangles appear in the following *basic* commutative diagram, for $s + 1 \leqslant q - 1$:

$$\begin{tikzcd}
G_{s+1,q} \arrow[dr, squiggly] & F_s \arrow[d] \arrow[dr] & G_q \arrow[d, squiggly] & \\
G_{s+1,q-1} \arrow[r, squiggly] & F_{s+1} \arrow[r] \arrow[d] \arrow[dr] & F_{q-1} \arrow[r] \arrow[d] \arrow[dr] & G_{s+1,q-1} \\
& G_{s+1} & F_q & G_{s,q-1} .
\end{tikzcd}$$

Note that each of the sequences in the diagram, one horizontal, two vertical and two diagonal sequences, is formed by consecutive morphisms of an exact triangle.

Fix a pair p, q of integers, let $n := p + q$, and define

$$E^{p,q} = E_2^{p,q} := T^n(G_q). \tag{6.21.2}$$

Consider for $2 \leqslant r \leqslant \infty$ the two compositions, with G_q as source and target respectively,

$$G_q \rightsquigarrow F_{q-1} \longrightarrow G_{q-r+1,q-1} \quad \text{and} \quad G_{q,q+r-2} \rightsquigarrow F_q \longrightarrow G_q .$$

The first appears in the basic diagram for $s := q - r$, the second appears in the basic diagram for $s := q - 1$ and $q := q + r - 1$. Define, for $2 \leqslant r \leqslant \infty$,

$$Z_r^{p,q} := \operatorname{Ker}(T^n G_q {\to} T^{n+1} G_{q-r+1,q-1}) \quad \text{and} \quad \overline{Z}_r^{p,q} := \operatorname{Cok}(T^{n-1} G_{q,q+r-2} {\to} T^n G_q). \tag{6.21.3}$$

The definitions, for $r=\infty$, yield the following equations:

$$Z_\infty^{p,q} := \operatorname{Ker}(T^n G_q \to T^{n+1} F_q) \quad \text{and} \quad \overline{Z}_\infty^{p,q} := \operatorname{Cok}(T^{n-1}\overline{F}_{q+1} \to T^n G_q).$$

Consider the basic diagram, with $s := q-r$, and apply T^n, that is, apply T^n to the sources of the twisted morphisms and apply T^{n+1} to the remaining vertices. The result is an exact commutative diagram in $\mathfrak{A}$. By the Spectral Lemma (6.2), we obtain the exact sequence, for $2 \leqslant r < \infty$,

$$0 \to Z_{r+1}^{p,q} \to Z_r^{p,q} \xrightarrow{\ \delta\ } \overline{Z}_r^{p+r,q-r+1} \to \overline{Z}_{r+1}^{p+r,q-r+1} \to 0. \tag{6.21.4}$$

So the spectral shooting is defined. To define the "bridge" isomorphisms, consider the following commutative diagram in $\mathfrak{K}$:

$$\begin{tikzcd}
F_q \arrow[dr, equal] & F_{q-1} \arrow[d] \arrow[dr] & X \arrow[d, equal] & \\
\overline{F}_{q+1} \arrow[r, squiggly] & F_q \arrow[r] \arrow[d] \arrow[dr] & X \arrow[r] \arrow[d] \arrow[dr] & \overline{F}_{q+1} \\
& G_q & 0 & \overline{F}_q\,.
\end{tikzcd}$$

Its sequences are consecutive morphisms of exact triangles. By the Spectral Lemma (6.2), the following sequence is exact:

$$\begin{aligned}
0 \to \operatorname{Ker}(T^n X \to T^n \overline{F}_q) &\longrightarrow \operatorname{Ker}(T^n X \to T^n \overline{F}_{q+1}) \longrightarrow \\
&\longrightarrow \operatorname{Cok}(T^{n-1}\overline{F}_{q+1} \to T^n G_q) \longrightarrow \operatorname{Cok}(T^n F_q \to T^n G_q) \to 0.
\end{aligned}$$

As T is exact, the two kernels in the sequence are, respectively, equal to the images $\operatorname{Im}(T^n F_{q-1} \to T^n X)$ and $\operatorname{Im}(T^n F_q \to T^n X)$. The first cokernel in sequence is $\overline{Z}_\infty^{p,q} = E^{p,q}/B_\infty^{p,q}$. Again, since T is exact, the last cokernel is the quotient of $T^n G_q$ modulo $\operatorname{Ker}(T^n G_q \to T^{n+1} F_{q-1})$, and hence equal to $E^{p,q}/Z_\infty^{p,q}$. So the exact sequence is the following:

$$0 \to F_{q-1}T^n X \to F_q T^n \to E^{p,q}/B_\infty^{p,q} \to E^{p,q}/Z_\infty^{p,q} \to 0.$$

In particular, we obtain the "bridge" isomorphism asserted in the Spectral sequence (6.21.1):

$$F_q T^n X / F_{q-1} T^n X = Z_\infty^{p,q}/B_\infty^{p,q}. \tag{6.21.5}$$

Note. The 3-term of the spectral sequence is the cohomology of the complexes, for all p, formed by the 2-terms:

$$\cdots \longrightarrow E_2^{p-2,-p+1} \longrightarrow E_2^{p,-p} \longrightarrow E^{p+2,-p-1} \longrightarrow \cdots.$$

Clearly, for the spectral sequence (6.21.1), these complexes are obtained by applying shifts of T to the zero-sequence in $\mathfrak{K}$,

$$\cdots \rightsquigarrow G_1 \rightsquigarrow G_0 \rightsquigarrow G_{-1} \rightsquigarrow \cdots.$$

(6.22) Proposition. *The spectral sequence* (6.21.1) *is finitely convergent if, for all n and q, the morphisms $T^nF_s \to T^nF_q$ for $s \ll 0$ and $T^n\overline{F}_q \to T^n\overline{F}_s$ for $s \gg 0$ are equal to zero.*

The sequence if finite, if and only if, for all n, the morphisms $T^nF_{q-1} \to T^nF_q$ are isomorphisms for $q \ll 0$ and for $q \gg 0$. In particular, if $T^nF_q = 0$ for $q \ll 0$ and $T^nF_q \to T^nX$ is an isomorphism for $q \gg 0$, then the sequence is finite and convergent.

Proof. The morphism $T^nF_s \to T^nF_{q-1}$ (for $s < q$) is equal to zero if and only if the morphism $T^nF_{q-1} \to T^nG_{s,q-1}$ is a monomorphism. If the latter morphism, with $n := n+1$, is a monomorphism, then the morphisms, $T^nG_q \to T^{n+1}F_{q-1}$ and $T^nG_q \to T^{n+1}G_{s,q-1}$, have the same kernel. In other words, with $r := q - s + 1$ we have the equality,

$$Z_r^{p,q} = Z_\infty^{p,q}. \tag{6.22.1}$$

Clearly, if $T^nF_s \to T^nF_q$ is the zero morphism, then the image of $T^nF_s \to T^nX$ is equal to zero. In particular,

$$F_sT^nX = 0 \quad \text{for } s \ll 0. \tag{6.22.2}$$

Apply the octahedral axiom to the decomposition $F_q \to F_s \to X$, for $q < s$. In particular, we obtain a diagram,

$$\begin{tikzcd}
 & & \overline{F}_{q+1} \arrow[lld, squiggly] \arrow[d] & \\
F_q & G_{q,s} \arrow[l, squiggly] \arrow[ru] & \overline{F}_{s+1} \arrow[l, squiggly] & X, \arrow[l] \arrow[llu]
\end{tikzcd}$$

with an exact triangle in the middle, and two extreme commutative triangles. Apply T^{n-1} to the left commutative triangle. Consider the two morphisms $T^{n-1}G_{q,s} \to T^nF_q$ and $T^{n-1}\overline{F}_{q+1} \to T^nF_q$. When composed with $T^nF_q \to T^nG_q$ their images are $B_r^{p,q}$ and $B_\infty^{p,q}$ (where $s = q + r - 1$). Therefore, the equality,

$$B_r^{p,q} = B_\infty^{p,q}, \tag{6.22.3}$$

holds if $T^{n-1}\overline{F}_{q+1} \to T^nF_q$ and $T^{n-1}G_{q,s} \to T^nF_q$ have the same image. In particular, the equality (6.22.3) holds if the morphism $T^{n-1}G_{q,s} \to T^{n-1}\overline{F}_{q+1}$ is epic. As the middle triangle is exact, the latter morphism is epic, if and only if the morphism $T^{n-1}\overline{F}_{q+1} \to T^{n-1}\overline{F}_{s+1}$ is zero. So, if we assume that $T^{n-1}\overline{F}_{q+1} \to T^{n-1}\overline{F}_{s+1}$ is zero, then (6.22.3) holds. Moreover, then $T^{n-1}X \to T^{n-1}F_s$ is zero and, from the exact triangle connecting F_s, X, $\overline{F}_{s+1}$, it follows that the $T^{n-1}F_s \to T^{n-1}X$ is an epimorphism. In particular,

$$F_sT^{n-1}X = T^{n-1}X \quad \text{for } s \gg 0. \tag{6.22.4}$$

Hence, by (6.22.1) and (6.22.3) the shooting is finitely convergent, and by (6.22.2) and (6.22.4) the filtration on the abutment is finitely convergent.

The last assertion of the Proposition is easily verified. ∎

(6.23) Decreasing triangular filtrations. Consider, in the triangulated category $\mathfrak{K}$, a *decreasing* filtration *from* X:

$$\begin{tikzcd}
 & & & G^p \arrow[dl] & & & \\
X \arrow[r] & \dots \arrow[r] & F^p \arrow[rr] & & F^{p-1} \arrow[ul, rightsquigarrow] \arrow[r] & \dots \arrow[r] & 0.
\end{tikzcd}$$

For any exact functor $T : \mathfrak{K} \to \mathfrak{A}$ *there is an induced* 1*-spectral sequence,*

$$E_1^{pq} = T^{p+q}(G^p) \underset{p}{\Longrightarrow} T^n X, \text{ with } F^p T^n X := \operatorname{Ker}(T^n X \to T^n F^{p-1}).$$

The assertion follows from (6.21), or directly by a similar argument. The decreasing filtration on the abutment corresponds to the increasing filtration given by $F_q T^n X = F^{n-q} T^n X$.

(6.24) Examples. (1) Consider a complex X in $\mathfrak{A}^\bullet$. For an integer p, define the pth *right truncation* of X as the complex,

$$X^{p]}: \qquad \cdots \to X^{p-2} \to X^{p-1} \to X^p \to 0 \to 0 \to \cdots .$$

The *left truncation* $X^{[p}$ is defined similarly. Note that $X^{p]}$ is the quotient complex of X corresponding to the subcomplex $X^{[p+1}$. So, in the derived category there is an exact triangle,

$$\begin{tikzcd}
 & X^{p]} \arrow[dl, rightsquigarrow] & \\
X^{[p+1} \arrow[rr] & & X. \arrow[ul]
\end{tikzcd} \tag{6.24.1}$$

In fact, the morphism $X^{p]} \rightsquigarrow X^{[p+1}$ is given by a morphism of complexes: it is equal to $-\partial : X^p \to X^{p+1}$ in degree p and (necessarily) equal to zero in all other degrees. Is is easy to see *that the triangle is exact in the homotopy category*, that is, the triangle is homotopy equivalent to the cone of the inclusion $X^{[p+1} \to X$.

The right truncations $X^{p]}$ form a *decreasing* filtration *from* X in the homotopy category:

$$\begin{tikzcd}
 & & & X^p(-p) \arrow[dl] & & & \\
X \arrow[r] & \cdots \arrow[r] & X^{p]} \arrow[rr] & & X^{p-1]} \arrow[ul, rightsquigarrow] \arrow[r] & \cdots \arrow[r] & 0,
\end{tikzcd}$$

and the left truncations form a decreasing filtration *to* X:

$$\begin{tikzcd}
 & & & X^p(-p) \arrow[dl, rightsquigarrow] & & & \\
0 \arrow[r] & \cdots \arrow[r] & X^{[p+1} \arrow[rr] & & X^{[p} \arrow[ul] \arrow[r] & \cdots \arrow[r] & X.
\end{tikzcd}$$

Note that the exact triangles in the two filtrations are special case of the exact triangle (6.24.1): the definitions of the truncations yield $X^{[p]} = X^p(-p)$, and the triangle in, for instance, the last filtration is obtained from (6.24.1) with $X := X^{[p}$.

(2) For an integer q, define the qth *right cycle truncation* as the complex,

$$F_qX: \quad \cdots \to X^{q-2} \to X^{q-1} \to Z^q \to 0 \to 0 \to \cdots,$$

where Z^q is the kernel of $X^q \to X^{q+1}$. Note that F_qX is a subcomplex of X corresponding to the quotient complex,

$$\cdots \to 0 \to 0 \to B^{q+1} \to X^{q+1} \to X^{q+1} \to \cdots,$$

where $B^{q+1} = X^q/Z^q$ is the image of $X^q \to X^{q+1}$. There is a natural quasi-isomorphism from this quotient complex to following quotient complex:

$$\overline{F}_{q+1}: \quad \cdots \to 0 \to 0 \to \overline{Z}^{q+1} \to X^{q+2} \to X^{q+3} \to \cdots,$$

where $\overline{Z}^{q+1}$ (in degree $q+1$) is the cokernel, X^{q+1}/B^{q+1}, of $X^q \to X^{q+1}$. The latter complex is the $(q+1)$th *left cocycle truncation* of X. So, in the derived category of $\mathfrak{A}$, there is an exact triangle,

$$\begin{tikzcd} & \overline{F}_{q+1} \arrow[dl, squiggly] & \\ F_qX \arrow[rr] & & X. \arrow[ul] \end{tikzcd} \tag{6.24.2}$$

The F_qX form an *increasing* filtration *to* X,

$$\begin{tikzcd} & & & H^q(X)(-q) \arrow[dl, squiggly] & & & \\ 0 \arrow[r] & \cdots \arrow[r] & F_{q-1}X \arrow[rr] & & F_qX \arrow[ul] \arrow[r] & \cdots \arrow[r] & X, \end{tikzcd}$$

and the $\overline{F}_qX$ form the corresponding increasing filtration *from* X:

$$\begin{tikzcd} & & & H^q(X)(-q) \arrow[dl] & & & \\ X \arrow[r] & \cdots \arrow[r] & \overline{F}_qX \arrow[rr] & & \overline{F}_{q+1} \arrow[ul, squiggly] \arrow[r] & \cdots \arrow[r] & 0. \end{tikzcd}$$

(6.25) The spectral sequences of hyper cohomology. Let $T: \mathfrak{A} \to \mathfrak{B}$ be a derivable functor. Consider a right complex $X \in \mathfrak{A}^+$, and the two filtrations of (6.23). The (hyper) derived $RT(X)$ is an object in $D^+(\mathfrak{B})$, and its nth cohomology is the $n$the derived $R^nT(X)$. In particular, evaluated on objects of $\mathfrak{A}$ (as complexes concentrated in degree 0), it defines an additive functor $R^nT: \mathfrak{A} \to \mathfrak{B}$. As usual, denote by $(R^nT)^\bullet$ its extension to a functor of complexes $\mathfrak{A}^\bullet \to \mathfrak{B}^\bullet$.

From the spectral sequences in (6.21) and (6.23) we obtain:

(1) *an induced* 1*-spectral sequence,*

$$E_1^{pq} = R^qT(X^p) \underset{p}{\Longrightarrow} R^nTX, \ \textit{with } F^pR^nTX := \operatorname{Ker} R^nT(X \to X^{p-1]});$$

its 2*-term is* $E_2^{pq} = H^p((R^qT)^\bullet(X))$,

(2) *and an induced* 2*-spectral sequence,*

$$E_2^{pq} = R^pT(H^qX) \Longrightarrow R^nT(X), \ \textit{with } F_qR^nT(X) := \operatorname{Im} R^nT(F_qX \to X).$$

(6.26) The Spectral sequence of a composite functor. Consider derivable functors of abelian categories, $T: \mathfrak{A} \to \mathfrak{B}$ and $S: \mathfrak{B} \to \mathfrak{C}$. Assume that there is a class $\mathfrak{Q}$ of objects $Q \in \mathfrak{A}$ with the following properties:

(1) For every object $A \in \mathfrak{A}$ there is a monomorphism $A \to Q$ into an object $Q \in \mathfrak{Q}$.

(2) For every $Q \in \mathfrak{Q}$, Q is T-acyclic and TQ is S-acyclic.

Then, as is well-known, the composition $ST: \mathfrak{A} \to \mathfrak{C}$ is derivable, and

$$R(ST) = (RS)(RT). \tag{6.26.1}$$

As a consequence, for every complex $X \in \mathfrak{A}^+$, there is a 2-spectral sequence,

$$R^pS(R^qTX) \implies R^n(ST)(X). \tag{6.26.2}$$

Indeed, by (6.26.1), the abutment is $R^nS(TX)$, and the spectral sequence is that of (6.24)(1), with $T := S$ and $X := TX$.

Note. The spectral sequence for hyper Ext is obtained as follows: Recall that for arbitrary objects X, Y in a triangular category $\mathfrak{K}$, there are Ext-groups defined by

$$\operatorname{Ext}^n(Y, X) := \operatorname{Hom}_{\mathfrak{K}}(Y, X(n)).$$

The functor $TX = \operatorname{Hom}_{\mathfrak{K}}(Y, X)$, for a fixed object Y of $\mathfrak{K}$, is an exact functor $T: \mathfrak{K} \to (\mathbf{Ab})$. Hence, when a filtration of X is given, (6.21.1) yields a spectral sequence with abutment $\operatorname{Ext}^n(Y, X)$.

The Ext-groups of an abelian category $\mathfrak{A}$ are obtained by taking $\mathfrak{K} := D(\mathfrak{A})$. In particular: Let $T: \mathfrak{A} \to \mathfrak{B}$ be a derivable functor. Then, for complexes complex $X \in \mathfrak{A}^+$ and $Y \in \mathfrak{B}^\bullet$, there is an induced 2-spectral sequence,

$$E_2^{pq} = \operatorname{Ext}^p_{\mathfrak{B}}(Y, R^qTX) \implies \operatorname{Ext}^n_{\mathfrak{B}}(Y, RTX).$$

(6.27) Double complexes. Consider the category $(\mathfrak{A}^\bullet)^\bullet$ of complexes of complexes. An object X in $(\mathfrak{A}^\bullet)^\bullet$ is complex such that each component X^p is a complex (whose qth component is denoted X^{pq}); the differential in X is denoted $\partial': X^p \to X^{p+1}$ and the differential in the component X^p is denoted $\partial'': X^{pq} \to X^{p,q+1}$. Note that we have two shift operators: The shift $X \mapsto X(n,0)$ is the shift of the *complex* X; it shifts the position of the components X^p and multiplies ∂' by $(-1)^n$. The shift $X \mapsto X(0,n)$ shifts the position of the compentents of each X^p and multiplies *both* types of differentials by $(-1)^n$.

Recall that there is an identification of $(\mathfrak{A}^\bullet)^\bullet$ with the category $\mathfrak{A}^{\bullet\bullet}$ of bicomplexes. It identifies X with the bicomplex obtained by multiplying the differential in X^p by $(-1)^p$. We will write $(\mathfrak{A}^\bullet)^\bullet_\sharp$ for the subcategory of $(\mathfrak{A}^\bullet)^\bullet$ consisting of complexes of complexes X with only finitely many nonzero X^{pq} on each diagonal $p + q = n$. So, under the identification, $(\mathfrak{A}^\bullet)^\bullet_\sharp = \mathfrak{A}^{\bullet\bullet}_\sharp$. Note that $(\mathfrak{A}^\bullet)^\bullet_\sharp$ contains the subcategories

$$(\mathfrak{A}^\bullet)^{\text{bnd}}, \quad (\mathfrak{A}^{n]})^+, \qquad (\mathfrak{A}^{[m,n]})^\bullet.$$

The subcategory $(\mathfrak{A}^\bullet)^\bullet_\sharp$ gives rise to a triangulated homotopy category $\mathrm{Hot}_\sharp(\mathfrak{A}^\bullet)$ and a derived category $D_\sharp(\mathfrak{A}^\bullet)$. The natural functor Tot on $\mathfrak{A}^{\bullet\bullet}_\sharp$, may be viewed as a funtor,

$$\mathrm{Tot}\colon (\mathfrak{A}^\bullet)^\bullet_\sharp \to \mathfrak{A}^\bullet.$$

It respects the two shifts:

$$\mathrm{Tot}\big(X(n,0)\big) = (\mathrm{Tot}\,X)(n), \qquad \mathrm{Tot}\big(X(0,n)\big) = (\mathrm{Tot}\,X)(n),$$

and it preserves homotopy and cones. As a consequence, it defines an exact functor of triangulated categories,

$$\mathrm{Tot}\colon \mathrm{Hot}_\sharp(\mathfrak{A}^\bullet) \to \mathrm{Hot}(\mathfrak{A}).$$

Let X be a complex in $(\mathfrak{A}^\bullet)^\bullet_\sharp$, and consider the decreasing filtration of Example (6.24)(1). It induces the 1-spectral sequence of (6.23) for $T = H^0\,\mathrm{Tot}$. As $\mathrm{Tot}\,X^{[p]} = X^p(-p)$, the result is *a natural* 1*-spectral sequence,*

$$E_1^{pq} = H^q(X^p) \underset{p}{\Longrightarrow} H^n\,\mathrm{Tot}\,X, \text{ with } F^pH^n\,\mathrm{Tot}\,X = \mathrm{Ker}\,H^n\,\mathrm{Tot}(X \to X^{p-1]}). \quad (6.27.1)$$

The 2*-term is given by*

$$E_2^{pq} = H^p(H^q_{\prime\prime}X), \qquad (6.27.2)$$

where the notation for the inner qth cohomology indicates the complex with pth component equal to $H^q(X^p)$. It is easy to see that the spectral sequence is finitely convergent. In particular:

If a complex X in $(\mathfrak{A}^\bullet)^\bullet_\sharp$ has all components X^p acyclic, then Tot X *is acyclic.*

From this result, it follows in particular that the functor Tot takes quasi-isomorphisms into quasi-isomorphisms. Hence it extends to a functor,

$$\mathrm{Tot}\colon D_\sharp(\mathfrak{A}^\bullet) \to D(\mathfrak{A}).$$

This functor may be applied to the filtration in Example (6.24)(2) (note that H^qX, the qth cohomology of the complex $X \in (\mathfrak{A}^\bullet)^\bullet$, is an object in $\mathfrak{A}^\bullet$; it may also be denoted $H^q_\prime X$). The result is a 2-spectral sequence,

$$E_2^{pq} = H^p(H^qX) \implies H^n\,\mathrm{Tot}\,X \text{ with } F_qH^n\,\mathrm{Tot}\,X := \mathrm{Im}\,H^n\,\mathrm{Tot}(F_qX \to X). \quad (6.27.3)$$

This spectral sequence is different from the spectral 2-spectral sequence given in (6.27.2), and the two filtrations on their common abutment $H^h\,\mathrm{Tot}(X)$ are different. However, (6.27.2) is obtained by applying (6.27.3) to the transposed complex (of complexes) X^{tr}. Indeed, the two spectral sequences have the same 2-term. Moreover, under transposition, the cycle truncation $F_q(X^{\mathrm{tr}})$ of X^{tr} corresponds to the subcomplex $F_q''X$ of X obtained by applying the cycle truncation F_q to all components X^p of X. Note that

$$F_q''X \in (\mathfrak{A}^{q]})^\bullet, \qquad X^{[p} \in (\mathfrak{A}^\bullet)^{[p}.$$

These two families of subcomplexes of X define the same filtration on the target, that is, for all p, q and $n = p + q$,

$$\mathrm{Im}\,H^n\,\mathrm{Tot}(X^{[p} \to X) = \mathrm{Im}\,H^n\,\mathrm{Tot}(F_q''X \to X).$$

In fact, it is easy to see that the two complexes $\mathrm{Tot}\,X^{[p}$ and $\mathrm{Tot}\,F_q''X$, for $p + q = n$, have the same degree-n cohomomology. The general assertion, that the two sequences, with the same 2-term, are identical, at least up to automorphism, may be proved similarly.

7. Adjoint functors.

Fix a categori $\mathfrak{A}$.

8. Relative abelian categories.

(8.1) Definition. Let $\mathfrak{A}$ be a (right) *relative abelian category* (or (right) *relativized abelian category*?), that is, an abelian category $\mathfrak{A}$ with a given a (right) *allowable* class S of morphisms of $\mathfrak{A}$. A (right) allowable class S of morphisms is assumed at a minimum to satisfy the following two conditions:

(allow1) The class S is multiplicative, and closed under isomorphisms and (finite) direct sums.

(allow2) For any object A of $\mathfrak{A}$, the morphism $0 \to A$ belongs to S.

The morphisms in the given class S are also called the *relative monomorphisms* of $\mathfrak{A}$ (or said to be relatively monic). It follows from the conditions that every split monomorphism is relatively monic. It is not assumed that the relative monomorphisms are monic.

Several notions related to monomorphisms and exactness have relativized versions: A sequence in $\mathfrak{A}$,

$$A' \xrightarrow{u'} A \xrightarrow{u} A'', \tag{8.1.1}$$

is called *relatively exact* if it is a zero sequence and the induced morphism $\operatorname{Cok} u' \to A''$ is relatively monic. A *relatively short exact sequence* is a sequence,

$$0 \to A' \xrightarrow{u'} A \xrightarrow{u} A'' \to 0, \tag{8.1.2}$$

such that $u'\colon A' \to A$ is relatively monic and u is the cokernel of u' (that is, the sequence $A \to A' \to A'' \to 0$ is exact). Note that if the sequence (8.1.2) is relatively short exact, then it is relatively exact; the converse holds if the relative monomorphisms are monic.

Clearly, a complex

$$\cdots \longrightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \longrightarrow \cdots, \tag{8.1.3}$$

is *relatively exact* (or *relatively acyclic*) if and only if, for all n, the right exact sequence,

$$0 \to X^n/B^n \to X^{n+1} \to X^{n+1}/B^{n+1} \to 0, \tag{8.1.4}$$

(where $B^n := \operatorname{Im} d^{n-1}$) is relatively short exact. A morphism of complexes is a *relative quasi-isomorphism* if its mapping cone is relatively exact.

A functor $T\colon \mathfrak{A} \to \mathfrak{B}$, where also $\mathfrak{B}$ is a relative abelian category, is *relatively exact*, if it is additive and takes relatively short exact sequences of $\mathfrak{A}$ into relatively short exact sequences of $\mathfrak{B}$. The condition implies that T takes a relatively exact complex of $\mathfrak{A}$ into a relatively exact complex of $\mathfrak{B}$, but it does not imply in general that a relative exact sequence (8.1.1) is taken into a relative exact sequence of $\mathfrak{B}$.

An object Q of $\mathfrak{A}$ is called *relatively injective* if any pair of morphisms $f\colon A \to Q$ and $s\colon A \to A'$, where s is relatively monic, embed into a commutative diagram,

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "f"'] & B' \arrow[dl, dashed] \\ Q. \end{tikzcd}$$

Finally, a morphism $t\colon A \to Q$ in $\mathfrak{A}$ is said to be *relatively injective* if it is relatively monic and if any pair of morphisms $f\colon B \to A$ and $s\colon B \to B'$ with a relative monic s embed into a commutative diagram,

$$\begin{tikzcd} B \arrow[r, "s"] \arrow[d, "f"'] & B' \arrow[d, dashed] \\ A \arrow[r, "t"] & Q. \end{tikzcd}$$

According to the last definition, an object Q is relatively injective if and only if the identity of Q is relatively injective. A relative monomorphism $A \to Q$ into a relative injective object is relatively injective.

(8.2) Examples. A given abelian category $\mathfrak{A}$ may be relativized by taking as allowable class the class S_1 of all monics, or the class S_2 of all split monics, or the class of all morphisms.

Clearly, with respect to the class S_1, the relativized concepts are the usual concepts.

With respect to the class S_2, the sequence (8.1.1) is relatively exact if and only if there is a slitting $A = B \oplus C$, where $B = \operatorname{Im} u' = \operatorname{Ker} u$. The sequence (8.1.2) is relatively short exact if it is short exact and split, and the complex (8.1.3) is relatively exact if and only if it is contractible. In particular, a morphism of complexes is a relative quasi-isomorphism if and only if it is a homotopy equivalence. Finally, any object of $\mathfrak{A}$ is a relatively injective object, and the relative injective morphisms are the split monics.

With respect to the class S_3, any zero sequence (8.1.1) is relatively exact, and the sequence (8.1.2) is relatively short exact if and only if it is right exact. The only relatively injective object is the zero object, and the relative injective morphisms are the zero morphisms.

(8.3) Homotopy Lemma. *Let X be a relatively exact complex and let Q be a right complex consisting of relatively injective objects. Then any morphism of complexes $X \to Q$ is homotopic to zero.*

Proof. Insert 'relative' at appropriate places in the corresponding proof of the nonrelativized statement. □

(8.4) Corollary. *Let $s\colon X \to Y$ be a relative quasi-isomorphism of complexes and let $v\colon X \to Q$ be a morphism into a right complex Q consisting of relative injective objects. Then, in the homotopy category, there is a unique morphism $w\colon Y \to Q$ such that $v = ws$. In particular, any relative quasi-isomorphism $s\colon Q \to Q'$ between right complexes of relative injective objects is a homotopy equivalence.*

Proof. Let Z be the mapping cone of the morphism $s\colon X \to Y$. Then there is an exact triangle (of complexes of abelian groups),

$$\begin{tikzcd} & \operatorname{Hom}^\bullet(Z, Q) \arrow[dr] & \\ \operatorname{Hom}^\bullet(X, Q) \arrow[ur, squiggly] & & \operatorname{Hom}^\bullet(Y, Q), \arrow[ll] \end{tikzcd}$$

and part of its cohomology sequence is the exact sequence,

$$\operatorname{Hom}_{\mathrm{Hot}}(Z, Q) \to \operatorname{Hom}_{\mathrm{Hot}}(Y, Q) \to \operatorname{Hom}_{\mathrm{Hot}}(X, Q) \to \operatorname{Hom}_{\mathrm{Hot}}(Z(-1), Q).$$

The two extreme groups are zero by the Lemma. Hence, the middle homomorphism is bijective. Thus the first assertion holds. The second is an immediate consequence. □

(8.5) Definition. For any right exact functor $\lambda\colon \mathfrak{A} \to \mathfrak{A}_0$ there is an *induced relativization* of $\mathfrak{A}$: a morphism $u\colon A \to A'$ is a relative monomorphism if $\lambda(u)\colon \lambda A \to \lambda A'$ is a monomorphism in $\mathfrak{A}_0$. Alternatively, $\mathfrak{A}$ may be relativized via a left exact contravariant functor.

Clearly, if $\mathfrak{A}$ is relativized via λ, then the zero sequence (8.1.1) is relatively exact if and only if it becomes exact under λ, the sequence (8.1.2) is relatively short exact if and only if it is right exact and becomes short exact under λ, and the complex (8.1.3) is relatively exact if and only if it becomes exact under λ. In particular, a morphism of complexes is a relative quasi-isomorphism if and only if it becomes a quasi-isomorphism under λ.

Note that the three relativizations in Example (8.2) are of this form: The class S_1 with λ equal to the identity, the class S_3 with λ equal to the zero functor. Finally, the class S_2 is obtained with the contravariant functor,

$$\mathfrak{A} \to \mathrm{Funct}(\mathfrak{A}, (\mathrm{Ab})),$$

given by $A \mapsto \mathrm{Hom}(A, -)$. Indeed, $A \to A'$ is a split monomorphism if and only if $\mathrm{Hom}(A', X) \to \mathrm{Hom}(A, X)$ is surjective for all X in $\mathfrak{A}$.

It follows from this last observation that the *split relativization induced by* λ (for which the relative monomorphisms are the morphisms $s\colon A \to A'$ such that $\lambda(s)$ is a split monomorphism) may be induced by a right exact functor.

(8.6) Remark. In order to work properly with the homotopy categories of complexes over $\mathfrak{A}$, more properties of the class of relative monomorphisms are needed. Clearly, for complexes we need the following two properties:

(1) If $f\colon X \to Y$ is a homotopy equivalence and X is relatively acyclic, then Y is relatively acyclic.

(2) If, in an exact triangle,

$$\begin{tikzcd} & Z \arrow[dl, rightsquigarrow] & \\ X \arrow[rr] & & Y, \arrow[ul] \end{tikzcd}$$

two of the complexes are relative acyclic, then so is the third.

In turn, the two properties are consequences of the following:

(allow3) If Z is the mapping cone of a morphism $X \to Y$ of complexes, and X and Z are relatively acyclic, then so is Y.

Obviously, the properties hold if the relativization if induced by a right exact functor λ from $\mathfrak{A}$, and (hence) also for the split relativization induced by λ.

(8.7) Remark. The properties in (8.6) may be shown to hold for a *proper* relative abelian category, that is, an abelian category relativized by a class S of morphisms satisfying the following conditions:

(prop1) The class S is multiplicative, and $0 \to A$ belongs to S for every object A of $\mathfrak{A}$.

(prop2) The class S is stable under pushout.

(prop3) For any composition $s = vt$ of morphisms, $t\colon A \to A'$ and $v\colon A' \to A''$, if $s \in S$ and v is a monomorphism, then $t \in S$.

(8.8) Proposition. *Assume the properties of* (8.6). *Then the following assertions hold:*

(1) *If, for a composition $w = uv$ of morphisms of complexes, two of the morphisms u, v, and w, are relative quasi-isomorphisms, then so is the third.*

(2) *The class of relative quasi-isomorphisms is a denominator system in the homotopy category* $\mathrm{Hot}(\mathfrak{A})$.

(3) *If a bicomplex X in $\mathfrak{A}^\sharp$ (that is, a bicomplex with only finitely many nonzero X^{pq} on each diagonal $p + q = n$) has relatively acyclic columns, then* $\mathrm{Tot}\, X$ *is relatively acyclic.*

(4) *Let*

$$0 \to X \to Y \to U \to 0, \tag{8.8.1}$$

be a sequence of complexes which is relatively exact in each degree. Let Z be the mapping cone of $X \to Y$. Then the induced morphism $Z \to U$ is a relative quasi-isomorphism.

(5) *Let $\mathfrak{Q}$ be an additive class of objects of $\mathfrak{A}$ such that for any object A of $\mathfrak{A}$ there is a relative monomorphism $A \to Q$ into an object Q of $\mathfrak{Q}$. Then, for any right complex X in $\mathfrak{A}^+$, there is a relative quasi-isomorphism $X \to Q$ into a right complex Q in $\mathfrak{Q}^+$.*

Proof. Clearly, (1) and (2) follow from the properties of (8.6). Assertion (3) follows from the standard construction of the total complex of a bicomplex in $\mathfrak{A}^{[0,n],*}$ as an iterated cone. Assertion (4) follows from Assertion (3), since the cone of the morphism $Z \to U$ is the total complex of the bicomplex associated to (8.8.1) (with U as a column in degree 0).

Consider finally (5). Assume that $X \in \mathfrak{A}^{\geqslant 0}$. Chose for each n a relative monomorphism $\tilde{s}^n\colon X^n \to \widetilde{Q}^n$ with $\widetilde{Q}^n$ in $\mathfrak{Q}$. View the $\widetilde{Q}^n$ as a complex with zero differentials, and let Q^0 be the truncated mapping cone of the identity of this complex,

$$Q^0:\quad \cdots \to 0 \to \widetilde{Q}^0 \oplus \widetilde{Q}^1 \to \widetilde{Q}^1 \oplus \widetilde{Q}^2 \to \cdots .$$

Then $s = (\tilde{s}, \tilde{s}d)$ is a morphism of complexes $s\colon X \to Q^0$, and it is a relative monomorphism in each degree. Repeat the construction with $X := \mathrm{Cok}\, s$ to obtain a complex Q^1, and continue. The result is a complex of complexes in $\mathfrak{A}^{\geqslant 0}$,

$$\cdots \to 0 \to X \to Q^0 \to Q^1 \to Q^2 \to \cdots ,$$

which is relatively acyclic in each degree, and with Q^n in $\mathfrak{Q}^{\geqslant 0}$. Apply (3) to obtain a relative quasi-isomorphism $X \to \mathrm{Tot}\, Q$. □

(8.9) Derived functors. Assume the conditions of (8.6) for $\mathfrak{A}$ and for a second relative abelian category $\mathfrak{B}$. Let $D^+(\mathfrak{A})$ be the triangulated category obtained by localizing $\mathrm{Hot}^+(\mathfrak{A})$ at the relative quasi-isomorphisms. It follows, for an additive functor $T\colon \mathfrak{A} \to \mathfrak{B}$, that the functor $T\colon \mathrm{Hot}^+(\mathfrak{A}) \to \mathrm{Hot}^+(\mathfrak{B}) \to D^+(\mathfrak{B})$ is derivable with respect to the class of relative quasi-isomorphisms if there are sufficiently many relatively T-acyclic objects of $\mathfrak{A}$, that is, if there exists an additive class $\mathfrak{Q}$ of objects in $\mathfrak{A}$ such that,

(i) For any object $A \in \mathfrak{A}$ there is a relative monomorphism $A \to Q$ into an object $Q \in \mathfrak{Q}$.

(ii) If Q is a relatively acyclic complex in $\mathfrak{Q}^+$, then the complex TQ is relatively acyclic.

If $\mathfrak{A}$ has sufficiently many relatively injective objects (that is, for any object A there is a relative monomorphism into a relatively injective object), then in fact any functor $T\colon \operatorname{Hot}^+(\mathfrak{A}) \to \mathfrak{C}$ has a derived functor $RT\colon D^+(\mathfrak{A}) \to \mathfrak{C}$. The value $RT(X)$ at a complex X is equal to TQ, for any relative quasi-isomorphism $X \to Q$ into a right complex Q of relatively injective objects.

The theory resolvent complexes is relativized similarly: Consider a T-coaugmented complex of additive functors:

$$\overline{C}\colon\quad \cdots \to 0 \to T \to C^0 \to C^1 \to C^2 \to \cdots .$$

The nonaugmented complex of functors C may be viewed as a functor $C\colon \mathfrak{A} \to \mathfrak{B}^{\geqslant 0}$, and it extends to a functor on rigth complexes: the value on a complex X in $\mathfrak{A}^+$ is the complex $\operatorname{Tot} C(X)$ in $\mathfrak{B}^+$. As a result we obtain a functor $\operatorname{Tot} C\colon \operatorname{Hot}^+(\mathfrak{A}) \to \operatorname{Hot}^+(\mathfrak{B}) \to D^+(\mathfrak{B})$, and a transformation $\delta\colon T \to \operatorname{Tot} C$, induced by the co-augmentation $T \to C^0$.

Consider the following two conditions on the complex $\overline{C}$:

(i) For every object A of $\mathfrak{A}$ there exists a relative monomorphism $A \to Q$ into an object Q such that the complex $\overline{C}(Q)$ is relatively acyclic in $\mathfrak{B}$.
(ii) Each functor $C^n\colon \mathfrak{A} \to \mathfrak{B}$ is relatively exact.

Lemma on resolvent complexes. *Let $T \to C'$ be a second T-coaugmented complex of additive functors $\mathfrak{A} \to \mathfrak{B}$. Assume that the condition* (i) *holds for C and that* (ii) *holds for C'. Then, in the category of functors $\operatorname{Hot}^+(\mathfrak{A}) \to D^+(\mathfrak{B})$, there is a unique transformation $\operatorname{Tot} C \to \operatorname{Tot} C'$ extending the identity transformation of T.*

Proof. The (sketch of) proof uses two standard observations: Let $\mathfrak{Q}$ be the class of objects Q of $\mathfrak{A}$ such that $\overline{C}(Q)$ is relatively acyclic. Let X be a complex in $\mathfrak{A}^+$. It follows from (8.8)(5) that there is a relative quasi-isomorphism $s\colon X \to Q$ into a complex Q in $\mathfrak{Q}^+$. Moreover, it follows from condition (i) for C that the natural morphism $\delta\colon T(Q) \to \operatorname{Tot} C(Q)$ is a relative quasi-isomorphism in $\mathfrak{B}^+$. It follows from condition (ii) for C' that any relative quasi-isomorphism $X \to Y$ induces a relative quasi-isomorphism $C'X \to C'Y$.

Consider the following 'prism' diagram:

$$\begin{tikzcd}
\operatorname{Tot} CX \arrow[rr] \arrow[dd, dashed] & & \operatorname{Tot} CQ \arrow[dd, dashed] & \\
& TX \arrow[ul] \arrow[rr] \arrow[dl] & & TQ \arrow[ul, "\delta"'] \arrow[dl] \\
\operatorname{Tot} C'X \arrow[rr, "s"] & & \operatorname{Tot} C'Q &
\end{tikzcd}$$

The horizontal morphisms are induced by $X \to Q$. The skew morphisms are induced by the transformations of functors $T \to \operatorname{Tot} C$ and $T \to \operatorname{Tot} C'$. As observed above, the two morphisms s and δ in the diagram are relative quasi-isomorphisms, and hence isomorphisms in $D^+(\mathfrak{B})$. So the required morphism $CX \to C'X$ is equal to the clock wise composition (in $D^+(\mathfrak{B})$) of the four outer morphisms in the diagram. ∎

A T-coaugmented complex of functors C with both properties (i) and (ii) is called a *relatively exact resolvent complex* for T. It follows from the Lemma that resolvent complexes are unique, up to a unique isomorphism in $D^+(\mathfrak{B})$. Moreover, if $T \to C$ and $T \to C'$ are resolvent complexes, then any morphism $C \to C'$ extending the identity of C, is the unique isomorphism $C \to C'$. If T has a relatively exact resolvent complex $T \to C$, then T is relatively derivable, and there is a natural isomorphism $RTX \to \operatorname{Tot} CX$ in $D^+(\mathfrak{B})$.

Note the special case when $\mathfrak{B}$ is relativized with the split monomorphisms as relative monomorphisms. Then $D^+(\mathfrak{B})$ is simply $\operatorname{Hot}^+(\mathfrak{B})$, and relatively exact resolvent complexes are unique up to homotopy equivalence.

Consider in particular the case where the identity of $\mathfrak{A}$, as a functor,

$$1 : \mathfrak{A} \to \mathfrak{A}_{\text{split}},$$

from $\mathfrak{A}$ with a given to relativization to $\mathfrak{A}$ with the split relativization, has a relatively exact resolvent complex $1 \to C$. Thus it is required for the objects Q in (ii) that $\overline{C}(Q)$ is contractible, and it is required that each functor C^n turns relatively short exact sequences into split exact sequences (and hence relatively acyclic complexes into contractible complexes). Clearly, in this case, any additive functor $T : \mathfrak{A} \to \mathfrak{B}$, viewed as a functor of relative abelian categories $T : \mathfrak{A} \to \mathfrak{B}_{\text{split}}$ has a relatively resolvent complex, namely $T \to TC$.

(8.10) Remark. The Lemma in (8.9) has a flavor similar to the Theorem of Acyclic Models: Let $\mathfrak{K}$ be an arbitrary category, and let $\mathfrak{M}$ be a class of objects (the *models*) of $\mathfrak{K}$. Consider the category of functors $\mathfrak{K} \to \mathfrak{B}$ from $\mathfrak{K}$ to a given abelian category $\mathfrak{B}$. Relativize the category of functors: A transformation $\mathfrak{F} \to \mathfrak{G}$ is *relatively monic* (or $\mathfrak{M}$*-monic*), if $\mathfrak{F}M \to \mathfrak{G}M$ is monic in $\mathfrak{B}$ for every model M. (This the relativization induced by the exact funtor that restricts a functor $\mathfrak{F} : \mathfrak{K} \to \mathfrak{B}$ to the class $\mathfrak{M}$.) Acordingly there is a notion of $\mathfrak{M}$-acyclic complexes of functors and $\mathfrak{M}$-injective functors. If Q is an injective object of $\mathfrak{B}$ and M is any model, then the functor,

$$\rho_M Q = Q^{\operatorname{Hom}_{\mathfrak{K}}(-,M)},$$

is $\mathfrak{M}$-injective. Indeed, the assertion follows from the functorial isomorphism, for arbitrary objects M, Q in $\mathfrak{B}$:

$$\operatorname{Hom}_{\text{Funct}}(\mathfrak{F}, \rho_M Q) = \operatorname{Hom}_{\mathfrak{B}}(\mathfrak{F}M, Q).$$

The theorem of acyclic models. *Let $T : \mathfrak{K} \to \mathfrak{B}$ a functor and let $T \to C$ and $T \to C'$ be two T-coaugmented complexes of functors. Assume that $TM \to CM$ is an exact resolution for every model M and that each functor C'^n is $\mathfrak{M}$-injective. Then, in the homotopy category of complexes of functors, there is a unique morphism $C \to C'$ extending the identity of T.*

Proof. This is a special case of Corollary (8.4), with $X := T$, $Y := C$ and $Q := C'$. ∎

(8.11) Example. Take $\mathfrak{K} = \mathbf{Top}$ and as models the topological simplices Δ^p for $p = 0, 1, \dots$. Apply the dual concepts with $\mathfrak{B} := \mathbf{Ab}$. Let $C_n^{\text{sing}} : \mathbf{Top} \to \mathbf{Ab}$ be the functor,

$$C_n^{\text{sing}} X = \mathbb{Z}^{\oplus \operatorname{Hom}_{\mathbf{Top}}(\Delta^n, X)}.$$

Then C_n^{sing} is relatively projective, since $\mathbb{Z}$ is projective in **Ab**. The functors C_n^{sing} fit onto the *singular chain complex* C^{sing}, naturally augmented by the constant functor $\mathbb{Z}$. Its homology defines the *singular homology*,

$$H_n^{\text{sing}}(X) = H_n(C^{\text{sing}}X).$$

The *reduced singular homology* is the homology of the augmented complex $\overline{C}^{\text{sing}}$. It is well known, and easy to see, that the reduced singular homology $\overline{H}_n(Z)$ vanishes when Z is contractible. In particular, with $Z = \Delta^p$, it follows that the $\mathbb{Z}$-augmented complex $C^{\text{sing}} \to \mathbb{Z}$ satisfies both assumptions in (the dual version of) the Theorem of Acyclic models.

A classical application is the following:

Homotopy property of singular homology. *Homotopic maps $f_0, f_1 \colon X \to Y$ induce homotopic morphisms of chain complexes $C^{\text{sing}}X \to C^{\text{sing}}Y$.*

Proof. Let I be the unit interval. It suffices to prove that the two inclusions $i_0, i_1 \colon X \to X \times I$ induce homotopic morphisms of chain complexes,

$$C^{\text{sing}}(X) \to C^{\text{sing}}(X \times I).$$

View the two sides as $\mathbb{Z}$-augmented complexes of functors on **Top**. The left side consists of relatively projective functors, and the right side is relatively acyclic, since $\Delta^p \times I$ is contractible. So, by the Theorem, there is, up to homotopy, only one morphism from the left side to the right side. ∎

Limits

1. Direct systems; limits, colimits.

Fix a categories $\mathfrak{C}$ and $\mathfrak{D}$, and index categories I, J. (An *index category* is a category so small that some of the constructions below make sense; in particular, an index category may be a *small category*, that is, a category whose class of morphisms is a set.)

(1.1) Setup. A functor $\mathcal{X}\colon I \to \mathfrak{C}$ is also called a (*direct*) *system* in $\mathfrak{C}$ *indexed* by I, or an I*-system* in $\mathfrak{C}$, or a *co-*I*-object* in $\mathfrak{C}$. There is category of I-systems in $\mathfrak{C}$, with transformations of functors as morphisms; it is denoted $\mathfrak{C}^I$.

An I-system $\mathcal{X}\colon I \to \mathfrak{C}$ associates in a functorial way to every *index* i (that is, an object i of I) an object $\mathcal{X}_i$ in $\mathfrak{C}$ and to every morphism $\varphi\colon i \to j$ of indices a *transition morphism* $\mathcal{X}_i \to \mathcal{X}_j$, denoted $\mathcal{X}(\varphi)$, or $\varphi_{\mathcal{X}}$, or φ_*, or simply φ. A *morphism* of I-systems $u\colon \mathcal{X} \to \mathcal{Y}$ is a transformation of functors; in other words, it is a family of morphisms $u_i\colon \mathcal{X}_i \to \mathcal{Y}_i$, indexed by the objects i of I, and *compatible* with the transition morphisms, that is, for any morphism $\varphi\colon i \to j$ in I the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{X}_i & \xrightarrow{u_i} & \mathcal{Y}_i \\ \downarrow{\scriptstyle \varphi_{\mathcal{X}}} & & \downarrow{\scriptstyle \varphi_{\mathcal{Y}}} \\ \mathcal{X}_j & \xrightarrow{u_j} & \mathcal{Y}_j. \end{array}$$

An object A of $\mathfrak{C}$ defines a *constant* I-system, with $\mathcal{X}_i = A$ for all i and all transition morphisms equal to the identity 1_A; it is denoted $\mathrm{const}_I(A)$ (or simply A). A morphism $a\colon \mathcal{X} \to A$, from the I-system $\mathcal{X}$ to the constant I-system A, is a family $a = (a_i)$ of morphisms $a_i\colon A \to \mathcal{X}_i$ which is a *compatible family* in the sense that the following diagrams, for all morphisms $\varphi\colon i \to j$ in I, commute:

$$\begin{tikzcd} \mathcal{X}_i \arrow[r, "a_i"] \arrow[d, "\varphi_{\mathcal{X}}"'] & A \\ \mathcal{X}_j. \arrow[ur, "a_j"'] & \end{tikzcd}$$

We will say that the object A with the compatible family $a_i\colon \mathcal{X}_i \to A$ is a *common target* for the system $\mathcal{X}$. If A is a common target for $\mathcal{X}$ with the compatible family $a\colon \mathcal{X} \to A$ and $f\colon A \to B$ is a morphism in $\mathfrak{C}$, then B with the composition $fa\colon \mathcal{X} \to B$ as compatible family is a common target for $\mathcal{X}$; it is said to be *induced from a via the morphism* $f\colon A \to B$.

(1.2) Definition. A *colimit* of an I-system $\mathcal{X}\colon I \to \mathfrak{C}$ is a common target $q\colon \mathcal{X} \to Q$ with the following universal property: any common target $b\colon \mathcal{X} \to B$ is induced from q via a unique morphism $f\colon Q \to B$. It follows from the universal property that a colimit, if it exists, is unique: If $\tilde{q}\colon \mathcal{X} \to \tilde{Q}$ is a second colimit of $\mathfrak{X}$, then the unique morphism $s\colon \tilde{Q} \to Q$ such that $\tilde{q} = sq$ is a canonical isomorphism $\tilde{Q} \xrightarrow{\sim} Q$.

We will use the notation $Q = \varinjlim \mathcal{X}$, or $Q = \varinjlim_{i\in I} \mathcal{X}_i$, or similar, to indicate that Q is the colimit of the system $\mathcal{X}\colon I \to \mathfrak{C}$. The colimit is a common target for $\mathcal{X}$, and hence it is more than on object of $\mathfrak{C}$: it comes with a compatible family of morphisms $\mathcal{X} \to Q$; the i'th member of the family is the *i'th canonical injection*, usually denoted $\mathrm{in}_i^{\mathcal{X}}\colon \mathcal{X}_i \to Q$. The normal use of the symbols is an abuse of notation: We use $\varinjlim \mathcal{X}_i$ to denote at the same time the object with the compatible family of of injections in_i *and* the object of $\mathfrak{C}$ which is the common target of the injections. So we view the injections as a compatible family of morphisms,

$$\begin{tikzcd} \mathcal{X}_i \arrow[r, "\mathrm{in}_i"] \arrow[d, "\varphi_{\mathcal{X}}"'] & \varinjlim \mathcal{X} \\ \mathcal{X}_j . \arrow[ur, "\mathrm{in}_j"'] & \end{tikzcd} \tag{1.2.1}$$

The universal property may be rephrased as follows: For an object B of $\mathfrak{C}$ and a compatible family $b_i\colon \mathcal{X}_i \to B$ of morphisms there is a unique morphism

$$f\colon \varinjlim \mathcal{X} \to B \quad \text{such that } f\,\mathrm{in}_i = b_i\colon \mathcal{X}_i \to B \text{ for all } i \in I;$$

it is said to be defined by the equations $f\,\mathrm{in}_i = b_i$.

(1.3) Definition. The dual concepts lead to a dual type of limit: Let $\mathcal{X}\colon I \to \mathfrak{C}$ be an I-system. If B is an object of $\mathfrak{C}$, then a morphism $b\colon B \to \mathcal{X}$, from the constant I-system B to the I-system $\mathcal{X}$ is given by a *compatible family* of morphisms $b_i\colon B \to X_i$. We will say that B with the compatible family is a *common source* for the system $\mathcal{X}$. A *limit* (sometimes called an *inverse limit*) of the (direct) I-system $\mathcal{X}\colon I \to \mathfrak{C}$ is a common source $p\colon P \to \mathcal{X}$ from with the following universal property: any common source $a\colon A \to \mathcal{X}$ is induced from P via a unique morphism $g\colon A \to P$. We use the notation $P = \varprojlim \mathcal{X}$ or $P = \varprojlim_{i\in I} \mathcal{X}_i$ to indicate that P is the limit of the system $\mathcal{X}\colon I \to \mathfrak{C}$. It is an abuse of notation: $\varprojlim \mathcal{X}$ denotes at the same time a common source for $\mathfrak{X}$ with a compatible family of morphisms $\mathrm{pr}_i^{\mathcal{X}}\colon \varprojlim \mathcal{X} \to X_i$, called the canonical *projections*, *and* the object of $\mathfrak{C}$ which is the common source of the canonical projections. The projections form a compatible family of morphisms,

$$\begin{tikzcd} \varprojlim \mathcal{X} \arrow[r, "\mathrm{pr}_i"] \arrow[dr, "\mathrm{pr}_j"'] & \mathcal{X}_i \arrow[d, "\varphi_{\mathcal{X}}"] \\ & \mathcal{X}_j . \end{tikzcd} \tag{1.3.1}$$

The universal property may be rephrased as follows: For an object A of $\mathfrak{C}$ and compatible family $a_i\colon A \to \mathcal{X}_i$ of morphisms there is a unique morphism

$$g\colon A \to \varprojlim \mathcal{X} \quad \text{such that } \mathrm{pr}_i\, g = a_i\colon A \to \mathcal{X}_i \text{ for all } i \in I;$$

it is said to be defined by the equations $\mathrm{pr}_i\, g = a_i$.

Clearly, the limit and the colimit are functorial in the I-system $\mathcal{X}$. The category $\mathfrak{C}$ is said *to have* $\varprojlim_I$'s, resp. *to have* $\varinjlim_I$'s if every I-system $\mathcal{X}\colon I \to \mathfrak{C}$ has a limit, resp. a colimit; if this is the case, there is a well-defined functor,

$$\varprojlim_I : \mathfrak{C}^I \to \mathfrak{C}, \qquad \text{resp. } \varinjlim_I : \mathfrak{C}^I \to \mathfrak{C}.$$

(1.4) Terminology. A contravariant functor $\mathcal{Z}\colon I \to \mathfrak{C}$ is also called an *inverse I-system* in $\mathfrak{C}$ or an *I-object* in $\mathfrak{C}$. In questions related to limits and colimits an inverse system $\mathcal{Z}\colon I \to \mathfrak{C}$ is always replaced by the (direct) system ${}^{\mathrm{op}}\mathcal{Z}\colon I^{\mathrm{op}} \to \mathcal{Z}$, indexed by the opposite category I^{op}.

There is a tendency that limits often occur in connection with inverse systems and colimits occur in connection with (direct) systems. It is an old (and quite confusing) tradition that the limit of an inverse system is called an *inverse limit*, and the colimit of a direct system is called a *direct limit*.

(1.5) Product and coproduct. Assume that the category I is *discrete* (no morphisms except the identities). Then an I-system $\mathcal{X}$ is just a family $\mathcal{X}_i$ indexed by the objects of I. A common target for $\mathcal{X}$ is a family of morphisms $a_i\colon \mathcal{X}_i \to A$. A colimit of the system $\mathcal{X}$ is called a *coproduct* (or a *direct sum*) of the $\mathcal{X}_i$, and denoted

$$\coprod \mathcal{X}, \quad \text{or} \quad \coprod \mathcal{X}_i, \quad \text{or} \quad \coprod_{i\in I} \mathcal{X}_i, \quad \text{or} \quad \bigvee \mathcal{X}_i, \quad \text{or} \quad \bigoplus \mathcal{X}_i\,;$$

the last symbol is mainly used in additive categories. The coproduct comes with injections $\mathrm{in}_i\colon \mathcal{X}_i \to \coprod \mathcal{X}$, and the equations $g\,\mathrm{in}_i = a_i$ define a morphism $g\colon \coprod \mathcal{X} \to A$. In the finite case $I = \{1, \dots, n\}$ we may use notations like the following for the coproduct:

$$\mathcal{X}_1 \coprod \cdots \coprod \mathcal{X}_n \quad \text{or} \quad \mathcal{X}_1 \vee \cdots \vee \mathcal{X}_n \quad \text{or} \quad \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_n.$$

Dually, a common source for $\mathcal{X}$ is a family of morphisms $a_i\colon A \to \mathcal{X}_i$. A limit of the system $\mathcal{X}$ is called a *product* of the $\mathcal{X}_i$, and it may be denoted

$$\prod \mathcal{X}, \quad \text{or} \quad \prod \mathcal{X}_i, \quad \text{or} \quad \prod_{i\in I} \mathcal{X}_i, \quad \text{or} \quad \bigtimes \mathcal{X}_i\,.$$

The product comes with projections $\mathrm{pr}_i\colon \prod \mathfrak{X} \to \mathcal{X}_i$, and the equations $\mathrm{pr}_i\, g = a_i$ define a morphism $g\colon A \to \prod \mathcal{X}$. In the finite case $I = \{1, \dots, n\}$ we may use notations like the following for the product:

$$\mathcal{X}_1 \prod \cdots \prod X_n \quad \text{or} \quad \mathcal{X}_1 \times \cdots \times X_n.$$

The coproduct of the empty family ($I = \emptyset$) is the *cofinal* or *initial* object of $\mathfrak{C}$ and the product of the empty family is the final object.

When the family is constant, $\mathcal{X}_i = X$ for all i, we use the notation X^I for the product and $X^{\oplus I}$ for the coproduct. In the finite case, $I = \{1, \dots, n\}$, we write X^n and $X^{\oplus n}$. Note that the product X^I is a contravariant functor with respect to the set I and the coproduct $X^{\oplus I}$ is a covariant in I.

For an arbitrary index category J we say that $\mathfrak{C}$ *has* $\prod_J$'s, resp. *has* $\coprod_J$'s, if for any family in $\mathfrak{C}$ indexed by a set of cardinality at most equal to the cardinality of the class of morphisms in J, the product, resp. the coproduct, exists.

(1.6) Example. Assume that I has a final object i_0. Then, for any I-system $\mathcal{X}\colon I \to \mathfrak{C}$ the object $\mathcal{X}_{i_0}$ is the colimit of the system, $\varinjlim \mathcal{X} \xrightarrow{\sim} X_{i_0}$. More precisely, if $\mathcal{X}_i \to \mathcal{X}_{i_0}$ is the transition morphism corresponding to the unique morphism $i \to i_0$, then the object X_{i_0} with the morphisms $\mathcal{X}_i \to \mathcal{X}$ as injections is the direct limit.

Dually, if I has an initial object j_0, then for any I-system $\mathcal{X}$, we have $X_{j_0} = \varprojlim \mathcal{X}$.

Clearly, assuming the existence, the colimit of the identity functor is a final object of $\mathfrak{C}$ and a limit is an initial object of $\mathfrak{C}$.

(1.7) Fibered product and amalgamated coproduct. For the category $I := 1' \to 0 \leftarrow 1$ (three objects and two morphisms in addition to the identities), an I-system $\mathcal{X}$ is a diagram $\mathcal{X}_1' \to \mathcal{X}_0 \leftarrow \mathcal{X}_1$. A common source A for $\mathcal{X}$ is a commutative diagram,

$$\begin{tikzcd}
A \arrow[r, "a_1"] \arrow[d, "a_1'"'] & X_1 \arrow[d, "\varphi"] \\
X_1' \arrow[r, "\varphi'"'] & X_0,
\end{tikzcd}$$

with $a_0 = \varphi a_1 = \varphi' a_1'$. The limit, if it exists, is called the *fibered product* or the *pull-back* of the given diagram; it is denoted $\mathcal{X}_1' \prod_{\mathcal{X}_0} \mathcal{X}_1$ or $\mathcal{X}_1' \times_{\mathcal{X}_0} \mathcal{X}_1$.

Dually, for the category $J := 1' \leftarrow 0 \to 1$ a J-system $\mathcal{Y}$ is a diagram $\mathcal{Y}_1' \leftarrow \mathcal{Y}_0 \to \mathcal{Y}_1$, and a common target $\mathcal{Y} \to B$ is a pair of morphisms $b_1'\colon \mathcal{Y}_1' \to B$ and $b_1\colon \mathcal{Y}_1 \to B$ such that the compositions $\mathcal{Y}_0 \to \mathcal{Y}_1' \to B$ and $\mathcal{Y}_0 \to \mathcal{Y}_1 \to B$ are equal. The colimit, if it exists, is called the *amalgamated coproduct* or the *push-forward* of the given diagram; it is denoted $\mathcal{Y}_1' \coprod_{\mathcal{Y}_0} \mathcal{Y}_1$ or $\mathcal{Y}_1' \vee_{\mathcal{Y}_0} \mathcal{Y}_1$.

$$\begin{tikzcd}
A \arrow[dr, dashed] \arrow[drr, "a_1"] \arrow[ddr, "a_1'"'] & & \\
& \mathcal{X}_1' \times_{\mathcal{X}_0} \mathcal{X}_1 \arrow[r, "\mathrm{pr}"'] \arrow[d, "\mathrm{pr}'"] & \mathcal{X}_1 \arrow[d] \\
& \mathcal{X}_1' \arrow[r] & \mathcal{X}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{Y}_0 \arrow[r] \arrow[d] & \mathcal{Y}_1 \arrow[d, "\mathrm{in}"'] \arrow[ddr, "b_1"] & \\
\mathcal{Y}_1' \arrow[r, "\mathrm{in}'"] \arrow[drr, "b_1'"'] & \mathcal{Y}_1' \vee_{\mathcal{Y}_0} \mathcal{Y}_1 \arrow[dr, dashed] & \\
& & B
\end{tikzcd}$$

(1.8) Equalizer and coequalizer. Assume that I is the category $0 \rightrightarrows 1$, with two objects $0, 1$ and two morphisms $\varphi, \psi\colon 0 \to 1$ in addition to the two identities. Then an I-system $\mathcal{X}$ in $\mathfrak{C}$ is a diagram of two morphisms $\mathcal{X}_0 \overset{\varphi}{\underset{\psi}{\rightrightarrows}} \mathcal{X}_1$.

The limit of the system is the *equalizer* of the morphisms φ, ψ, denoted $\mathrm{Eq}(\varphi, \psi)$ or $\mathrm{Ker}(\varphi, \psi)$. The canonical projection $\mathrm{pr}_0\colon \mathrm{Ker}(\varphi,\psi) \to X_0$ is, in fact, a monomorphism (and $\mathrm{pr}_1 = \varphi\,\mathrm{pr}_0 = \psi\,\mathrm{pr}_0$).

The colimit of the system is the *coequalizer* of the morphisms φ, ψ, denoted $\mathrm{Coeq}(\varphi, \psi)$ or $\mathrm{Cok}(\varphi, \psi)$. The canonical injection $\mathrm{in}_1\colon X_1 \to \mathrm{Cok}(\varphi, \psi)$ is, in fact, an epimorphism (and $\mathrm{in}_0 = \mathrm{in}_1\,\varphi = \mathrm{in}_1\,\psi$).

$$\mathrm{Ker}(\varphi,\psi) \xrightarrow{pr_0} \mathcal{X}_0 \overset{\varphi}{\underset{\psi}{\rightrightarrows}} \mathcal{X}_1 \xrightarrow{\mathrm{in}_1} \mathrm{Cok}(\varphi,\psi).$$

(1.9) Limits in the category of sets. *If I is a small category, then any I-system $\mathfrak{X}$ of sets has a limit. In fact, let $\prod_{i\in I}\mathcal{X}_i$ be the product set, consisting of all element families $(x_i)_{i\in I}$ with $x_i \in \mathcal{X}_i$, and let P be the subset of all compatible families, that is, element families*

$$(x_i)_{i\in I} \quad \text{such that } \varphi_*(x_i) = x_j \text{ for any morphism } \varphi\colon i \to j \text{ in } I.$$

Then the set P with the i'th coordinate projection $(x_i)_{i\in I} \mapsto x_i$ as the canonical projection $P \to \mathcal{X}_i$ is the limit $\varprojlim_{i\in I}\mathcal{X}_i$.

Proof. This is just an observation: Let $P \subseteq \prod \mathcal{X}_i$ be the subset of compatible element families. Clearly, a family of maps $a_i\colon A \to \mathcal{X}_i$ defines a map $\alpha\colon A \to \prod\mathcal{X}_i$ from A to the product set, and α maps into the subset P if and only if $a_i\colon A \to \mathcal{X}_i$ is a compatible family of maps. □

(1.10) Main existence lemma. (1) *Assume that the category $\mathfrak{C}$ has equalizers and $\prod_I$'s. Then $\mathfrak{C}$ has $\varprojlim_I$'s. In fact, for any I-system $\mathcal{X}\colon I \to \mathfrak{C}$, the limit $\varprojlim_{i\in I}\mathcal{X}_i$ is the equalizer of the following pair of morphisms,*

$$\prod_{i\in I}\mathcal{X}_i \overset{s}{\underset{t}{\rightrightarrows}} \prod_{\varphi\colon i\to j}\mathcal{X}_j\,.$$

In the product on the right side, the index set is the class of all morphisms $\varphi\colon i \to j$ in I, and the object corresponding to the index $\varphi\colon i \to j$ is the object $\mathcal{X}_j$ given by the target of φ. The two morphisms s, t into this product are given by their projections corresponding to an index $\varphi\colon i \to j$; they are given by the equations of morphisms $\prod\mathcal{X} \to \mathcal{X}_j$:

$$\mathrm{pr}_\varphi\, s = \varphi_*\,\mathrm{pr}_i, \ \text{ and } \mathrm{pr}_\varphi\, t = \mathrm{pr}_j\,.$$

(2) *Assume that there is a class $J \subseteq I$ of objects such that for every object $i \in I$ there exists an object $j \in J$ and a morphism $j \to i$. Assume that the category $\mathfrak{C}$ has arbitrary intersections of subobjects, and equalizers, and $\prod_J$'s. Then $\mathfrak{C}$ has $\varprojlim_I$'s.*

Proof. Part (a) is just a game where you play with the universal properties of the product and the equalizer. You have to play it!

Part (b) is similar, knowing the rules: Consider the product $P := \prod_{j\in J}\mathcal{X}_j$. For any morphism $\varphi\colon j \to i$ in I, with $j \in J$, let $p_\varphi\colon P \to \mathcal{X}_i$ be the morphism $\varphi_*\,\mathrm{pr}_j$. For any pair of morphisms $\varphi\colon j \to i$ and $\varphi'\colon j' \to i$ with $j, j' \in J$, the equalizer $\mathrm{Ker}(p_\varphi, p_{\varphi'})$ is a subobject of P. Now check that the intersection of all these equalizers, with obvious projections, is the limit $\varprojlim_{i\in I}\mathcal{X}_i$. □

(1.11) Limits and colimits of sets. Let I be a small category and let $\mathcal{X}\colon I \to \mathbf{Sets}$ be an I-system of sets. Clearly, the general description in (1.10) leads to the description of the limit $\varprojlim_I \mathcal{X}_i$ in (1.9).

The category of sets has coproducts, given by the disjoint union $\coprod_I \mathcal{X}$, and it has coequalizers: If $Z \underset{t}{\overset{s}{\rightrightarrows}} X$ is a pair of maps, then the coequalizer $\operatorname{Cok}(s,t)$ is the quotient $X/\sim$ of X modulo the equivalence relation $\sim$ generated by the relations $s(z) \sim t(z)$ for $z \in Z$. In even more detail, for elements x, x' write $x \leftrightarrow x'$ if either $x' = x$, or there exists an element $z \in Z$ such that $(x, x') = (s,t)(z)$ or $(x, x') = (t,s)(z)$. Then,

$$x \sim x' \iff \text{there exists a finite string } x = x_0 \leftrightarrow x_1 \leftrightarrow \cdots \leftrightarrow x_n = x'.$$

So the dual version of Lemma (1.10) applies. It follows that the category **Sets** has $\varinjlim_I$'s, and that the colimit $\varinjlim_I \mathcal{X}$ may be described as the quotient of $X := \coprod_{i\in I} \mathcal{X}_i$ modulo the equivalence relation generated by the following relation: Two elements x, x' in X, say $x \in \mathcal{X}_i$ and $x' \in \mathcal{X}_j$ are related if there is a morphism $\varphi\colon i \to j$ such the $\varphi_*(x) = x'$.

(1.12) Examples. The construction in (1.10)(1) applies to many classical categories of sets with an extra structure. Of special interest is the following result:

Observation. *The category* **Ab** *of abelian groups has arbitrary small limits. Moreover, if $\mathcal{X}\colon I \to$* **Ab** *is a system of abelian groups, then the set underlying the group $\varprojlim_i \mathcal{X}_i$ is the limit of the underlying sets.*

Proof. Clearly, if $\mathcal{X}_i$ is a family of abelian groups indexed by a set I, then the product group is the product set $\prod_{i\in I} \mathcal{X}_i$ with coordinate-wise composition; if $f, g\colon X \to Y$ are homomorphisms of abelian groups, then the set equalizer is a subgroup of X and hence, with its subgroup structure, equal to the equalizer in the category **Ab**.

So, by (1.11), for any system $\mathcal{X}_i$ of abelian groups indexed by a small category I, the limit $\varprojlim_{i\in I} \mathcal{X}_i$ exists and it may be obtained by giving the limit of underlying sets a natural induced structure as an abelian group. $\square$

Almost identical observations may be made for most other categories of sets with extra structure, for instance for the following categories:

$\mathfrak{C} =$ **Gr** is the category of groups,
$\mathfrak{C} = k$-**Alg** is the category of commutative algebras over a commutative ring k,
$\mathfrak{C} = k$-**Mod** is the category of k-modules,
$\mathfrak{C} =$ **Top** is the category of topological spaces,
$\mathfrak{C} =$ **POS** is the category of partially ordered sets.

(1.13) Note. It follows from Lemma (1.10) that $\mathfrak{C}$ has arbitrary finite limits, resp. finite colimits, if and only if $\mathfrak{C}$ has finite products and equalizers, resp. finite coproducts and coequalizers. In particular, an abelian category has arbitrary finite limits and finite colimits.

Again, by the same Lemma, $\mathfrak{C}$ has arbitrary small limits (i.e., limits of systems indexed by small categories), resp. small colimits, if and only if $\mathfrak{C}$ has small products and equalizers, resp. small coproducts and coequalizers.

Note also that the category $\mathfrak{Funct}(\mathfrak{I}, \mathfrak{C})$ of all functors $F\colon \mathfrak{I} \to \mathfrak{C}$ has the same limits and colimits as the category $\mathfrak{C}$. The limit of a system $i \mapsto F_i$ of functors $F_i\colon \mathfrak{I} \to \mathfrak{C}$ is determined "argument by argument": $(\varprojlim_I F_i)(Z) = \varprojlim_I (F_i Z)$ for $Z \in \mathfrak{I}$.

(1.14) Definition. Let $F\colon \mathfrak{C} \to \mathfrak{D}$ be a functor. If $\mathcal{X}\colon I \to \mathfrak{C}$ is an I-system in $\mathfrak{C}$, then the composition $F\mathcal{X}\colon I \to \mathfrak{D}$ is an I-system in $\mathfrak{D}$. So F induces a functor $\mathfrak{C}^I \to \mathfrak{D}^I$. In particular, if $a_i\colon A \to \mathcal{X}_i$ is a compatible family in $\mathfrak{C}$, corresponding to a morphism from the constant I-system A to $\mathcal{X}$, then $Fa_i\colon FA \to F\mathcal{X}_i$ is a compatible family in $\mathfrak{D}$.

Assume that the limits exist. The projections $\mathrm{pr}_i^{\mathcal{X}}\colon \varprojlim \mathcal{X} \to \mathcal{X}_i$ form a compatible family in $\mathfrak{C}$. So the images under F form a compatible family $F(\mathrm{pr}_i^{\mathcal{X}})\colon F(\varprojlim \mathcal{X}) \to F\mathcal{X}_i$ in $\mathfrak{D}$. So there is an induced morphism,

$$F(\varprojlim_I \mathcal{X}) \xrightarrow{u} \varprojlim_I F\mathcal{X}, \text{ defined by the equations } \mathrm{pr}_i^{F\mathcal{X}}\, u = F(\mathrm{pr}_i^{\mathcal{X}}). \tag{1.14.1}$$

The functor is said to *commute with* $\varprojlim_I$'s if, whenever the limit $\varprojlim_I \mathcal{X}$ of an I-system $\mathcal{X}$ exists, the compatible family $F(\mathrm{pr}_i)\colon F(\varprojlim \mathcal{X}) \to F\mathcal{X}_i$ is the limit of the I-system $F\mathcal{X}$. Assuming the existence of both limits in (1.14.1), the functor F commutes with $\varprojlim_I$'s if the morphism u in (1.14.1) is an isomorphism for any I-system $\mathcal{X}$.

The functor $F\colon \mathfrak{C} \to \mathfrak{D}$ is said to be *left exact*, resp. *right exact*, if it commutes with all finite limits, resp. colimits.

It follows from Lemma (1.10) that a functor F is left exact if and only if it commutes with equalizers and with finite products; it commutes with all small limits if and only if it commutes with equalizers and arbitrary products indexed by sets.

(1.15) Proposition. *The functor* $\mathrm{Hom}_{\mathfrak{C}}(-,-)$ *from* $\mathfrak{C}^{\mathrm{op}} \times \mathfrak{C}$ *to* **Sets** *commutes with* $\varprojlim$ *in the second variable and, when viewed as a covariant functor* $\mathfrak{C}^{\mathrm{op}} \to$ **Sets** *in the first variable, also in the first variable. More precisely, for a given system* $\mathcal{X}\colon I \to \mathfrak{C}$*, the following assertions hold: Let* $p_i\colon P \to \mathcal{X}_i$ *be a common source and let* $q_i\colon \mathcal{X}_i \to Q$ *be a common target for the system* $\mathcal{X}$*. Consider, for objects* $A, B \in \mathfrak{C}$ *the maps of sets induced by* p *and* q*:*

$$(1)\quad \mathrm{Hom}_{\mathfrak{C}}(A, P) \xrightarrow{u_p} \varprojlim_{i\in I} \mathrm{Hom}_{\mathfrak{C}}(A, \mathcal{X}_i), \qquad (2)\quad \mathrm{Hom}_{\mathfrak{C}}(Q, B) \xrightarrow{v_q} \varprojlim_{i\in I} \mathrm{Hom}_{\mathfrak{C}}(\mathcal{X}_i, B).$$

Then $P = \varprojlim_i \mathcal{X}_i$ *if and only if* (1) *is bijective for all* A*, and* $\varinjlim_i \mathcal{X}_i = Q$ *if and only if* (2) *is bijective for all* B*.*

Proof. Indeed, the precise result is a reformulation of the universal properties of limits and colimits. □

(1.16) Definition. Consider a functor $\Phi\colon J \to I$. Then for any I-system $\mathcal{X}$ in $\mathfrak{C}$ the composition $\mathcal{X}\Phi$ is a J-system in $\mathfrak{C}$; it is said to be obtained from $\mathcal{X}$ by *restriction* to J (via Φ), and may be denoted $\mathcal{X}_{|\Phi}$ or $\mathcal{X}|J$. Restriction defines a functor $\mathfrak{C}^I \to \mathfrak{C}^J$. Assume that the colimit $\varinjlim_{i\in I} \mathcal{X}_i$ exists. Then the canonical injections define a compatible family of morphisms $\mathrm{in}_i\colon \mathcal{X}_i \to \varinjlim_I \mathcal{X}$, and it restricts to a compatible family of morphisms

$\mathcal{X}_{\Phi j} \to (\varinjlim_I \mathcal{X})$ from the J-system $\mathcal{X}\Phi$ to the constant J-system $\varinjlim_I \mathcal{X}$. So, if the colimit $\varinjlim_J \mathcal{X}\Phi = \varinjlim_{j\in J} \mathcal{X}_{\Phi j}$ exists, the first of the following morphisms is a canonical morphism in $\mathfrak{C}$,

$$\varinjlim_{j\in J} \mathcal{X}_{\Phi j} \longrightarrow \varinjlim_{i\in I} \mathcal{X}_i; \qquad \varprojlim_{i\in I} \mathcal{X}_i \to \varprojlim_{j\in J} \mathcal{X}_{\Phi j} \tag{1.16.1}$$

the second morphism is obtained similarly, assuming that the limits exist.

Recall that the *right fiber* of the functor $\Phi\colon J \to I$ at an object i of the target category, denoted i/Φ (or i/J), is the following category: The objects of i/Φ are the pairs (j, φ) consisting of an object $j \in J$ and a morphism $\varphi\colon i \to \Phi j$ in I. The morphisms in i/Φ from (j, φ) to (j', φ') are the the morphisms $\psi\colon j \to j'$ in J for which the following diagram is commutative:

$$\begin{tikzcd} i \arrow[d, "\varphi"'] \arrow[dr, "\varphi'"] & \\ \Phi j \arrow[r, "\Phi(\psi)"] & \Phi j'. \end{tikzcd}$$

The functor $\Phi\colon J \to I$ is called a *final* or *terminal functor* if its right fibers are non-empty and connected.

Example. The category I is said to be *filtering*, if the following two conditions are satisfied:

(FILT 1) For any pair of objects $j, j' \in I$ there exists an object $k \in I$ and morphisms $j \to k \leftarrow j'$.

(FILT 2) Any pair of morphisms $i \rightrightarrows j$ can be coequalized by a morphism $j \to k$.

It is said to be *pseudo-filtering* if (FILT 2) holds and (FILT 1) holds in the week form:

(PS-FILT 1) For any pair of morphisms $j \leftarrow i \to j'$ there exists an object $k \in I$ and morphisms $j \to k \leftarrow j'$, and

$$\begin{tikzcd}[row sep=small] & j \arrow[dr, dashed] & \\ i \arrow[ur] \arrow[dr] & & k, \\ & j' \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} i \arrow[r, shift left] \arrow[r, shift right] & j \arrow[r, dashed] & k. \end{tikzcd}$$

Of course, assuming (FILT 2), the square in (PS-COFILT 1) may be assumed commutative.

It is easy to see that I is pseudo-filtering if and only if the connected components of I are filtering.

If I is pseudo-filtering, then the inclusion of a subcategory $J \subseteq I$ is final, if and only if for every object $i \in I$ there is an object $j \in J$ and a morphism $i \to j$.

(1.17) Proposition. *Assume that the functor $\Phi\colon J \to I$ is final. Then, if one of the two colimits $\varinjlim_{i\in I} \mathcal{X}_i$ and $\varinjlim_{j\in J} \mathcal{X}_{\Phi j}$ exists, then so does the other, and the canonical morphism of* (1.16.1) *is an isomorphism,*

$$\varinjlim_{j\in J} \mathcal{X}_{\Phi j} \xrightarrow[u]{\sim} \varinjlim_{i\in I} \mathcal{X}_i. \tag{1.17.1}$$

Hint. To define an inverse of u, consider the family of morphisms $v_i\colon \mathcal{X}_i \to \varinjlim_J \mathcal{X}\Phi$ defined for $i \in I$ as follows: Chose an object $j \in J$ and a morphism $i \to \Phi j$ in I. Let v_i be the composition,

$$\mathcal{X}_i \xrightarrow{\varphi} \mathcal{X}_{\Phi j} \xrightarrow{\mathrm{in}_j^{\mathcal{X}\Phi}} \varinjlim \mathcal{X}\Phi.$$

Now check that the v_i are well-defined, that they form a compatible family $v\colon \mathcal{X} \to \varinjlim_J \mathcal{X}\Phi$ from the I-system $\mathcal{X}$ to the constant I-system $\varinjlim_J \mathcal{X}\Phi$, and that the induced morphism $\varinjlim_I \mathcal{X} \to \varinjlim_J \mathcal{X}\Phi$ is the inverse of (1.17.1). □

(1.18) Proposition. *If a functor $F\colon \mathfrak{C} \to \mathfrak{D}$ has a left adjoint then it commutes with all limits. In particular, then F is left exact.*

Proof. By assumption there is a functor $\lambda\colon \mathfrak{D} \to \mathfrak{C}$ and a functorial adjunction bijection, for objects $D \in \mathfrak{D}$ and $C \in \mathfrak{C}$,

$$\mathrm{Hom}_{\mathfrak{C}}(\lambda D, C) = \mathrm{Hom}_{\mathfrak{D}}(D, FC). \tag{1.18.1}$$

Consequently, for any I-system $\mathcal{X}$ such that the limit $\varprojlim_I \mathcal{X}$ exists, there is a commutative diagram of sets,

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{D}}(D, F(\varprojlim_{i\in I} \mathcal{X}_i)) \arrow[r, equal] \arrow[d] & \mathrm{Hom}_{\mathfrak{C}}(\lambda D, \varprojlim_{i\in I} \mathcal{X}_i) \arrow[d, "\wr"] \\
\varprojlim_{i\in I} \mathrm{Hom}_{\mathfrak{D}}(D, F(\mathcal{X}_i)) \arrow[r, equal] & \varprojlim_{i\in I} \mathrm{Hom}_{\mathfrak{C}}(\lambda D, \mathcal{X}_i).
\end{tikzcd}$$

The horizontal bijections are induced by the adjunction bijections. The right vertical map is a bijection, reflecting the universal property of $\varprojlim_I$, cf. (1.15). The left vertical map is induced by the compatible family of morphisms $F(\mathrm{pr}_i^{\mathcal{X}})\colon F(\varprojlim_I \mathcal{X}) \to F(X_i)$. As the map is bijective, it follows that the compatible family has the universal property. Hence $F(\varprojlim_I \mathcal{X})$ is the limit of the system $F\mathcal{X}$. □

(1.19) Examples. As noted in (1.12) several classical categories $\mathfrak{C}$ have as objects sets with an extra structure, and the forgetful functor $\square\colon \mathfrak{C} \to \mathbf{Sets}$ commutes with $\varprojlim$'s. In fact, if $\mathfrak{C}$ is such a category, there if a functor $W\colon \mathbf{Sets} \to \mathfrak{C}$ which is left adjoint to $\square$. It associates with a set T an object $W(T)$ which you may think of as the free object in $\mathfrak{C}$ generated by T; it is defined by the equation $\mathrm{Hom}_{\mathfrak{C}}(W(T), A) = \mathrm{Hom}_{\mathbf{Sets}}(T, \square A)$.

Here are the examples corresponding to the categories in (1.12):

$\mathfrak{C} = \mathbf{Ab}$, $W(T) := \mathbb{Z}^{\oplus T}$ is the free abelian group generated by T.

$\mathfrak{C} = \mathbf{Gr}$, $W(T)$ is the free group of words in letters corresponding to the elements of $T \cup T^{-1}$.

$\mathfrak{C} = k$-$\mathbf{Alg}$, $W(T) := k[T]$ is the polynomial algebra in variables corresponding to the elements of T.

$\mathfrak{C} = k$-$\mathbf{Mod}$, $W(T) := k^{\oplus T}$ is the free k-module on generators corresponding to the elements of T.

$\mathfrak{C} = \mathbf{Top}$, $W(T) = T^{\mathrm{discr}}$ is the discrete topological space T.

$\mathfrak{C} = \mathbf{POS}$, $W(T) := T^{\mathrm{discr}}$ is the discrete partially ordered set T.

(1.20). Assuming the existence of the limits and the colimits we have functors $\varprojlim_I\colon \mathfrak{C}^I \to \mathfrak{C}$ and $\varinjlim_I\colon \mathfrak{C}^I \to \mathfrak{C}$. By the universal property of limits, we have the bijection, for any object $A \in \mathfrak{C}$ and any I-system $\mathcal{X}\colon I \to \mathfrak{C}$:

$$\mathrm{Hom}_{\mathfrak{C}^I}(\mathrm{const}_I\, A, \mathcal{X})) = \mathrm{Hom}_{\mathfrak{C}}(A, \varprojlim_I \mathcal{X}).$$

In other words, the limit functor $\varprojlim_I$ is right adjoint to $A \mapsto \mathrm{const}_I\, A$ is left adjoint to $\varprojlim_I$. Similarly, the colimit is left adjoint to $A \mapsto \mathrm{const}_I\, A$:

$$\mathrm{Hom}_{\mathfrak{C}^I}(\mathcal{X}, \mathrm{const}_I\, A) = \mathrm{Hom}_{\mathfrak{C}}(\varinjlim_I \mathcal{X}, A).$$

Therefore, the following result is an immediate consequence of Proposition (1.18).

Proposition. *If the category $\mathfrak{C}$ has $\varprojlim_I$'s then the limit functor $\varprojlim_I : \mathfrak{C}^I \to \mathfrak{C}$ commutes with arbitrary limits; in particular, the limit functor is left exact. If the category $\mathfrak{C}$ has $\varinjlim_I$'s then the colimit functor $\varinjlim_I : \mathfrak{C}^I \to \mathfrak{C}$ commutes with all colimits; in particular, the colimit functor is right exact.*

(1.21) Remark. Proposition (1.18) is morally an "if and only if" statement. The more precise result is the following:

Theorem. *A functor $F : \mathfrak{C} \to \mathfrak{D}$ has a left adjoint if and only if every system in $\mathfrak{C}$ indexed by a right fiber of F has a limit and F commutes with these limits.*

Proof. Fix D in $\mathfrak{D}$. Recall that the right fiber of F at D, denoted $D/\mathfrak{C}$, is the following category: Its objects are pairs (X, f), with an object $X \in \mathfrak{C}$ and morphism $f : D \to FX$ in $\mathfrak{D}$. A morphism in $D/\mathfrak{C}$, say from (X', f') to (X, f), is a morphism $h : X' \to X$ in $\mathfrak{C}$ such that $f = F(h) f'$:

$$\begin{tikzcd} D \arrow[d, "f'"'] \arrow[dr, "f"] & \\ FX' \arrow[r, "F(h)"] & FX. \end{tikzcd} \tag{1.19.1}$$

Assume that $\lambda : \mathfrak{D} \to \mathfrak{C}$ is a left adjoint of F. Then the adjunction bijection, say

$$p : \mathrm{Hom}_{\mathfrak{D}}(D, FX) \to \mathrm{Hom}_{\mathfrak{C}}(\lambda D, X), \tag{1.19.2}$$

functorial in $D \in \mathfrak{D}$ and in $X \in \mathfrak{C}$, associates with each object $(X, f) \in D/\mathfrak{C}$ a morphism $p(g) : \lambda D \to X$ which is an object in the category $\lambda D/\mathfrak{C}$ of all morphisms with source λD. It is obviously an isomorphism of categories from $D/\mathfrak{C}$ to $\lambda D/\mathfrak{C}$. The latter category has an initial object, which is the identity $1 : \lambda D \to \lambda D$. Hence the right fiber $D/\mathfrak{C}$ has an initial object (which is the morphism $\varepsilon : D \to F\lambda D$ corresponding to the identity of λD under the bijection p). Therefore, any system indexed by $D/\mathfrak{C}$ has a limit.

Conversely, assume the conditions for the 'if' part. The functor $D/\mathfrak{C} \to \mathfrak{C}$ determined by $(X, f) \mapsto X$ is a $D/\mathfrak{C}$-system in $\mathfrak{C}$. Define λD as its limit:

$$\lambda D := \varprojlim_{(X,f) \in D/\mathfrak{C}} X.$$

The limit comes with a compatible family of canonical projections: For any index (X, f) in $D/\mathfrak{C}$ there is a morphism $\mathrm{pr}_{X,f} : \lambda D \to X$, and this family is compatible, that is, for any commutative diagram (1.19.1), the following diagram is commutative:

$$\begin{tikzcd} \lambda D \arrow[d, "\mathrm{pr}_{X',f'}"'] \arrow[dr, "\mathrm{pr}_{X,f}"] & \\ X' \arrow[r, "h"] & X. \end{tikzcd} \tag{1.19.3}$$

Now, for fixed $X \in \mathfrak{C}$, define the map p as in (1.19.2) by $p(f) := \mathrm{pr}_{X,f}$. It follows from the compatibility of the family of projections that the map p is functorial with respect to X.

If $g\colon D' \to D$ is a morphism in $\mathfrak{D}$, there is an obvious functor of the fibers $D/\mathfrak{C} \to D'/\mathfrak{C}$, determined by $(X, f) \mapsto (X, fg)$. So there is a natural morphism $\lambda D' \to \lambda D$ of limits, see (1.16), commuting with the projections. It follows that $D \mapsto \lambda D$ is functor and that the map p in (1.19.2) is functorial in D.

It remains to prove that p is bijective. By assumption, the functor F commutes with limits. So, since λD is the limit of the system $(X, f) \mapsto X$, it follows that $F\lambda D$, with the compatible family of morphisms $F(\mathrm{pr}_{X,f})\colon F\lambda D \to F(X)$, is the limit of the image system $(X, f) \mapsto FX$. By definition of the right fiber $D/\mathfrak{C}$, there is an obvious compatible family of morphisms from the object D to the image system $(X, f) \mapsto FX$. Hence, by the existence part of the universal property of limits, this compatible family is induced by a morphism from D to the limit $F\lambda D$. So there is a morphism $\varepsilon\colon D \to F\lambda D$ such that, for all $(X, f) \in D/\mathfrak{C}$, the first of the following diagrams is commutative:

$$\begin{tikzcd} D \arrow[d, "\varepsilon"'] \arrow[dr, "f"] & \\ F\lambda D \arrow[r, "F(\mathrm{pr}_{X,f})"] & FX, \end{tikzcd} \qquad \begin{tikzcd} D \arrow[d, "\varepsilon"'] \arrow[dr, "q(g)"] & \\ F\lambda D \arrow[r, "F(g)"] & FX. \end{tikzcd} \tag{1.19.4}$$

In the second diagram, if $g\colon \lambda D \to X$ is a morphism, we let $q(g) := F(g)\varepsilon$. Then, by definition of q, the second diagram is commutative. The map $g \mapsto q(g)$ is a map of the Hom-sets in (1.19.2), from the right to the left. We claim that the map q is the inverse of p.

First, by the commutativity of the first diagram in (1.19.4), it follows for any morphism $f\colon D \to FX$ that $qp(f) = f$.

So it remains to prove, for a given morphism $g\colon \lambda D \to X$ that $pq(g) = g$. Now, the morphism $\varepsilon\colon D \to F\lambda D$ defines an object $(\lambda D, \varepsilon)$ in $D/\mathfrak{C}$. Let (X, f) be an arbitrary object in $D/\mathfrak{C}$. It follows from the commutative diagrams in (1.19.4) that $\mathrm{pr}_{X,f}$ is a morphism $(\lambda D, \varepsilon) \to (X, f)$ and g is a morphism $(\lambda D, \varepsilon) \to (X, q(g))$. So the following two diagrams, corresponding to (1.19.3), are commutative:

$$\begin{tikzcd} \lambda D \arrow[d, "\mathrm{pr}_{\lambda D,\varepsilon}"'] \arrow[dr, "\mathrm{pr}_{X,f}"] & \\ \lambda D \arrow[r, "\mathrm{pr}_{X,f}"] & X, \end{tikzcd} \qquad \begin{tikzcd} \lambda D \arrow[d, "\mathrm{pr}_{\lambda D,\varepsilon}"'] \arrow[dr, "\mathrm{pr}_{X,q(g)}"] & \\ \lambda D \arrow[r, "g"] & X. \end{tikzcd} \tag{1.19.5}$$

In the second diagram, $pq(g) = \mathrm{pr}_{X,q(g)}$. So, by the commutativity of the second diagram, to prove the equation $pq(g) = g$ it it suffices to prove that $\mathrm{pr}_{\lambda D,\varepsilon} = 1_{\lambda D}$.

To prove that latter equation note that the first diagram in (1.19.5) is commutative for all $(X, f) \in D/\mathfrak{C}$. Hence, since λD is the limit of the system $(X, f) \mapsto X$, it follows from the uniqueness part of the universal property for limits that $\mathrm{pr}_{\lambda D,\varepsilon}$ is the identity morphism of λD. □

(1.20) Exercises.

1. Let $T\colon \mathfrak{A} \to \mathfrak{B}$ be a functor between abelian categories. Assume that T is right exact. Prove that T is left exact if and only if T preserves monomorphisms.

2. The category of direct systems.

(2.1) Setup. Fix the category $\mathfrak{C}$. Let $\mathcal{Y}\colon J \to \mathfrak{C}$ be a J-system in $\mathfrak{C}$. Consider, for an object $A \in \mathfrak{C}$, the colimit,

$$\varinjlim_{j\in J} \operatorname{Hom}_{\mathfrak{C}}(A, \mathcal{Y}_j). \tag{2.1.1}$$

Recall, cf. (1.11), that the colimit of sets is a quotient of the disjoint union $\coprod \operatorname{Hom}_{\mathfrak{C}}(A, \mathcal{Y}_j)$ modulo an equivalence determined from the transition morphisms. Write $\psi\colon j_1 \longleftrightarrow j_2$ to indicate that ψ is either a morphism $\psi\colon j_1 \to j_2$ or a morphism $\psi\colon j_2 \to j_1$. Then, more explictly, an element of the colimit (2.1.1) is given by a *representative* $f\colon A \to \mathcal{Y}_j$ for some $j \in J$, and two representatives $f\colon A \to \mathcal{Y}_j$ and $f'\colon A \to \mathcal{Y}_{j'}$ are equivalent if there is a path of morphisms in J: $j = j_0 \overset{\psi_1}{\longleftrightarrow} j_1 \overset{\psi_2}{\longleftrightarrow} j_2 \overset{\psi_3}{\longleftrightarrow} \cdots \overset{\psi_n}{\longleftrightarrow} j_n = j'$ and a sequence of morphisms $f_\nu\colon A \to \mathcal{Y}_{j_\nu}$ for $\nu = 1, \dots, n-1$ such that the following diagram is commutative:

$$\begin{tikzcd}
 & & A \arrow[dll, "f"'] \arrow[dl, "f_1"] \arrow[d, "f_2"] & \cdots & \arrow[from=ull, drr, "f'"] & \\
\mathcal{Y}_j \arrow[r, leftrightarrow, "\psi_1"'] & \mathcal{Y}_{j_1} \arrow[r, leftrightarrow, "\psi_2"'] & \mathcal{Y}_{j_2} \arrow[r, leftrightarrow, "\psi_3"'] & \cdots \arrow[r, leftrightarrow, "\psi_n"'] & \mathcal{Y}_{j'}.
\end{tikzcd}$$

(2.2) Definition. The *category of all* (direct) *systems* in $\mathfrak{C}$, denoted dir-$\mathfrak{C}$, has as objects the systems in $\mathfrak{C}$ for all possible index categories. If $\mathcal{X}\colon I \to \mathfrak{C}$ and $\mathcal{Y}\colon J \to \mathfrak{C}$ are objects of dir-$\mathfrak{C}$, then the set of morphisms in dir-$\mathfrak{C}$, from $\mathcal{X}$ to $\mathcal{Y}$, is the following set:

$$\operatorname{Hom}_{\text{dir-}\mathfrak{C}}(\mathcal{X}.\mathcal{Y}) := \varprojlim_{i\in I} \varinjlim_{j\in J} \operatorname{Hom}_{\mathfrak{C}}(\mathcal{X}_i, \mathcal{Y}_j); \tag{2.1.1}$$

the limit and colimit on the right are in the category of sets. So, according to the description above, a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is a compatible element family $f = (f_i)$ of elements $f_i \in \varinjlim_{j\in J} \operatorname{Hom}_{\mathfrak{C}}(\mathcal{X}_i, \mathcal{Y}_j)$. Such a family is determined by selecting for each index $i \in I$ an index $j = j(i) \in J$ and a morphism *representing* f_i:

$$f_{ij}\colon \mathcal{X}_i \to \mathcal{Y}_j;$$

to such morphisms $f_{ij}\colon \mathcal{X}_i \to \mathcal{Y}_j$ and $f_{ij'}\colon \mathcal{X}_i \to \mathcal{Y}_{j'}$ determine the same f_i if they are equivalent. Compatibility of the f_i means that for any morphism $\varphi\colon i \to i'$ in I, the morphisms $f_{i'j'}\varphi_*\colon \mathcal{X}_i \to Y_{j'}$ and $f_{ij}\colon \mathcal{X}_i \to \mathcal{Y}_j$ (where $j = j(i)$ and $j' = j(i')$) are equivalent:

$$\begin{tikzcd}
\mathcal{X}_i \arrow[r, "f_{ij}"] \arrow[d, "\varphi_*"'] \arrow[dr] & \mathcal{Y}_j \\
\mathcal{X}_{i'} \arrow[r, "f_{i'j'}"] & \mathcal{Y}_{j'}.
\end{tikzcd}$$

Composition of morphisms in dir-$\mathfrak{C}$ is defined on the representatives: Let $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$ be morphisms in dir-$\mathfrak{C}$, say f is represented by morphisms $f_{ij(i)}\colon \mathcal{X}_i \to \mathcal{Y}_{j(i)}$ with a selection $j = j(i)$ and g is represented by morphisms $g_{jk(j)} : \mathcal{Y}_j \to \mathcal{Z}_{k(j)}$ with a selection $k = k(j)$, then the composition gf is represented by the morphisms $g_{jk} f_{ij} : \mathcal{X}_i \to \mathcal{Y}_j \to \mathcal{Z}_k$ where $j = j(i)$ and $k = k(j(i))$.

It is easy to see that composition is associative. Obviously, for any system $\mathcal{X} : I \to J$, the family $1 : \mathcal{X}_i \to \mathcal{X}_i$ represents a morphism $\mathcal{X} \to \mathcal{X}$ which is a unit for the composition. Hence dir-$\mathfrak{C}$ is a category.

Clearly, a functor $F : \mathfrak{C} \to \mathfrak{D}$ extends in a natural way to a functor dir-$\mathfrak{C} \to$ dir-$\mathfrak{D}$ between the categories of systems.

(2.3). Associated with every object X of $\mathfrak{C}$ there is a system in $\mathfrak{C}$:

$$X_1 : \mathbf{1} \to \mathfrak{C},$$

indexed by the one point category $\mathbf{1}$ and with the (constant) value X. Clearly, the association defines an embedding,

$$(\,)_1 : \mathfrak{C} \subseteq \text{dir-}\mathfrak{C}, \tag{2.3.1}$$

of $\mathfrak{C}$ as a full subcategory of the category of systems in $\mathfrak{C}$. More generally, if $\mathcal{Y} : J \to \mathfrak{C}$ and $\mathcal{Z} : K \to \mathfrak{C}$ are objects of dir-$\mathfrak{C}$, then

$$\mathrm{Hom}_{\text{dir-}\mathfrak{C}}(\mathcal{Z}, X_1) = \varprojlim_{k \in K} \mathrm{Hom}_{\mathfrak{C}}(\mathcal{Z}_k, X), \quad \mathrm{Hom}_{\text{dir-}\mathfrak{C}}(X_1, \mathcal{Y}) = \varinjlim_{j \in J} \mathrm{Hom}_{\mathfrak{C}}(X, \mathcal{Y}_j). \tag{2.3.2}$$

It follows from the first equation that it is equivalent to give a morphism $f : \mathcal{Z} \to X_1$ and to give a common target X for the system $\mathcal{Z} : K \to \mathfrak{C}$.

Via the inclusion (2.3.1) an I-system $\mathcal{X} : I \to \mathfrak{C}$ in $\mathfrak{C}$ may be viewed as an I-system $\mathcal{X}_1 : i \mapsto (\mathcal{X}_i)_1$ in dir-$\mathfrak{C}$. For the I-system $\mathcal{X}_1 : I \to$ dir-$\mathfrak{C}$, the object $\mathcal{X} \in$ dir-$\mathfrak{C}$ is a common target: For every $i \in I$, the identity of $\mathcal{X}_i$ represents an element in $\mathrm{Hom}_{\text{dir-}\mathfrak{C}}(\mathcal{X}_i, \mathcal{X}) = \varinjlim_{j \in I} \mathrm{Hom}_{\mathfrak{C}}(\mathcal{X}_i, \mathcal{X}_j)$. From the compatible family $(\mathcal{X}_i)_1 \to \mathcal{X}$ of morphisms in dir-$\mathfrak{C}$, we obtain a natural map of sets, for $\mathcal{Y}$ in dir-$\mathfrak{C}$,

$$\mathrm{Hom}_{\text{dir-}\mathfrak{C}}(\mathcal{X}, \mathcal{Y}) \longrightarrow \varprojlim_{i \in I} \mathrm{Hom}_{\text{dir-}\mathfrak{C}}((\mathcal{X}_i)_1, \mathcal{Y}), \tag{2.3.3}$$

and it follows from (2.3.2) that the map (2.3.3) is bijective for all $\mathcal{Y}$. Therefore, by Proposition (1.15), the system $\mathcal{X}$, as an object in dir-$\mathfrak{C}$, is the colimit of the system $(\mathcal{X})_1$,

$$\mathcal{X} = \varinjlim_{i \in I} (\mathcal{X}_i)_1.$$

Warning. The embedding $(\,)_1 : \mathfrak{C} \to$ dir-$\mathfrak{C}$ does not commute with colimits. So, even if the colimit $X := \varinjlim \mathcal{X}_i$ exists in $\mathfrak{C}$, it does not follow that X_1 is the colimit of the system $(\mathcal{X})_1 : I \to$ dir-$\mathfrak{C}$.

(2.4). Let $\mathcal{X}\colon I \to \mathfrak{C}$ and $\mathfrak{Y}\colon J \to \mathfrak{C}$ be objects of dir-$\mathfrak{C}$. Assume that the colimit $Y := \varinjlim_j \mathcal{Y}_j$ exists. Then, for every object $Z \in \mathfrak{C}$, there is a canonical map of sets

$$\varinjlim_j \operatorname{Hom}_{\mathfrak{C}}(Z, \mathcal{Y}_j) \to \operatorname{Hom}_{\mathfrak{C}}(Z, Y). \tag{2.4.1}$$

Assume that the colimit $X := \varinjlim_i \mathcal{X}_i$ exists. Set $Z = \mathcal{X}_i$ in (2.4.1) and pass to the limit over i. On the left you get $\operatorname{Hom}_{\text{dir-}\mathfrak{C}}(\mathcal{X}, \mathcal{Y})$ and on the right you get $\operatorname{Hom}_{\mathfrak{C}}(X, Y)$ since $\operatorname{Hom}_{\mathfrak{C}}(-, Y)$ commutes with limits in the first variable. So the result is a canonical map,

$$\operatorname{Hom}_{\text{dir-}\mathfrak{C}}(\mathcal{X}, \mathcal{Y}) \to \operatorname{Hom}_{\mathfrak{C}}(X, Y). \tag{2.4.2}$$

It associates with a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of systems a morphism $\varinjlim_I \mathcal{X} \to \varinjlim_J \mathcal{Y}$; it is natural to denote it $\varinjlim f$. It is functorial in $\mathcal{X}$ and $\mathcal{Y}$ when the colimits are defined.

Lemma. *Consider a morphism of systems $f\colon \mathcal{X} \to \mathcal{Y}$. Then, if the colimits exist, the induced morphism $\varinjlim \mathcal{X} \to \varinjlim \mathcal{Y}$ is an isomorphism if and only if the following maps, for all objects Z if $\mathfrak{C}$, are bijective,*

$$\operatorname{Hom}_{\text{dir-}\mathfrak{C}}(\mathcal{Y}, Z_1) \to \operatorname{Hom}_{\text{dir-}\mathfrak{C}}(\mathcal{X}, Z_1). \tag{2.4.3}$$

Proof. Indeed, by definition of the morphisms in dir-$\mathfrak{C}$ and by the universal property of colimit, the map (2.4.2) identifies with the map induced by $\varinjlim f$,

$$\operatorname{Hom}_{\mathfrak{C}}(\varinjlim \mathcal{Y}, Z) \to \operatorname{Hom}_{\mathfrak{C}}(\varinjlim \mathcal{X}, Z),$$

and the latter map is bijective for all Z if and only if $\varinjlim f$ is an isomorphism. ▯

(2.5) Note. For a small category J, the identity functor $1_J\colon J \to J$ is a system in J and hence an object in dir-J.

Observation 1. *The system $1_J\colon J \to J$, given by the identity, is the final object of* dir-J.

Proof. Denote by $\{*\}$ the one-point-set. We have to prove, for any system $\Phi\colon I \to J$ that

$$\operatorname{Hom}_{\text{dir-}J}(\Phi, 1_J) = \{*\}.$$

By (2.2.1), the Hom-set is a limit (over $i \in I$), and the limit of one-point-sets is a one-point-set. Hence we may assume that $I = \mathbf{1}$ and that Φ has the form k_1 for some element k of J. So we have to establish the equation,

$$\varinjlim_{j \in J} \operatorname{Hom}_J(k, j) = \{*\}.$$

This equation follows from the description in (2.1): A representative for an element in the colimit is a morphism $\varphi\colon k \to j$ for some j, and clearly $\varphi\colon k \to j$ is equivalent to the identity $1\colon k \to k$. ▯

Observation 2. *More generally, a functor $\Phi\colon I \to J$ is a final object in the category* dir- J *if an only if the right fibers of Φ are non-empty and connected, that is, if and only if Φ is final as defined in* (1.16).

Proof. As in the previous proof, Φ is a final object in dir- J if and only if for every $k \in J$ we have the equation,

$$\varinjlim_{i\in I} \operatorname{Hom}_J(k, \Phi i) = \{*\}.$$

So the assertion follows from the description in (1.11). ∎

Now, let $\Phi\colon I \to J$ be any functor. Then, since 1_J is the final object of dir- J, there is a canonical morphism $\Phi \to 1_J$ in dir- J. So, applying the functor $\mathcal{Y}\colon J \to \mathfrak{C}$, the result is a morphism in dir- $\mathfrak{C}$:

$$\mathcal{Y}\Phi \to \mathcal{Y}. \tag{2.5.1}$$

It is the morphism obtained by selecting $j(i) := \Phi i$ and with the identities $\mathcal{Y}_{\Phi i} \to \mathcal{Y}_{\Phi i}$ as representatives. Since the colimit is functorial on dir- $\mathfrak{C}$, the morphism (2.5.1) in dir- $\mathfrak{C}$ induces a morphism in $\mathfrak{C}$,

$$\varinjlim_{i\in I} \mathcal{Y}_{\Phi i} \to \varinjlim_{j\in J} \mathcal{Y}_j. \tag{2.5.2}$$

We recover the result of Proposition (1.17): If $\Phi\colon I \to J$ is a final functor, then (2.5.1) is an isomorphism; hence, so is (2.5.2).

(2.6) Dir-representable functors. Consider the category $\mathfrak{Funct} = \mathfrak{Funct}(\mathfrak{C}^{\mathrm{op}}, \mathbf{Sets})$ of contravariant set-valued functors on $\mathfrak{C}$. For a fixed object $Y \in \mathfrak{C}$, the Hom-functor in the first variable, $\operatorname{Hom}_{\mathfrak{C}}(\ ,Y)$ belongs to $\mathfrak{Funct}(\mathfrak{C}^{\mathrm{op}}, \mathbf{Sets})$. We write $Y(Z)$ for its value at $Z \in \mathfrak{C}$:

$$Y(Z) := \operatorname{Hom}_{\mathfrak{C}}(Z, Y);$$

clearly $Y \mapsto Y(\,)$ is a covariant functor $\mathfrak{C} \to \mathfrak{Funct}(\mathfrak{C}^{\mathrm{op}}, \mathbf{Sets})$. It is a full embedding. In fact, by the Yoneda representation theorem, for any contravariant functor $T \in \mathfrak{Funct}(\mathfrak{C}^{\mathrm{op}}, \mathbf{Sets})$ there is a natural bijection of sets, functorial in $X \in \mathfrak{C}$,

$$\operatorname{Hom}_{\mathfrak{Funct}}(X(\,), T) = T(X).$$

It is determined by $\Psi \mapsto \Psi_X(1_X)$ for transformations $\Psi\colon X(\,) \to T$; the transformation $\Phi(\xi)$ corresponding to an element $\xi \in T(X)$ is given by $f \mapsto T(f)(\xi)$ for $f\colon Z \to X$.

In particular, for $T := Y(\,)$, we obtain the bijection $\operatorname{Hom}_{\mathfrak{Funct}}(X(\,), Y(\,)) = \operatorname{Hom}_{\mathfrak{C}}(X, Y)$, and hence $X \mapsto X(\,)$ is an embedding,

$$\mathfrak{C} \subseteq \mathfrak{Funct}(\mathfrak{C}^{\mathrm{op}}, \mathbf{Sets}),$$

of $\mathfrak{C}$ as a full subcategory of the functor category. A contravariant functor of the form $X(\,)$ with $X \in \mathfrak{C}$ is said to be *representable*.

Let $\mathcal{Y}\colon J \to \mathfrak{C}$ be a J-system in $\mathfrak{C}$. The functor category $\mathfrak{Funct}(\mathfrak{C}^{\mathrm{op}}, \mathbf{Sets})$ has the same colimits as the category of sets. So, in the functor category we may form the colimit $\varinjlim_j \mathcal{Y}_j(\,)$. It is denoted $\mathcal{Y}(\,)$, and its value at the object $Z \in \mathfrak{C}$ is the colimit of sets,

$$\mathcal{Y}(Z) = \varinjlim_j \mathrm{Hom}_{\mathfrak{C}}(Z, \mathcal{Y}_j).$$

A contravariant functor of the form $\mathcal{Y}(\,)$ with a system $Y\colon J \to \mathfrak{C}$ is said to be *dir-representable*. If $\mathcal{X}\colon I \to \mathfrak{C}$ is a second system in $\mathfrak{C}$, then we have the following four equalities:

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{Funct}}(\mathcal{X}(\,), \mathcal{Y}(\,)) &= \varprojlim_i \mathrm{Hom}_{\mathfrak{Funct}}(\mathcal{X}_i, \mathcal{Y}(\,)) = \varprojlim_i \mathcal{Y}(\mathcal{X}_i) = \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathfrak{C}}(\mathcal{X}_i, \mathcal{Y}_j))\\ &= \mathrm{Hom}_{\text{dir-}\mathfrak{C}}(\mathcal{X}, \mathcal{Y}).\end{aligned}$$

Indeed, the first holds since $\mathcal{X}(\,)$ is the colimit of the $\mathcal{X}_i(\,)$ in the functor category, the second holds by Yoneda, the third by definition of the functor $\mathcal{Y}(\,)$, and the last by (2.2.1). It follows that $\mathcal{X} \mapsto \mathcal{X}(\,)$ is an equivalence from the category dir-$\mathfrak{C}$ of direct systems in $\mathfrak{C}$ to the full subcategory of $\mathfrak{Funct}(\mathfrak{C}^{\mathrm{op}}, \mathbf{Sets})$ consisting of dir-representable functors.

(2.7) Definition. A system $\mathcal{X}\colon I \to \mathfrak{C}$ is said to be *essentially constant* if the following two (equivalent) conditions hold:

(i) There is an object $X \in \mathfrak{C}$ and an isomorphism in dir-$\mathfrak{C}$,

$$\mathcal{X} \xrightarrow{\sim} X_1. \tag{2.7.1}$$

(ii) The colimit $\varinjlim_{i\in I} \mathcal{X}_i = X$ exists, and any functor $F\colon \mathfrak{C} \to \mathfrak{D}$ commutes with this limit: $\varinjlim_{i\in I} F\mathcal{X}_i = FX$.

The equivalence is almost obvious: A common target $x\colon \mathcal{X} \to X$ corresponds to the morphism (2.7.1) in dir-$\mathfrak{C}$. If (2.7.1) is an isomorphism, then so is the morphism $F\mathcal{X} \to (FX)_1$ in dir-$\mathfrak{D}$; apply the functor $\varinjlim$ to obtain the isomorphism $\varinjlim F\mathcal{X} = FX$. Conversely, if (ii) holds, then the isomorphism in (2.7.1) is obtained by taking as F the inclusion $(\,)_1\colon \mathfrak{C} \to$ dir-$\mathfrak{C}$.

Clearly, under the correspondence between systems and dir-representable functors, the system $\mathcal{X}$ is essentially constant if and only if the contravariant functor $\mathcal{X}(\,)$ is representable.

(2.8). Consider a system in dir-$\mathfrak{C}$ indexed by an index category I, say $i \mapsto \mathcal{X}^{(i)}$ for $i \in I$. Then each $\mathcal{X}^{(i)}$ is a system $\mathcal{X}^{(i)}\colon J_i \to \mathfrak{C}$ with an index category J_i.

If the colimit,

$$\mathcal{Y}_i := \varinjlim_{j\in J_i} \mathcal{X}^{(i)}_j,$$

exists for each index $i \in I$, then by the functorial properties of $\varinjlim$ in (2.4) the $\mathfrak{Y}_i$ form an I-system $\mathcal{Y}$ in $\mathfrak{C}$, and hence an object in $\mathcal{Y} \in$ dir-$\mathfrak{C}$. Moreover, for each $i \in I$ there is a morphism $\kappa_i\colon \mathcal{X}^{(i)} \to \mathcal{Y}$ represented by the morphism $\mathrm{in}_j\colon \mathcal{X}^{(i)}_j \to \mathcal{Y}_i$, and the κ_i is a compatible family of morphisms in dir-$\mathfrak{C}$,

$$\kappa_i\colon \mathcal{X}^{(i)} \to \mathcal{Y},$$

making $\mathcal{Y}$ a common target of the system $\mathcal{X}^{(i)}$. This common target is in general not universal, that is, $\mathcal{Y}$ is not the colimit of the system $\mathcal{X}^{(i)}$; if the colimit $\mathcal{X} := \varinjlim \mathcal{X}^{(i)}$ exists in dir-$\mathfrak{C}$, then the κ_i determine a canonical morphism $\mathcal{X} \to \mathcal{Y}$.

Proposition. *Consider an I-system $i \mapsto \mathcal{X}^{(i)}$ in* dir-$\mathfrak{C}$ *such that*

(i) *The colimit $\mathcal{X} := \varinjlim_{i\in I} \mathcal{X}^{(i)}$ exists in* dir-$\mathfrak{C}$, *and*

(ii) *the colimit $\mathcal{Y}_i := \varinjlim_{j\in J_i} \mathcal{X}^{(i)}_j$ exists in $\mathfrak{C}$ for every $i \in I$.*

Then the canonical morphism $\mathcal{X} \to \mathcal{Y}$ induces, for every object Z of $\mathfrak{C}$, a bijection,

$$\mathrm{Hom}_{\text{dir-}\mathfrak{C}}(\mathcal{Y}, Z_1) \to \mathrm{Hom}_{\text{dir-}\mathfrak{C}}(\mathcal{X}, Z_1). \tag{2.8.1}$$

Proof. The system $\mathcal{Y}$ in dir-$\mathfrak{C}$ is the colimit of the systems $(Y_i)_1$ and the system $\mathcal{X}$ is the colimit of the systems $\mathcal{X}^{(i)}$. Therefore, since Hom commutes with limits in the first variable, it suffices to prove that the following map is bijective for any i:

$$\mathrm{Hom}_{\text{dir-}\mathfrak{C}}((\mathcal{Y}_i)_1, Z_1) \to \mathrm{Hom}_{\text{dir-}\mathfrak{C}}(\mathcal{X}^{(i)}, Z_1).X, Z_1). \tag{2.8.2}$$

Identify the map (2.8.2) with the following map:

$$\mathrm{Hom}_{\mathfrak{C}}(\mathcal{Y}_i, Z) \to \varprojlim_i \mathrm{Hom}_C(\mathcal{X}^{(i)}, Z).X, Z_1). \tag{2.8.3}$$

The map (2.8.3) is bijective because $\mathcal{Y}_i = \varinjlim \mathcal{X}^{(i)}$. ∎

(2.9) Corollary. *Under the conditions (i) and* (ii) *of* (2.8), *we have an isomorphism in $\mathfrak{C}$,*

$$\varinjlim \mathcal{X} \xrightarrow{\sim} \varinjlim \mathcal{Y},$$

provided that one of the two colimits exists.

Proof. The assertion follows from Lemma (2.4). ∎

Remark. If every system $\mathcal{X}^{(i)}$ is essentially constant, i.e., $\mathcal{X}^{(i)} \xrightarrow{\sim} (\mathcal{Y}_i)_1$ in dir-$\mathfrak{C}$, then we have the equations,

$$\mathcal{X} = \varinjlim \mathcal{X}^{(i)} = \varinjlim (\mathcal{Y}_i)_1 = \mathcal{Y}.$$

In particular, in this case $\mathcal{X}$ is essentially constant if and only if $\mathcal{Y}$ is essentially constant.

(2.10). Consider a direct system $\mathcal{X}\colon I \times J \to \mathfrak{C}$ defined on a product category.

⋮

(2.11). An abelian category $\mathfrak{A}$ has equalizers and coequalizers. Hence it has $\varprojlim_I$'s if and only if it has $\prod_I$'s, and it has $\varinjlim'_I$ if and only if it has $\coprod_I$'s.

Proposition. *Let $\mathfrak{A}$ be an abelian category with exact $\coprod_I$. Then the functor $\varinjlim_I\colon \mathfrak{A}^I \to \mathfrak{A}$ is right exact.*

Proof. … ∎

3. Inductive and projective limits.

(3.1) Definition. A system $\mathcal{X}\colon I \to \mathfrak{C}$ is called an *inductive system* if the index category I is filtering. An inverse system $\mathcal{Y}\colon I \to \mathfrak{C}$, that is, a contravariant functor $\mathcal{Y}\colon I \to \mathfrak{C}$, is called a *projective system* if the index category I is filtering.

A colimit of an inductive system is often called an *inductive limit*, and a limit of a projective system is often called a *projective limit*.

(3.2) Observation. *Let $i \mapsto \mathcal{X}_i$ be an inductive system of sets. Consider the inductive limit,*

$$\varinjlim_{i\in I} \mathcal{X}_i = \bigvee_{i\in I} \mathcal{X}_i / \sim,$$

as the quotient of the disjoint union $\bigvee \mathcal{X}_i$ modulo the equivalence relation $\sim$, cf. (1.11)*. Then:*

(1) *Two elements $x \in \mathcal{X}_i$ and $x' \in \mathcal{X}_{i'}$ are equivalent if and only if there are morphisms $\varphi\colon i \to j$ and $\varphi'\colon i' \to j$ such that $\varphi_*(x) = \varphi_*(x')$ in $\mathcal{X}_j$.*

(2) *For any two elements in the inductive limit there is an index i such that the elements have representatives in $\mathcal{X}_i$.*

(3) *Two elements $x, x' \in \mathcal{X}_i$ are equivalent if and only if there is a morphism $\varphi\colon i \to j$ such that $\varphi_*(x) = \varphi_*(x')$.*

The assertions are easy consequences of the filtering conditions on the index set.

Proposition. *Let I be a filtering category. Then the inductive limit $\varinjlim_I\colon \mathbf{Sets}^I \to \mathbf{Sets}$ commutes with finite limits and arbitrary colimits.*

Proof. The second assertion is obvious since a colimit commutes with arbitrary colimits. To prove the first assertion is suffices to prove that $\varinjlim_I$ commutes with equalizers and final element and with the product $\mathcal{X} \times \mathcal{Y}$ of two I-systems.

□

(3.3) Construction. Let $i \to \mathcal{X}_i$ be an inductive system of abelian groups. Consider the inductive limit of the underlying sets,

$$E := \bigvee_i \mathcal{X}_i / \sim .$$

Let $a, b \in E$ be elements of the quotient, and chose an index i and representatives $x, y \in \mathcal{X}_i$ for a and b. Let $a + b$ be the element in the quotient E represented by the sum $x + y \in \mathcal{X}_i$. It is easy to see, using the description in (3.2) that the element $a+b$ is independent of the choice of i and of the representatives x, y. So we have obtained a composition $(a, b) \mapsto a + b$ in the set E. Furthermore, it is easy to see that E with this composition is an abelian group, that the canonical injections of sets $\mathcal{X}_i \to E$ are homomorphisms of groups (and hence E is a common target of the given system in $\mathbf{Ab}$), and that E is, in fact, the colimit in $\mathbf{Ab}$ of $\mathcal{X}$. Hence E is the inductive limit of the $\mathcal{X}_i$. The construction may be stated as the following result:

Corollary. *The forgetful functor* $\square$: **Ab** $\to$ **Sets** *commutes with inductive limits.*

(3.4) Corollary. *Let I be a filtering category. Then the inductive limit* $\varinjlim_I \colon \mathbf{Ab}^I \to \mathbf{Ab}$ *commutes with finite limits and arbitrary colimits.*

Proof. We know that the inductive limit, as a colimit, commutes with arbitrary colimits. To prove the first assertion consider a finite category J and a J-system $j \mapsto \mathcal{X}^{(j)}$ of I-systems $i \mapsto \mathcal{X}_i^{(j)}$ in **Ab**. We have to prove that the canonical homomorphism,

$$\varprojlim_{j\in J} \varinjlim_{i\in I} \mathcal{X}_i^{(j)} \to \varinjlim_{i\in I} \varprojlim_{j\in J} \mathcal{X}_i^{(j)}, \tag{3.4.1}$$

is an isomorphism of abelian groups. It suffices to prove that it is a bijective map of the underlying sets. So apply the forgetful functor $\square$: **Ab** $\to$ **Sets** to (3.4.1). The functor commutes with arbitrary limits by the results in Section 1, and it commutes $\lim_I$ by the corollary in (3.3). Therefore, the bijectivity of (3.4.1) follows from the Proposition in (3.2). $\square$

Similar considerations apply to categories of sets with an algebraic structure, like **Gr**, ${}_k$**Mod**, **Rings**, etc.

(3.5) Note. It is easy to see that the assertions (3.2) (i) and (iii) and the conclusion in Corollary (3.4) hold if I is only assumed to be pseudo-filtering. ????

(3.6) Definition. The category ind-$\mathfrak{C}$ of *ind-objects* of $\mathfrak{C}$ is the full subcategory of dir-$\mathfrak{C}$ determined by the inductive systems in $\mathfrak{C}$. So for inductive systems $\mathcal{X}\colon I \to \mathfrak{C}$ and $\mathcal{Y}\colon J \to \mathcal{C}$, the set of morphisms from $\mathcal{X}$ to $\mathcal{Y}$ is the following set,

$$\mathrm{Hom}_{\text{ind-}\mathfrak{C}}(\mathcal{X}.\mathcal{Y}) := \varprojlim_{i\in I} \varinjlim_{j\in J} \mathrm{Hom}_{\mathfrak{C}}(\mathcal{X}_i, \mathcal{Y}_j). \tag{3.6.1}$$

As J is filtering, the colimit on the right hand side is an inductive limit, and so the observations in (3.2) apply.

The one-point-category **1** is filtering and so the constant system X_1 defined by an object $X \in \mathfrak{C}$ is an inductive system. In other words, the the functor $X \mapsto X_1$ of (2.3) is a full embedding of $\mathfrak{C}$ into the ind-category,

$$\mathfrak{C} \subseteq \text{ind-}\mathfrak{C}.$$

If $\mathcal{X}\colon I \to \mathfrak{C}$ is an inductive system, an J is a final subcategory of I, then J filtering and the restriction $\mathcal{X}|J\colon J \to \mathfrak{C}$ is an inductive system. Moreover, since the inclusion $J \subseteq I$ is final, it follow that the natural morphism is an isomorphism,

$$\mathcal{X}|J \xrightarrow{\sim} \mathcal{X}.$$

(3.7) Note. Under duality, direct systems and inductive systems correspond to inverse systems and projective systems. An inverse system in $\mathfrak{C}$ is a contravariant functor $\mathcal{X}\colon I \to \mathfrak{C}$, and it corresponds to a covariant functor $\mathcal{X}^{\mathrm{op}}\colon I \to \mathfrak{C}^{\mathrm{op}}$, and hence to a direct system in $\mathfrak{C}^{\mathrm{op}}$. The category of inverse systems in $\mathfrak{C}$, denoted inv- $\mathfrak{C}$, has as objects the inverse systems in $\mathfrak{C}$. The set of morphisms in the inv-category from the inverse I-system $\mathcal{X}$ to the inverse J-system $\mathcal{Y}$ is the set,

$$\mathrm{Hom}_{\text{inv-}\mathfrak{C}}(\mathcal{X}.\mathcal{Y}) := \varprojlim_{j\in J} \varinjlim_{i\in I} \mathrm{Hom}_{\mathfrak{C}}(\mathcal{X}_i, \mathcal{Y}_j);$$

The category pro- $\mathfrak{C}$ is the full subcategory of inv- $\mathfrak{C}$ determined by the projective systems. Clearly, the functor $\mathcal{X} \to \mathcal{X}^{\mathrm{op}}$ determines isomorphisms of categories,

$$\big(\text{inv-}\mathfrak{C}\big)^{\mathrm{op}} = \text{dir-}\mathfrak{C}^{\mathrm{op}}, \qquad \big(\text{pro-}\mathfrak{C}\big)^{\mathrm{op}} = \text{ind-}\mathfrak{C}^{\mathrm{op}}.$$

(3.8) Lemma. *Let $i \mapsto \mathcal{X}_i$ be an inductive I-system which is local, that is, for every morphism $\varphi\colon i \to j$ in I the transition morphism $\varphi\colon \mathcal{X}_i \to \mathcal{X}_j$ is an isomorphism in $\mathfrak{C}$. Then the inductive limit $\varinjlim_I \mathcal{X}$ exists and, for every index $i_0 \in I$, the i_0'th injection is an isomorphism,*

$$\mathrm{in}_{i_0}\colon \mathcal{X}_{i_0} \xrightarrow{\sim} \varinjlim_{i\in I} \mathcal{X}.$$

Proof. It follows from (FILT 2) that the transition morphism $\varphi\colon \mathcal{X}_i \to \mathcal{X}_j$, corresponding to a morphism $\varphi\colon i \to j$, is independent of φ; call it $\varphi_{i,j}$. So, if there exists a morphism $\varphi\colon i \to j$, there is a well-defined isomorphism $\varphi_{ij}\colon \mathcal{X}_i \to \mathcal{X}_j$. Now, if i is an arbitrary index there is an index j and morphisms $i \to j$ and $i_0 \to j$. So there is a well-defined isomorphism $\varphi_i := \varphi_{i_0 j}^{-1}\varphi_{ij}\colon \mathcal{X}_i \to \mathcal{X}_{i_0}$. Now check that $\mathcal{X}_{i_0}$ is a common target of $\mathcal{X}$ with the φ_i as compatible family and that this makes $\mathcal{X}_{i_0}$ the colimit of the $\mathcal{X}_i$. ▯

(3.9) Lemma. *Let $\mathcal{X}\colon I \to \mathfrak{C}$ be an ind-object. Assume that there exists a final subcategory $J \subseteq I$ such that for any morphism $j \to j'$ in J, the corresponding transition morphism $\mathcal{X}_j \to \mathcal{X}_{j'}$ is an isomorphism. Then $\mathcal{X}$ is essentially constant.*

Proof. The morphism $\mathcal{X}|J \to \mathcal{X}$ is an isomorphism in ind- $\mathfrak{C}$. Hence the assertion of the Lemma follows from Lemma (3.8). ▯

(3.10) Observation. Assume that the category $\mathfrak{C}$ has an additive hom-structure. Then, in equation (3.6.1), the set $\varinjlim_j \mathrm{Hom}_{\mathfrak{C}}(\mathcal{X}_i, \mathcal{X}_j)$ is an inductive limit of abelian groups, and hence an abelian group, see (3.3). Moreover, the transition morphisms in the (inverse) system $i \mapsto \varinjlim_j \mathrm{Hom}_{\mathfrak{C}}(\mathcal{X}_i, \mathcal{Y}_j)$ are homomorphisms of abelian groups; so the limit on the right side of (3.6.1) is an abelian group, see the Observation in (1.12).

So the equality (3.6.1) gives the set $\mathrm{Hom}_{\text{ind-}\mathfrak{C}}(\mathcal{X}.\mathcal{Y})$ the structure of an abelian group; it defines an additive Hom-structure in $\mathfrak{C}$.

D R A F T

4. Localization.

Fix categories $\mathfrak{C}$ and $\mathfrak{D}$, and a class S of morphisms of $\mathfrak{C}$. From Section (4.3) S is assumed to be a left denominator system.

(4.1) Definition. The class S of morphisms in $\mathfrak{C}$ is called a *left denominator system* if the following conditions are satisfied:

(LOC 0) S is closed under composition and contains all identities.

(LOC 1) Every pair of morphisms $s\colon X \to X'$ and $f\colon X \to Y$ with $s \in S$ can be embedded into a commutative diagram with $t \in S$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[d, dashed, "t"] \\ X' \arrow[r, dashed, "f'"] & Y'. \end{tikzcd}$$

(LOC 2) Every pair of morphisms $X \rightrightarrows Y$ which is equalized by a morphism $s\colon X' \to X$ in S can be coequalized by a morphism $t\colon Y \to Y'$ in S.

The condition (LOC 0) is the *multiplicative condition*. Conditions (LOC 1) and (LOC 2) are the *left Ore conditions*. If S satisfies, in addition, the (dual) right Ore conditions, then S is called a *denominator system* in $\mathfrak{C}$.

A left denominator system is called *saturated* if the following condition holds:

(SAT) Let $f\colon X \to Y$ be a morphism. Assume that there are morphisms $g\colon Y \to Z$ and $h\colon Z \to W$ such that $gf \in S$ and $hg \in S$. Then $f \in S$.

(4.2) Definition. Let $F\colon \mathfrak{C} \to \mathfrak{D}$ be a functor. If S is any class of morphisms in $\mathfrak{C}$, we will say that F is *S-local* or *S-localizing* if it transforms morphisms in S into isomorphisms in $\mathfrak{D}$.

Clearly, for a given functor $F\colon \mathfrak{C} \to \mathfrak{D}$, the class T of morphisms t in $\mathfrak{C}$ such the $F(t)$ is an isomorphism in $\mathfrak{D}$ satisfies the multiplicative condition and the saturation condition (but not nessecarily the denominator conditions); moreover, F is T-localizing.

(4.3) Definition. Fix a left denominator system S in $\mathfrak{C}$. Let X be an object of $\mathfrak{C}$ and denote by X/S the following category: The objects of X/S are the morphisms $s\colon X \to U$ with $s \in S$ (source X and arbitrary target U); if $s\colon X \to U$ and $t\colon X \to V$ are objects of X/S, then the set of morphisms $s \to t$ is the set of morphisms $f\colon U \to V$ in $\mathfrak{C}$ making the following diagram commutative:

$$\begin{tikzcd} U \arrow[r, "f"] & V \\ X. \arrow[u, "s"] \arrow[ur, "t"'] & \end{tikzcd}$$

There is an obvious "target" functor $(s : X \to X') \mapsto X'$ from X/S to $\mathfrak{C}$; it is denoted

$$X_S\colon X/S \to \mathfrak{C}.$$

The target of an object $s \in X/S$ will be denoted X_s; so an object $s \in X/S$ is a morphism $s\colon X \to X_s$ in S, and the target functor X_S associates with $s\colon X \to X_s$ the target X_s.

Observation. *The category X/S is filtering. As a consequence, the system $X_S\colon X/S \to \mathfrak{C}$ is an inductive system and it may be viewed as an object in the ind-category:*

$$X_S \in \text{ind-}\mathfrak{C}.$$

The assertion is an immediate consequence of the left denominator conditions.

Definition. The *localization* of $\mathfrak{C}$ with respect to S, also called the category of *left fractions* of $\mathfrak{C}$ with denominators in S is the full subcategory of ind-$\mathfrak{C}$ determined by the inductive systems of the form X_S for $X \in \mathfrak{C}$. The category of left fractions is denoted $S^{-1}\mathfrak{C}$ or $\mathfrak{C}_S$, and it is also said to by obtained from $\mathfrak{C}$ by *inverting the morphisms of S*.

For every morphism $f\colon X \to Y$ in $\mathfrak{C}$ there is a natural induced morphism $f_S\colon X_S \to Y_S$ in $S^{-1}\mathfrak{C}$ described as follows: For each index $s \in X/S$ use (LOC 1) to obtain an index $t \in Y/S$ and a commutative diagram,

$$\begin{tikzcd} X_s \arrow[r, dashed, "f'"] & Y_t \\ X \arrow[u, "s"] \arrow[r, "f"] & Y; \arrow[u, dashed, "t"'] \end{tikzcd}$$

the morphism f' represents an element $f_s \in \varinjlim_{t\in Y/S} \operatorname{Hom}_{\mathfrak{C}}(X_s, Y_t)$, and the f_s, for objects $s \in X/S$, define the induced morphism $f_S\colon X_S \to Y_S$ in ind-$\mathfrak{C}$. It is easy to see with this definition that $X \mapsto X_S$ is a functor,

$$(\ )_S\colon \mathfrak{C} \to S^{-1}\mathfrak{C}.$$

The rules for manipulations with left fractions and other properties of the category of left fractions and the functor $(\ )_S$ will be developed in the following.

(4.4) The rules. In the sequel we will repeatedly meet inductive limits of sets of the following form, for objects $X, Y \in \mathfrak{C}$:

$$\varinjlim_{s\in Y/S} \operatorname{Hom}_{\mathfrak{C}}(X, Y_s). \tag{4.4.1}$$

Recall that an element in the inductive limit is given by a representative $a\colon X \to Y_s$ for some index $s\colon Y \to Y_s$ in Y/S; the pair (s, a) may be visualized as a diagram,

$$\begin{tikzcd} & Y_s & \\ X \arrow[ur, "a"] & & Y. \arrow[ul, "s"'] \end{tikzcd} \tag{4.4.2}$$

Two pairs (s, a) and (t, b) define the same element in the inductive limit if there is an index $u\colon Y \to Y_u$ in Y/S and morphisms $f\colon s \to u$ and $g\colon t \to u$ (that is, f, g are morphisms $f\colon Y_s \to Y_u$ and $g\colon Y_t \to Y_u$ such that $fs = u = gt$) such $fa = gb =: c$:

$$\begin{tikzcd} & Y_s \arrow[d, dashed] & \\ X \arrow[ur, "a"] \arrow[r, "c"] \arrow[dr, "b"'] & Y_u & Y. \arrow[ul, "s"'] \arrow[l, "u"'] \arrow[dl, "t"] \\ & Y_t \arrow[u, dashed] & \end{tikzcd}$$

Again, by the filtering properties of the index category, any two elements of the inductive limit have representatives of the form (s, a) and (s, b) (with the same index $s \in Y/S$), and two pairs of this form represent the same element in the limit, if there is an index $t \in Y/S$ and a morphism $f: t \to s$ such that $a, b: X \to Y_t$ are equalized by the morphism $f: Y_t \to Y_s$.

A very good question: *The index category* X/S *in* (4.4.1) *is, in general, not a small category; so, does the inductive limit* (4.4.1) *exist in the category of sets?*

The answer is simple: No, why should it! – in general. The negative answer represents a problem, with several solutions: (1) Enlarge the concept of a category; (2) Add conditions on the setup ensuring the existence of the inductive limit; or, (3) ignore the problem and pretend that the limit exists. Our choice is the path described in (3).

(4.5) Observation. *Let* $s: X \to X'$ *be a morphism in* S*, and* Y *an object in* $\mathfrak{C}$*. Then the canonical map induced by* s *is a bijection:*

$$\varinjlim_{t \in Y/S} \operatorname{Hom}_{\mathfrak{C}}(X', Y_t) \xrightarrow{\sim} \varinjlim_{t \in Y/S} \operatorname{Hom}_{\mathfrak{C}}(X, Y_t).$$

Proof. Use the denominator conditions. ∎

(4.6). For every object $X \in \mathfrak{C}$, the identity $X \xrightarrow{1} X$ is the initial object in X/S. In particular, if $F: \mathfrak{C} \to \mathfrak{D}$ is any functor such the the inductive limit $\varinjlim FX_S = \varinjlim_{s \in X/S} FX_s$ exists, there is a canonical injection morphism $FX = FX_1 \to \varinjlim FX_S$.

Lemma. *Let* $F: \mathfrak{C} \to \mathfrak{D}$ *be an* S*-localizing functor. Then, for any object* $X \in \mathfrak{C}$*, the composition* $FX_S: X/S \to \mathfrak{D}$ *is a local system, and the canonical morphism is an isomorphism,*

$$FX \xrightarrow{\sim} \varinjlim_{s \in X/S} FX_s. \tag{4.6.1}$$

Proof. Consider in X/S objects $s: X \to X_s$ and $t: X \to X_t$ and a morphism $f: s \to t$. Then f is a morphism $f: X_s \to X_t$ and $fs = t$. Hence the following diagram in $\mathfrak{D}$ is commutative:

$$\begin{tikzcd} FX_s \arrow[r, "F(f)"] & FX_t \\ X. \arrow[u, "F(s)"] \arrow[ur, "F(t)"'] & \end{tikzcd}$$

As the morphisms $F(s)$ and $F(t)$ are isomorphisms of $\mathfrak{D}$, then so is $F(f)$. Consequently, the functor $FX_S: X/S \to \mathfrak{D}$ transforms any morphism in X/S into an isomorphism; hence it is a local system.

It follows from Lemma (3.8) that the inductive limit exists and that any of the canonical injections is an isomorphism. ∎

(4.7) Observation. *For every X, Y in $\mathfrak{C}$ the natural map induced by $X_1 \to X_S$ is a bijection,*

$$\mathrm{Hom}_{\text{ind-}\mathfrak{C}}(X_S, Y_S) \xrightarrow{\sim} \mathrm{Hom}_{\text{ind-}\mathfrak{C}}(X_1, Y_S) = \varinjlim_{t \in Y/S} \mathrm{Hom}_{\mathfrak{C}}(X, Y_t). \tag{4.7.1}$$

Proof. Fix Y and consider the functor $F := \varinjlim_{t\in Y/S} \mathrm{Hom}_{\mathfrak{C}}(\ , Y_t)$. By definition of the ind-category, the left side of (4.7.1) is the projective limit of the system FX_S, and the map in (4.7.1) is the projection corresponding to the index $1 \in X/S$:

$$\varprojlim_{s \in X/S} F(X_s) \to F(X_1). \tag{4.7.2}$$

The functor F is a contravariant functor from $\mathfrak{C}$ to **Sets**, and may be viewed as a covariant functor $F \colon \mathfrak{C} \to \mathbf{Sets}^{\mathrm{op}}$. It is S-local by Observation (4.5). So the lemma in (4.6) applies, and it yields an isomorphism in the dual category $\mathbf{Sets}^{\mathrm{op}}$. In the category **Sets**, it is the map (4.7.2). Hence (4.7.2) is bijective. □

(4.8). The result in (4.7) is fundamental for manipulations in the category $S^{-1}\mathfrak{C}$. The set of morphisms from X_S to Y_S in $S^{-1}\mathfrak{C}$ will allways be described via the bijection (4.7.1):

$$\mathrm{Hom}_{S^{-1}\mathfrak{C}}(X_S, Y_S) = \varinjlim_{t \in Y/S} \mathrm{Hom}_{\mathfrak{C}}(X, Y_t). \tag{4.8.1}$$

The inductive limit on the right side is determined in (4.4). Accordingly, a morphism $\varphi \colon X_S \to Y_S$ in $S^{-1}\mathfrak{C}$ is represented by pair (t, f) consting of an index $t \in Y/S$ and a morphism $f \colon X \to Y_t$; it may visualized by the diagram (4.4.2). Composition in $S^{-1}\mathfrak{C}$ is determined as follows: Consider in $S^{-1}\mathfrak{C}$ morphisms $\varphi \colon X_S \to Y_S$ and $\psi \colon Y_S \to Z_S$, represented by pairs (t, f) and (u, g), with $t, u \in S$,

$$\begin{tikzcd}[column sep=small] & Y' & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "t"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] & Z' & \\ Y \arrow[ur, "g"] & & Z. \arrow[ul, "u"'] \end{tikzcd}$$

To represent the composition $\psi\varphi$, embedd to two morphisms t, g with source Y in a commutative square where the edge t' opposite of t belongs to S:

$$\begin{tikzcd}[column sep=small] & & Z'' & & \\ & Y' \arrow[ur, dashed, "g'"] & & Z' \arrow[ul, dashed, "t'"'] & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "t"'] \arrow[ur, "g"] & & Z. \arrow[ul, "u"'] \end{tikzcd}$$

Then the composition $\psi\varphi$ is represented by the pair $(t'u, g'f)$.

Clearly, for a morphism $f \colon X \to Y$, the morphism $f_S \colon X_S \to Y_S$ defined in (4.3) is represented by the pair $(1, f)$:

$$\begin{tikzcd}[column sep=small] & Y \arrow[dr, equal] & \\ X \arrow[ur, "f"] & & Y \end{tikzcd}$$

The morphism $f_S\colon X_S \to Y_S$ is the unique morphism such that the following diagram in ind-$\mathfrak{C}$ is commutative:

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d] & Y_1 \arrow[d] \\ X_S \arrow[r, "f_S"] & Y_S. \end{tikzcd}$$

(4.9) Observation. *For any morphism $u\colon X \to U$ in S, the morphism $u_S\colon X_S \to U_S$ in $S^{-1}\mathfrak{C}$ is invertible. Its inverse is the morphism $U_S \to X_S$ represented by $(u, 1)$. In general, the morphism $X_S \to Y_S$ in $S^{-1}\mathfrak{C}$ represented by the pair (t, f) where $t \in Y/S$ and $f\colon Y \to Y_t$ is equal to $(t_S)^{-1} f_S$.*

Proof. Let $\sigma\colon U_S \to X_S$ be the morphism in $S^{-1}\mathfrak{C}$ represented by the pair $(u, 1)$. The pairs $(u, 1)$, $(1, u)$ and $(u, 1)$ are the three pairs at the base of the following diagram:

$$\begin{tikzcd} & & U \arrow[dr, dash, "1"] & & U \arrow[dr, dash, "1"] & & \\ & U \arrow[ur, dash, "1"] & & U \arrow[ur, dash, "1"] & & U & \\ U \arrow[ur, dash, "1"] & & X \arrow[ul, "u"'] \arrow[ur, "u"] & & U \arrow[ul, dash, "1"] \arrow[ur, dash, "1"'] & & X. \arrow[ul, "u"'] \end{tikzcd}$$

The pair in the middle represents u_S. The diagram is obiously commutative. By the rule of composition, it follows from the left part of the diagram that $u_S\sigma$ is represented by the pair $(1_U, 1_U)$; hence $u_S\sigma$ is the identity of U_S in $S^{-1}\mathfrak{C}$. Again, by the rule of composition, it follows from the left part of the diagram that σu_S is represented by the pair (u, u). Clearly, (u, u) is equivalent to $(1_X, 1_X)$. Hence σu_S is the identity of X_S.

Let φ be the morphism $X_S \to Y_S$ represented by (t, f). It follows easily from the rule of computation that $t_S\varphi$ is represented by the pair $(1, f)$; hence $t_S\varphi = f_S$. As t_S is invertible, it follows that $\varphi = (t_S)^{-1} f_S$. ▯

(4.10) The universal property of localization. *Let S be a left denominator system in $\mathfrak{C}$. Then every S-localizing functor $F\colon \mathfrak{C} \to \mathfrak{D}$ has a unique extension to a functor $\tilde{F}\colon S^{-1}\mathfrak{C} \to \mathfrak{D}$. In other words, there is a unique functor $\tilde{F}\colon S^{-1}\mathfrak{C} \to \mathfrak{D}$ such that $\tilde{F}(\ )_S = F$:*

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "F"] \arrow[d, "(\ )_S"'] & \mathfrak{D} \\ S^{-1}\mathfrak{C}. \arrow[ur, dashed, "\tilde{F}"'] & \end{tikzcd}$$

Proof. Uniquenes is obvious: The morphism $(\ )_S\colon \mathfrak{C} \to S^{-1}\mathfrak{C}$ is bijective on objects, and so the equation $\tilde{F}(X_S) = F(X)$ determines $\tilde{F}$ on objects. By the observation in (4.8) any morphism $\varphi\colon X_S \to Y_S$ is of the form $\varphi = (t_S)^{-1} f_S$, and so the equation $\tilde{F}(\varphi) = F(s)^{-1} F(f)$ determines $\tilde{F}$ on morphisms.

To prove existence consider this diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{S^{-1}\mathfrak{C}}(X_S, Y_S) \arrow[r, dashed, "\tilde{F}"] \arrow[d, "\wr"'] & \mathrm{Hom}_{\mathfrak{D}}(F(X), F(Y)) \arrow[d, "\wr"] \\
\varinjlim_{t\in Y/S} \mathrm{Hom}_{\mathfrak{C}}(X, Y_t) \arrow[r, "F"] & \varinjlim_{t\in Y/S} \mathrm{Hom}_{\mathfrak{D}}(F(X), F(Y_t)).
\end{tikzcd}$$

The left vertical map is the bijection (4.8.1). The bottom horizontal map is the natural morhisms of inductive limits induced by F. The right vertical map is the canonical injection into the inductive limit determined by the index $1 \in Y/S$. The map is the bijection (4.6.1) applied with the functor $Y \mapsto \mathrm{Hom}_{\mathfrak{D}}(F(X), F(Y))$. So, commutativity of the diagram defines the top horizontal map $\tilde{F}$.

It is easy to see that there is a functor $\tilde{F}$ defined by the top horizontal map of the diagram, and that this functor has the required properties. □

(4.11) Note. If S is a right denominator system, then a dual construction leads to the category $\mathfrak{C}S^{-1} = {}_S\mathfrak{C}$ of *right fractions* and a functor ${}_S(\,)\colon \mathfrak{C} \to \mathfrak{C}S^{-1}$, where

$$\mathrm{Hom}_{\mathfrak{C}S^{-1}}({}_SX, {}_SY) = \varinjlim_{s\in S/X} \mathrm{Hom}_{\mathfrak{C}}({}_sX, Y);$$

it satisfies the universal property (4.10).

It follows in particular that if a system S is a denominator system (left and right), then there is a unique isomorphism, from the category of left fractions to the category of right fractions making the following diagram commutative:

$$\begin{tikzcd}
 & \mathfrak{C} \arrow[dl, "(\,)_S"'] \arrow[dr, "{}_S(\,)"] & \\
S^{-1}\mathfrak{C} \arrow[rr, equal] & & \mathfrak{C}S^{-1}.
\end{tikzcd}$$

(4.12) Definition. Let S be a left denominator system in $\mathfrak{C}$. Let X, Y be objects of $\mathfrak{C}$. The set of morphism in the category $S^{-1}\mathfrak{C}$ from X_S to Y_S is often denoted $\mathrm{Ext}_S(X, Y)$:

$$\mathrm{Ext}_S(X, Y) := \mathrm{Hom}_{S^{-1}\mathfrak{C}}(X_S, Y_S) = \varinjlim_{t\in Y/S} \mathrm{Hom}_{\mathfrak{C}}(X, Y_t).$$

The elements of $\mathrm{Ext}_S(X, Y)$ may be called *S-extensions* of X by Y. Note that Ext_S is functor in two variables,

$$\mathrm{Ext}_S\colon \mathfrak{C}^{\mathrm{op}} \times \mathfrak{C} \to \mathbf{Sets},$$

and S-localizing in each variable. By the universal property, an S-localizing functor $F\colon \mathfrak{C} \to \mathfrak{D}$ induces a transformation of functors,

$$\mathrm{Ext}_S(X, Y) \to \mathrm{Hom}_{\mathfrak{D}}(FX, FY),$$

called the *Yoneda transformation*. If $\mathfrak{D}$ is the category of sets, the transformation may be viewed as a pairing,

$$\mathrm{Ext}_S(X, Y) \times FX \to FY.$$

(4.13) Note. Let S be a left denominator system in $\mathfrak{C}$. Let $f\colon X \to Y$ be a morphism in $\mathfrak{C}$. It is easy to see that $f_S\colon X_S \to Y_S$ is an isomorphism in $S^{-1}\mathfrak{C}$ if and only if there are morphisms $g\colon Y \to X'$ and $f'\colon X' \to Y'$ in $\mathfrak{C}$ such that gf and $f'g$ belong to S. In particular, if S is saturated, then f_S is invertible in $S^{-1}\mathfrak{C}$ if and only if $f \in S$.

(4.14). For an arbitrary class T of morphisms in $\mathfrak{C}$, an object $Q \in \mathfrak{C}$ is called T-*injective* if every morphism $X \to X'$ in T induces a surjective map,

$$\mathrm{Hom}_{\mathfrak{C}}(X', Q) \to \mathrm{Hom}_{\mathfrak{C}}(X, Q). \tag{4.14.1}$$

Consider a left denominator system S in $\mathfrak{C}$.

Observation 1. *If Q is an S-injective object, then every morphism $s\colon X \to X'$ induces a bijection,*

$$\mathrm{Hom}_{\mathfrak{C}}(X', Q) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{C}}(X, Q). \tag{4.14.2}$$

Proof. The map (4.14.2) is surjective since the object Q is S-injective. To prove that the map is injective, consider two morphisms $f, g\colon X' \to Q$ having the same image under the map (4.14.2). Then they are equalized by s. Hence, by (LOC 2), they are coequalized by a morphism $t\colon Q \to Q'$ with $t \in S$, that is, $tf = tg$. Since the map $\mathrm{Hom}_{\mathfrak{C}}(Q', Q) \to \mathrm{Hom}_{\mathfrak{C}}(Q, Q)$ is surjective, there is a morphism $p\colon Q' \to Q$ such that $pt = 1_Q$. Then, clearly, the equation $tf = tg$ implies that $f = g$. So the map (4.14.2) □

The observation may be rephrased as follows: An object $Q \in \mathfrak{C}$ is S-injective if and only if the functor $\mathrm{Hom}_{\mathfrak{C}}(\ , Q)$ is S-localizing.

Note that it follows from the result that if a morphism $f\colon Q \to Q'$ between S-injective objects belong to S, then it is an isomorphism.

Observation 2. *If Q is S-injective, then the following map is a bijection, for any object $X \in \mathfrak{C}$:*

$$\mathrm{Hom}_{\mathfrak{C}}(X, Q) \xrightarrow{\sim} \mathrm{Hom}_{S^{-1}\mathfrak{C}}(X_S, Q_S) = \mathrm{Ext}_S(X, Q).$$

Proof. If $t\colon Q \to Z$ is a morphism in S, by Observation 1, there is a unique morphism $f\colon Z \to Q$ such that $ft = 1_Q$. In other words, the morphism $1_Q\colon Q \to Q$ as an object in Q/S is the final object in Q/S. Consequently, for any system $s \mapsto \mathcal{Z}_s$ indexed by the category Q/S, the canonical morphism $\mathcal{Z}_1 \to \varinjlim \mathcal{Z}_s$ is an isomorphism. So the bijection is a consequence of the definition of $\mathrm{Ext}_S(X, Q)$ as an inductive limit. □

(4.15) Remark. *If an S-localizing functor $F\colon \mathfrak{C} \to \mathfrak{D}$ has a right adjoint functor $\rho\colon \mathfrak{D} \to \mathfrak{C}$, then $\rho(D)$ is S-injective for any object $D \in \mathfrak{D}$.*

Indeed, the adjunction equation,

$$\mathrm{Hom}_{\mathfrak{C}}(X, \rho D) = \mathrm{Hom}_{\mathfrak{D}}(FX, D),$$

shows that the functor $\mathrm{Hom}_{\mathfrak{C}}(\ , \rho D)$ is S-localizing.

Proposition. *Let S be a left denominator system in $\mathfrak{C}$. Then the following three conditions are equivalent:*

(i) *The functor $(\;)_S \colon \mathfrak{C} \to S^{-1}\mathfrak{C}$ has a right adjoint functor ρ.*

(ii) *For every object X of $\mathfrak{C}$ there is an S-injective object Q and a morphism $f \colon X \to Q$ such that $f_S \colon X_S \to Q_S$ is an isomorphism in $S^{-1}\mathfrak{C}$.*

(iii) *The ind-object X_S is essentially constant for every object X of $\mathfrak{C}$.*

Moreover, if the conditions hold then the functor $\rho \colon S^{-1}\mathfrak{C} \to \mathfrak{C}$ of (i) *is fully faithful and $(\;)_S\rho \xrightarrow{\sim} 1$.*

Proof. (i) $\Rightarrow$ (ii): Assume that $\rho \colon S^{-1}\mathfrak{C} \to \mathfrak{C}$ is left adjoint to $(\;)_S$. Take objects X, Z of $\mathfrak{C}$. The adjunction bijection appears as the middle vertical map in the following diagram:

$$\begin{tikzcd}
& \mathrm{Hom}_{\mathfrak{C}}(X, \rho Z) \arrow[dl, "(\;)_S"'] \arrow[dd, equal] & \\
\mathrm{Hom}_{S^{-1}\mathfrak{C}}(X_S, (\rho Z)_S) \arrow[dr, "\eta"'] & & \mathrm{Hom}_{\mathfrak{C}}(\rho(X_S), \rho Z) \arrow[ul, "\epsilon"'] \\
& \mathrm{Hom}_{S^{-1}\mathfrak{C}}(X_S, Y_S). \arrow[ur, "\rho"'] &
\end{tikzcd}$$

The maps labeled ϵ and ρ are induced by the unit $\epsilon \colon X \to \rho(X_S)$ and the counit $\eta \colon (\rho Z)_S \to Z$. It follows from the functoriality of the adjunction map that the diagram is commutative.

By the remark, ρZ is S-injective. Hence the map $(\;)_S$ in the diagram is bijective by Observation 2. It follows that the map labeled η is bijective. Since any object of $S^{-1}\mathfrak{C}$ has the form X_S, it follows that

$$\eta_Z \colon (\rho Z)_S \to Z$$

is an isomorphism. As we have noticed, the object $Q := \rho(X_S)$ is S-injective. Hence, to finish the proof of (ii) it suffices to show for the morphism

$$\epsilon_X \colon X \to \rho(X_S)$$

that f_S is invertible. Now, by general properties of adjoint functors, we have the equality:

$$\varepsilon_{X_S}\big(\epsilon_X\big)_S = 1 \colon X_S \to \big(\rho(X_S)\big)_S \to X_S.$$

Since η_{X_S} is invertible, it follows that $(\varepsilon_X)_S$ is invertible. Hence (ii) has been proved. Morover, it follows that the map labeled ϵ in the diagram is bijective. Hence so is the map ρ in the diagram. Whence ρ is fully faithful.

(ii) $\Rightarrow$ (i): Choose for each object X of $\mathfrak{C}$ an S-injective object $\rho(X)$ and a morphism in $\mathfrak{C}$,

$$\epsilon \colon X \to \rho(X),$$

such that $(f_X)_S$ is an isomorphism. For each object $Y \in \mathfrak{C}$, there are bijections,

$$\mathrm{Hom}_{S^{-1}\mathfrak{C}}(Y_S, X_S) \xrightarrow{\sim} \mathrm{Hom}_{S^{-1}\mathfrak{C}}(Y_S, (\rho X)_S) = \mathrm{Hom}_{\mathfrak{C}}(Y, \rho X),$$

the first induced by the isomorphism $(f_X)_S$, the second by the (???), functorial in Y. Use the bijection to defning ρ is a functor $\rho \colon S^{-1}\mathfrak{C} \to \mathfrak{C}$, right adjoint to $(\;)_S$. $\square$

(4.16) Proposition. *Let S be a left denominator system in $\mathfrak{C}$. Then the functor $(\ )_S\colon \mathfrak{C} \to S^{-1}\mathfrak{C}$ is right exact.*

Proof. Let $\mathcal{X}\colon I \to \mathfrak{C}$ be system with a finite index category I, and assume that the colimit $X = \varinjlim_i \mathcal{X}_i$ exists in $\mathfrak{C}$. Every object of $S^{-1}\mathfrak{C}$ is of the form Y_S with an object $Y \in \mathfrak{C}$ so we have to prove, for any object $Y \in \mathfrak{C}$ that the map induced by the system is a bijecition:

$$\mathrm{Hom}_{S^{-1}\mathfrak{C}}(X_S, Y_S) \to \varprojlim_{i\in I} \mathrm{Hom}_{S^{-1}\mathfrak{C}}((\mathcal{X}_i)_S, Y_S).$$

The bijectivity follows from the fundamental description (4.8.1) of morphisms in $S^{-1}\mathfrak{C}$, as the inductive limit $\varinjlim_{t\in Y/S}$, in the category of sets, commutes with the finite limit $\varprojlim_I$, see (3.2). ∎

Corollary. *If $\mathfrak{C}$ has finite coproducts, or coequalizers, or finite colimits, then so has $S^{-1}\mathfrak{C}$.*

Proof. We use repeatedly that every object of $S^{-1}\mathfrak{C}$ is of the form X_S with $X \in \mathfrak{C}$.

First, it follows immediately from the proposition that if $\mathfrak{C}$ has finite coproducts, then so has $S^{-1}\mathfrak{C}$.

Next, assume that $\mathfrak{C}$ has coequalizers. Consider in $S^{-1}\mathfrak{C}$ a pair of morphisms $\varphi, \psi\colon X_S \to Y_S$. Then, by the rules in (4.4) there is a morphism $t\colon Y \to Y'$ in S and morphisms $f, g\colon X \to Y'$ such $f_S = t_S\varphi$ and $g_S = t_S\psi$. Let $h\colon Y' \to Z$ be a coequalizer for f, g. Then, by the proposition, h_S is a coequalizer for f_S, g_S. As t_S is an isomorphism, it follows that $h_S t_S$ is a coequalizer for φ, ψ. Therefore, $S^{-1}\mathfrak{C}$ has coequalizers.

Clearly, the third assertion is a consequence of the first two assertions. ∎

(4.17) Proposition. *Assume that S is a left denominator system in $\mathfrak{C}$. Then:*

(1) *If $\mathfrak{C}$ has an additive Hom-structure, then there is unique additive Hom-structure on $S^{-1}\mathfrak{C}$ such that the functor $(\ )_S\colon \mathfrak{C} \to S^{-1}\mathfrak{C}$ is additive.*
(2) *If $\mathfrak{C}$ is an additive or semi-additive category, then so is $S^{-1}\mathfrak{C}$, and the functor $(\ )_S\colon \mathfrak{C} \to S^{-1}\mathfrak{C}$ is additive.*
(3) *Assume that X is a denominator system. If $\mathfrak{C}$ is an exact category or an abelian category, then so is $S^{-1}\mathfrak{C}$, and the functor $(\ )_S\colon \mathfrak{C} \to S^{-1}\mathfrak{C}$ is exact.*

Proof. Assume the conditions in (1). An additive Hom-structure on $S^{-1}\mathfrak{C}$ such that the functor $(\ )_S$ is additive, is necessarily unique: Indeed, two morphisms $\varphi, \psi\colon X_S \to Y_S$ have representatives (t, f) and (t, g) with the same index $t \in Y/S$. Then $\varphi = (t_S)^{-1} f_S$ and $\psi = (t_S)^{-1} g_S$, and hence $\varphi + \sigma = (t_S)^{-1}(f_S + g_S) = (t_S)^{-1}(f + g)_S$. Conversely, it is obvious that the additive Hom-structure on the category ind-$\mathfrak{C}$, as defined in (3.??), determines an additive Hom-structure on the subcategory $S^{-1}\mathfrak{C}$ satifying the requirement.

Assume the conditions in (2). By (4.16), the functor $(\ )_S\colon \mathfrak{C} \to S^{-1}\mathfrak{C}$ commutes with finite coproducts. Therefore, since the functor is surjective on objects, it follows that $S^{-1}\mathfrak{C}$ has finite coproduct. The remaining assertions follow from (1).

Assume the conditions in (3). Consider a morphism $\varphi\colon X_S \to Y_S$ in $S^{-1}\mathfrak{C}$, say represented by the pair (t, f) where $t \in Y/S$ and $f\colon X \to Y_t$. Then $\varphi = (t_S)^{-1} f_S$. The isomorphism

$(t_S)^{-1}$ induces an isomorphism from the cokernel of f_S to the cokernel of φ. As f_S has a cokernel by (4.16), so has φ. The existence of kernels holds by duality, since S is also a right denonminator system. The rest of the assertions are left as exercises. □

(4.18) Definition. A full subcategory $\mathfrak{C}_0$ of $\mathfrak{C}$ is called *localizing* (with respect to the given left denominator system S) if the following condition holds: For any object $Q \in \mathfrak{C}_0$ and any morphism $s\colon Q \to X$ in S there exists a morphism $f\colon X \to Q'$ with $Q' \in \mathfrak{C}_0$ such that $fs \in S$. Equivalently, if $S_0 := S \cap \mathfrak{C}_0$, then the conditions means for every object $Q \in \mathfrak{C}_0$ that the natural inclusion $Q/S_0 \subseteq Q/S$ is final. It is easy to see for a localizing subcategory $\mathfrak{C}_0$ that the system S_0 is a left denominator system in $\mathfrak{C}_0$ and that the natural functor ind- $\mathfrak{C}_0 \to$ ind- $\mathfrak{C}$ induces a fully faithful embedding,

$$(S_0)^{-1}\mathfrak{C}_0 \subseteq S^{-1}\mathfrak{C}.$$

(4.19) Exercises.
1. Let $f\colon X \to Y$ be a morphism in S. Prove that composition with f is a natural functor $\Phi_f\colon Y/S \to X/S$, and prove that the restricted system $X_S\Phi_f$ is equal to Y_S. Prove that the functor Φ_f is final, and conclude that the restriction morphism $Y_S \to X_S$ in ind- $\mathfrak{C}$ is an isomorphism. Prove that this isomorphism is the inverse of f_S.

2. Let S be a left denominator system in $\mathfrak{C}$. Let f be a morphism in $\mathfrak{C}$ such that $fs \in S$ for some morphism $s \in S$. Prove that there is morphism g and a morphism $t \in S$ such that (gf and gt are defined and) $gf \in S$ and $gt \in S$.

Assume that the left denominator system S is saturated. Repeat the argument with f replaced by g, and conclude that $f \in S$. Assume for three composable morphisms f, g, h that $hg \in S$ and $gf \in S$. Prove that $f \in S$, $g \in S$, and $h \in S$.

DRAFT

5. Localization in triangulated categories.

Fix a triangulated category $\mathfrak{K}$ and in K a system S of morphisms. From section 5.4, S is assumed to be a triangular system and, in particular, a left and right denominator system.

(5.1) Setup. Let $\mathfrak{K}$ be a triangulated category, and S a class of morphisms in $\mathfrak{D}$. It is natural to require conditions on S making it compatible with the triangular operations:

(LOC 3) The class S is stable under shifts, that is, $s \in S$ if and only if $s(1) \in S$.

Consider the following commutative diagram with two exact triangles:

$$\begin{tikzcd}
& X \arrow[rr, "s"] \arrow[dd] & & X' \arrow[dd] \\
Z \arrow[ur, squiggly] \arrow[rr, dashed, "u"] & & Z' \arrow[ur, squiggly] & \\
& Y \arrow[ul] \arrow[rr, "t"'] & & Y'. \arrow[ul]
\end{tikzcd} \tag{5.1.1}$$

By the prism axiom, the pair s, t may be extended with a a morphism $u\colon Z \to Z'$ to a morphism (s, t, u) of triangles. We consider the following condition on S:

(LOC 4) In the setup of the commutative diagram (5.1.1), if $s, t \in S$, then the pair s, t may be extended with a $u \in S$ to a morphism (s, t, u) of triangles.

The system of morphisms $S \subseteq \mathfrak{K}$ will be called a *triangular system of morphisms* if it contains all identities and satisfies conditions (LOC 3) and (LOC 4). As we shall see, the conditions imply the multiplicativity condition (LOC 0). and the left and the right Ore conditions (LOC 1) and (LOC 2); in particular, a triangular system of morphisms is a denominator system.

We will prove later that if a triangular system is saturated, then the following condition holds:

(LOC 4*) In the setup of the commutative diagram (5.1.1), if (s, t, u) is a morphism of triangles, and $s, t \in S$, then $u \in S$.

(5.2) Definition. If S is a system of morphisms of $\mathfrak{K}$ then an object Z of $\mathfrak{K}$ is called S-acyclic if the zero morphism $Z \to 0$ belongs to S. If $\mathfrak{M}$ is a class of objects of $\mathfrak{K}$, then a morphism $s\colon X \to Y$ is called an $\mathfrak{M}$-*isomorphism* if the cone of s is isomorphic to an object of $\mathfrak{M}$.

(5.3) Proposition. *Let S be a triangular system of morphisms of $\mathfrak{K}$. Then S is a multiplicative denominator system. and all isomorphisms are in S. Moreover, the class $\mathfrak{M} := \mathfrak{M}(S)$ of all S-acyclic objects is a triangular subclass of $\mathfrak{K}$, and S is the class of $\mathfrak{M}$-isomorphisms.*

Conversely, if $\mathfrak{M} \subseteq \mathfrak{K}$ is a triangular subclass, then the system $S := S(\mathfrak{M})$ is a triangular system of morphisms in $\mathfrak{K}$, and $\mathfrak{M}$ is the class of S-acyclic objects.

Proof. We will first prove the asserted bijective correspondence between triangular systems S of morphisms and triangular subclasses $\mathfrak{M} \subseteq \mathfrak{K}$. Consider for a morphism $s\colon {}_X \to Y$ the following diagram:

$$\begin{tikzcd}
 & X \arrow[rr, "s"] \arrow[dd, "s"] & & Y \arrow[dd, equal] \\
Z \arrow[ur, squiggly] \arrow[rr] & & 0 \arrow[ur, squiggly] & \\
 & Y \arrow[ul] \arrow[rr, equal] & & Y. \arrow[ul]
\end{tikzcd} \tag{5.2.1}$$

The left triangle is the cone of s. So the two triangles are exact.

Assume first that S is a given triangular system of morphisms, and let $\mathfrak{M} := \mathfrak{M}(S)$ be the class of S-acyclic objects. Consider an exact triangle $Z \to Z' \to Z'' \to Z(1)$ and the unique morphism from it into the zero triangle. Then, clearly, it follows from (LOC 4) that if $Z, Z' \in \mathfrak{M}$, then $Z'' \in \mathfrak{M}$. Moreover, it follows from (LOC 3) that the class $\mathfrak{M}$ is stable under shifts, and it contains the zero object, because the identity of the zero object is in S. Therefore $\mathfrak{M}$ is a triangular subclass of $\mathfrak{K}$. Moreover, in the diagram (5.2.1) the identity of Y is in S. Therefore, by (LOC 4), s is in S if and only if $Z \in 0$ is in S, that is, if and only if s is an $\mathfrak{M}$-isomorphism.

Conversely, assume that $\mathfrak{M}$ is a given triangular class of objects, and let $S := S(\mathfrak{M})$ be the system of $\mathfrak{M}$-isomorphisms. The zero object is in $\mathfrak{M}$, and $\mathfrak{M}$ is stable under shifts; it follows immediately that every isomorphism is in S, and that (LOC 3) holds. Moreover, if Z is any object of $\mathfrak{K}$, then $Z(1)$ is the cone of the zero morphism $Z \to 0$; hence Z is in $\mathfrak{M}$ if and only if Z is $\mathfrak{M}$-acyclic.

To finish the proof of the bijectivity of the correspondence, we have to show that the system $S(\mathfrak{M})$ satisfies (LOC 4). In fact, to finish the proof of the proposition we have to show that the system S satisfies the multiplicativity condition (LOC 0) and the left and right Ore conditions (LOC 1) and (LOC 2) as well.

First, the multiplicativity condition follows from the Octahedral Axiom. In fact, if $h = st$ is a composition, then there is an exact triangle connecting the cones of s, t, and st. Hence, if two of the cones belong to $\mathfrak{M}$, then so does the third. Hence, if two of the morphisms s, t, and st belong to S, then so does the third. In particular, (LOC 0) holds.

The condition (LOC 4) follows from the cone of the cones construction. Indeed, in the setup of (5.1.1), assume that s, t are $\mathfrak{M}$-isomorphisms. Then their cones, X'' and Y'' belong to $\mathfrak{M}$. There are morphisms $u\colon {}'_Z \to Z''$ and $X'' \to Y''$ having the same cone Z''. This common cone belongs to $\mathfrak{M}$, because Z' and Z'' belong to $\mathfrak{M}$. Therefore, the cone of u belongs to $\mathfrak{M}$, that is, u is an $\mathfrak{M}$-isomorphism.

For the Ore conditions, it suffices to verify the right conditions (LOC 1) and (LOC 2), because the assumptions are self dual. In the proof, we only use that $\mathfrak{M}$ is stable under shifts. Let there be given two morphisms $f\colon X \to Y$ and $s\colon Y'toY$ with $s \in S$. Consider the following diagram:

$$\begin{tikzcd}
& X' \arrow[rr, dashed, "g"] \arrow[dd, "t"] & & Y' \arrow[dd, "s"] \\
Y'' \arrow[ur, squiggly] \arrow[rr, equal] & & Y'' \arrow[ur, squiggly] & \\
& X \arrow[ul, "hf"] \arrow[rr, dashed, "g"'] & & Y. \arrow[ul, "hg"]
\end{tikzcd} \tag{5.2.1}$$

The right triangle is obtained by embedding s into an exact triangle where $h\colon Y \to Y''$ is the morphism into the cone. The left triangle is obtained by embedding $hf\colon X \to Y''$ into an exact triangle which is then rotated. The pair of morphisms 1, f is then completed with a morphism $g\colon X' \to Y'$ to a morphism of triangles. In particular, the g, f completes the given morphisms to a commutative square, and t is in $S(\mathfrak{M})$ because the cone of t is Y, and hence equal to the cone of s which is in $\mathfrak{M}$.

To verify (LOC 2) consider morphisms $f\colon X \to Y$ and $s\colon Y \to Y'$ with $s \in S$ such that $sf = 0$. Consider the following diagram:

$$\begin{tikzcd}
X' \arrow[d, "t"'] & Y'' \arrow[l, squiggly] \arrow[d, "h"] & \\
X \arrow[ur, dashed, "g"] \arrow[r, "f"'] & Y \arrow[r, "s"] & Y'. \arrow[ul, squiggly]
\end{tikzcd}$$

It obtained as follows. First, the triangle to the right is exact; it is obtained by rotating the cone of s. Since s is in S, the vertex Y'' is in $\mathfrak{M}$. Next, the morphism g is a lift of f; such a lift exists, since $sf = 0$ and the right triangle is exact. Finally, the triangle to the left is a rotation of the cone of g. So the triangle is exact, and the cone of t is the vertex Y''. As $Y'' \in \mathfrak{M}$ it follows that $t \in S$. Moreover, since t and g are consecutive morphisms in an exact triangle, it follows that $gt = 0$. Hence $ft = hgt = 0$.

Hence all the properties of the system S of $\mathfrak{M}$-isomorphisms have been justified. ∎

(5.4). In the rest of Section 5 we consider a fixed triangular system S in $\mathfrak{K}$. By the Proposition, it is a denominator system, and it is characterized by the class S-acyclic objects.

Corollary 1. *A triangular functor $F\colon \mathfrak{K} \to \mathfrak{K}'$ from $\mathfrak{K}$ to a triangulated category $\mathfrak{K}$ (or to an abelian category $\mathfrak{A}$) is S-local if an only if $F(Z) = 0$ for every S-acyclic object Z.*

Proof. ∎

Corollary 2. *An object Q of $\mathfrak{K}$ is S-injective if and only if* $\mathrm{Hom}_{\mathfrak{K}}(Z, Q) = 0$ *for every S-acyclic object Q.*

Proof. ∎

(5.5) Lemma. *Let $F\colon \mathfrak{K} \to \mathfrak{K}'$ be a triangular functor from $\mathfrak{K}$ to a triangulated category $\mathfrak{K}'$. Then the system T of morphisms s in $\mathfrak{K}$ such that $F(s)$ is an isomorphism in $\mathfrak{K}'$ is a triangular saturated system.*

Similarly, if $F\colon \mathfrak{K} \to \mathfrak{A}$ is a triangular functor from $\mathfrak{K}$ to an abelian category $\mathfrak{A}$, then the system T of morphisms s in $\mathfrak{K}$ such that $(F(s(n)))$ is an isomorphism in $\mathfrak{A}$ for all $n \in \mathbb{Z}$ is a triangular saturated system in $\mathfrak{K}$.

Proof. □

(5.6) Proposition. *The localization $S^{-1}\mathfrak{K}$ has a unique structure as a triangulated category such that the the functor $(\ )_S\colon \mathfrak{K} \to S^{-1}\mathfrak{K}$ is triangular.*

Proof. The shifts in $S^{-1}\mathfrak{K}$ are clearly determined by $(X_S)(n) := (X(n))_S$, and the class of exact triangles is necessarily the the triangles in $S^{-1}\mathfrak{K}$ isomorphic to the image of an exact triangle of $\mathfrak{K}$.

It is easy to verify the axioms of a triangulated category. □

(5.7) Note. It is immediate from the definition that the functor $(\ )_S\colon \mathfrak{K} \to S^{-1}\mathfrak{K}$ has the following universal property:

Any triangular functor defined on $\mathfrak{K}$ (with target a triangular category or an abelian category) has a unique extension to a triangular functor defined on $S^{-1}\mathfrak{K}$

(5.8). The localized category is in particular an additive category. Hence, for any two objects X and Y of $\mathfrak{K}$. the Hom-set is an abelian group, the *ext-group* (with respect to S)

$$\mathrm{Ext}_S(X, Y) := \mathrm{Hom}_{S^{-1}\mathfrak{K}}(X_S, Y_S).$$

For $n \in \mathbb{Z}$ we define the n'th ext-group,

$$\mathrm{Ext}_S^n(X, Y) := \mathrm{Ext}_S(X, Y(n)) = \mathrm{Ext}_S(X(-n), Y);$$

The second equality indicates the isomorphism obtained by applying the shift automorphism $X \mapsto X(-n)$ in $S^{-1}\mathfrak{K}$. The same automorphism induces an isomorphism,

$$\mathrm{Ext}_S^{n+p}(X, Y) = \mathrm{Ext}_S^p(X(-n).Y).$$

Clearly, if $F\colon \mathfrak{K} \to \mathfrak{K}'$ is a triangular S-local functor from $\mathfrak{K}$ to a triangulated category $\mathfrak{K}'$, then the Yoneda transformation is a homomorphism of abelian groups,

$$\mathrm{Ext}_S^n(X, Y) \to \mathrm{Hom}_{\mathfrak{K}'}(F(X, F(Y)(n)).$$

Similarly, if $H\colon \mathfrak{K} \to \mathfrak{A}$ is a triangular S-local functor from $\mathfrak{K}$ to an abelian category $\mathfrak{A}$, and $H^n(X) := H(X(n))$, there is an induced transformation of abelian groups,

$$\mathrm{Ext}_S^n(X, Y) \to \mathrm{Hom}_{\mathfrak{A}}(H^p(X), H^{p+n}(Y)). \tag{5.8.1}$$

If $\mathfrak{A} = \mathbf{Ab}$ the transformation may be viewed as a pairing,

$$H^n \otimes \mathrm{Ext}_S^p(X, Y) H^n(X) \to H^{n+p}(Y). \tag{5.8.2}$$

In fact, in any abelian category $\mathfrak{A}$ with coproducts there is definition of the tensor above such that the Yoneda transformation (5.8.1) corresponds to the Yoneda pairing (5.8.2).

In the special case of the functor $H(X) = \operatorname{Ext}_S(Z, X)$ with a fixed object Z, the Yoneda pairing,

$$\operatorname{Ext}^n_S(Z, X) \otimes \operatorname{Ext}^p_S(X, Y) \to \operatorname{Ext}^{n+p}_S(Z, Y)$$

is given by composition in $S^{-1}\mathfrak{K}$:

$$\varphi \otimes \psi \mapsto \psi(n)\varphi.$$

D R A F T

6. Derivable functors.

Fix categories $\mathfrak{C}$ and $\mathfrak{D}$, and in $\mathfrak{C}$ a left denominator system S of morphisms.

(6.1) Definition. Let $F\colon \mathfrak{C} \to \mathfrak{D}$ be a functor. For an object $X \in \mathfrak{C}$ we write $R_S F(X)$ or $RF(X)$ for the inductive limit,

$$RF(X) := \varinjlim_{s \in X/S} F(X_s),$$

provided that the inductive limit exists in $\mathfrak{D}$. If $RF(X)$ exists for every $X \in \mathfrak{C}$, we say that the (*right*) *derived functor* RF *exists* (with respect to S); clearly, RF is a functor,

$$RF\colon \mathfrak{C} \to \mathfrak{D}.$$

Moreover, since a morphism $s\colon X \to Y$ in S induces an isomorphism of ind-objects $X_S \to Y_S$, and hence an isomorphism

$$RF(X) = \varinjlim F(X_S) \xrightarrow{\sim} \varinjlim F(Y_S) = RF(Y),$$

it follows that the functor RF is S-local. Consequently, RF has a unique extension to a functor (also denoted) RF from the localized category,

$$\begin{tikzcd}
\mathfrak{C} \arrow[r, "RF"] \arrow[d, "(\,)_S"'] & \mathfrak{D} \\
S^{-1}\mathfrak{C}. \arrow[ur, dashed, "RF"'] &
\end{tikzcd}$$

(6.2). The transformation $X_1 \to X_S$ induced a transformation $F(X)_1 = F(X_1) \to F(X_S)$ in ind-$\mathfrak{D}$. Consequently, if RF exists, there is a natural transformation $F \to RF$ of functors $\mathfrak{C} \to \mathfrak{D}$.

Proposition. *Let $F\colon \mathfrak{C} \to \mathfrak{D}$ be a functor such that the derived functor RF with respect to S exists. Then for every transformation $F \to G$ from F to an S-local functor $G\colon \mathfrak{C} \to \mathfrak{D}$ there is a unique transformation of functors $RF \to G$ making the following diagram commutative:*

$$\begin{tikzcd} F \arrow[r] \arrow[d] & RF \\ G. \arrow[ur, dashed] & \end{tikzcd}$$

Proof. In the ind-category ind-$\mathfrak{D}$ there is a commutative diagram

$$\begin{tikzcd} F(X_1) \arrow[r] \arrow[d] & F(X_S) \arrow[d] \\ G(X_1) \arrow[r, "\sim"] & G(X_S), \end{tikzcd}$$

where $G(X_1) \to G(X_S)$ is an isomorphism by Remark (5.?). Apply the colimit functor $\varinjlim$ to get the required morphism $RF(X) \to G(X)$. It is easily seen to be unique. ∎

(6.3) Definition. Let $F\colon \mathfrak{C} \to \mathfrak{D}$ be a functor and X an object of $\mathfrak{C}$. Then F is called (*right*) *derivable at X* (*with respect to S* if the ind-object $F(X_S)$ in ind-$\mathfrak{D}$ is essentially constant, that is, if the colimit $RF(X) = \varinjlim F(X_S)$ exists in $\mathfrak{D}$ and the induced morphism $F(X_S) \to RF(X)_1$ is an isomorphism in ind-$\mathfrak{D}$. If the functor F is *derivable everywhere*, the functor

$$RF\colon \mathfrak{C} \to \mathfrak{D},$$

is called the *right derived* functor of F.

(6.4) Proposition. *The following three conditions on the left denominator system $S \subseteq \mathfrak{C}$ are equivalent:*

(i) *The functor $(\ )_S\colon \mathfrak{C} \to S^{-1}\mathfrak{C}$ has a right adjoint.*
(ii) *For every object X in $\mathfrak{C}$ there exists an S-injective object Q and a morphism $f\colon X \to Q$ such that the induced morphism $f_S\colon X_S \to Q_S$ is an isomorphism in $S^{-1}\mathfrak{C}$.*
(iii) *The identity functor $\mathfrak{C} \to \mathfrak{C}$ is right derivable everywhere with respect to S.*

The three conditions are implied by any of the following two:

(iv) *The class of S-injective objects is S-dense.*
(v) *There exists an S-dense class $\mathfrak{Q}$ of objects in $\mathfrak{C}$ such that, for any commutative diagram,*

$$\begin{tikzcd} X \arrow[r, "s'"] \arrow[d, "s"'] & Q' \\ Q, \arrow[ur, "f"'] & \end{tikzcd}$$

if $Q, Q' \in \mathfrak{Q}$ and $s, s' \in S$, then $f \in S$.

If S is saturated then all five conditions are equivalent.

Proof. The equivalence of (i), (ii), and (iii) is the the result in Proposition (5.?). If S is saturated, (iv) is just a restatement of (ii). To prove (iv) $\Rightarrow$ (v), simply observe the the class of S-injective objects has the property in (v).

Finally, we prove the implication (v) $\Rightarrow$ (iii). Let X be an object in $\mathfrak{C}$, and denote by $X/S/\mathfrak{Q}$ the full subcategory of X/S consisting of morphisms $s\colon X \to Q$ with target $Q \in \mathfrak{Q}$. Since $\mathfrak{Q}$ is S-dense, the subcategory $X/S/\mathfrak{Q}$ is final in X/S, and the condition (v) means that the inductive system $X_S\colon X/S \to \mathfrak{C}$ restricts to a constant inductive system on $X/S/\mathfrak{Q}$; therefore X_S is essentially constant. ∎

(6.5) Definition. Let $F\colon \mathfrak{C} \to \mathfrak{D}$ be a functor. An object Q in $\mathfrak{C}$ is said to be (*right*) *unfolded* for F or *F-acyclic* (with respect to S) if the following canonical morphism is an isomorphism in ind-$\mathfrak{D}$:

$$(FQ)_1 = F(Q_1) \to F(Q_S).$$

Note that an object Q is F-unfolded if and only if F is derivable at Q and $FQ \xrightarrow{\sim} RF(Q)$. Note further that an F-unfolded object is unfolded for any composition GF of F with a functor $G\colon \mathfrak{D} \to \mathfrak{E}$. In particular, an object unfolded for the identity of $\mathfrak{C}$ is unfolded for any functor $F\colon \mathfrak{C} \to \mathfrak{D}$.

Observation. *With respect to the given denominator system an object Q of $\mathfrak{C}$ is unfolded for the identity functor of $\mathfrak{C}$ if and only if Q is S-injective.*

Proof. If Q is S-injective, then the morphism $1\colon Q \to Q$ is the final object in Q/S; hence $Q_1 \to Q_S$ is an isomorphism in ind-$\mathfrak{C}$. Conversely, assume that $Q_1 \to Q_S$ is an isomorphism in ind-$\mathfrak{C}$. Then, for any object X of $\mathfrak{C}$, we have isomorphisms,

$$\operatorname{Hom}_{\mathfrak{C}}(X, Q) = \operatorname{Hom}_{\text{ind-}\mathfrak{C}}(X_1, Q_1) = \operatorname{Hom}_{\text{ind-}\mathfrak{C}}(X_1, Q_S) = \operatorname{Hom}_{S^{-1}\mathfrak{C}}(X_S, Q_S).$$

It follows that the functor $\operatorname{Hom}_{\mathfrak{C}}(\ , Q)$ is S-local. Hence Q is S-injective. ∎

(6.6) Definition. A functor $F\colon \mathfrak{C} \to \mathfrak{D}$ is called *uniformly* (*right*) *derivable* (with respect to S) is there exists an S-dense class $\mathfrak{Q}$ of objects of $\mathfrak{C}$ with the following property for every object X:

($*_X$): For any commutative diagram in $\mathfrak{C}$,

$$\begin{tikzcd} X \arrow[r, "s'"] \arrow[d, "s"'] & Q' \\ Q, \arrow[ur, "f"'] & \end{tikzcd}$$

if $Q, Q' \in \mathfrak{Q}$ and $s, s' \in S$, then $Ff\colon FQ \to FQ'$ is an isomorphism in $\mathfrak{D}$.

An S-dense class with the property is said to be (*right*) *F-unfolding* (*with respect to S*.

Note that a uniformly derivable functor is derivable everywhere as it follows from (a generalization) the proof of (v) $\Rightarrow$ (iii) in (6.4) above.

(6.7) Unfolding Theorem. *Let $F\colon \mathfrak{C} \to \mathfrak{D}$ be a functor and let $\mathfrak{Q}$ be an S-dense class of objects of $\mathfrak{C}$. Then the following three conditions are equivalent:*

(i) *The class $\mathfrak{Q}$ is F-unfolding.*
(ii) *Every object $Q \in \mathfrak{Q}$ is F-unfolding.*
(iii) *For every morphism $s\colon Q \to Q'$ in S, with $Q, Q' \in \mathfrak{Q}$, the morphism $Fs\colon FQ \to FQ'$ is an isomorphism in $\mathfrak{D}$.*

If the three conditions are satisfied, then F is uniformly derivable, and the class of all F-unfolded objects is an F-unfolding class; moreover, for every object $X \in \mathfrak{C}$ and any morphism $s\colon X \to Q$ in S from X to an F-unfolded object Q there is a commutative diagram in $\mathfrak{D}$ with isomorphisms as indicated:

$$\begin{tikzcd} FX \arrow[r] \arrow[d] & FQ \arrow[d, "\wr"] \\ RF(X) \arrow[r, "\sim"] & RF(Q). \end{tikzcd}$$

Proof. Since $\mathfrak{Q}$ is S-dense, the condition (i) is that the property $(*_X)$ of (6.6) holds for every object X of $\mathfrak{C}$. It follows that (i) $\Rightarrow$ (iii), and further, that (iii) $\Rightarrow$ $(*_Q)$ for every object Q in $\mathfrak{Q}$; hence (iii) $\Rightarrow$ (ii). Finally, to prove that (ii) $\Rightarrow$ $(*_X)$ for every object X in $\mathfrak{C}$, consider a commutative diagram,

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[dr, "s"'] & Q \arrow[d, "f"] \\ & Q', \end{tikzcd} \qquad \text{with } Q, Q' \in \mathfrak{Q} \text{ and } s, s' \in S.$$

The diagram induces a commutative diagram in ind-$\mathfrak{C}$:

$$\begin{tikzcd} X_S \arrow[r, "\sim"] \arrow[dr, "\wr"'] & Q_S \arrow[d, "f_S"] & Q_1 \arrow[l] \arrow[d, "f_1"'] \\ & Q'_S & Q'_1, \arrow[l] \end{tikzcd}$$

and it follows that f_S is an isomorphism. Apply F to obtain a commutative diagram in ind-$\mathfrak{D}$:

$$\begin{tikzcd} F(Q_S) \arrow[d, "F(f_S)", "\wr"'] & FQ_1 \arrow[l, "\sim"'] \arrow[d, "F(f)_1"'] \\ F(Q'_S) & FQ'_1. \arrow[l, "\sim"'] \end{tikzcd}$$

It follows that $F(f)\colon FQ \to FQ'$ is an isomorphism in $\mathfrak{D}$.

The remaining assertions of the Theorem are easily proved. $\square$

(6.8). Consider composable functors, $F\colon \mathfrak{C} \to \mathfrak{D}$ and $G\colon \mathfrak{D} \to \mathfrak{E}$. Clearly, if F is derivable at X, then so is GF and $R(GF)(X) = G(RF(X))$. Similarly, it F is uniformly derivable, then GF is uniformly derivable, and any F-unfolded object is unfolded for GF.

Note that by Proposition (6.5), the identity functor of $\mathfrak{C}$ is uniformly derivable if and only if the class of S-injective objects is S-dense. In particular, if the class of S-injective objects is S-dense, then every functor $F\colon \mathfrak{C} \to \mathfrak{D}$ is uniformly derivable.

(6.9) Example. Consider for a fixed object $A \in \mathfrak{C}$ the functor,

$$H_A = \operatorname{Hom}_{\mathfrak{C}}(A,\ ) : \mathfrak{C} \to \mathbf{Sets}.$$

Then H_A is derivable everywhere with respect to S, and

$$RH_A(X) = \varinjlim_{s \in X/S} H_A(X_s) = \varinjlim_{s \in X/S} \operatorname{Hom}_{\mathfrak{C}}(A, X_s) = \operatorname{Ext}_S(A, X).$$

So $\operatorname{Ext}_S(A,\ )$ is the right derived of $\operatorname{Hom}_{\mathfrak{C}}(A,\ )$:

$$R\operatorname{Hom}_{\mathfrak{C}}(A,\ ) = \operatorname{Ext}_S(A,\ ).$$

Similarly, with respect to a given right denominator system T, the right derived of the functor $\operatorname{Hom}_{\mathfrak{C}}(\ , B)$, as a functor $\mathfrak{C}^{\mathrm{op}} \to \mathbf{Sets}$ is equal to $\operatorname{Ext}_T(A, B)$.

By the Unfolding Theorem (6.7)(iii), a class $\mathfrak{Q}$ in $\mathfrak{C}$ is unfolding for all the functors $\operatorname{Hom}_{\mathfrak{C}}(A,\ )$ for $A \in \mathfrak{C}$, if and only if it is unfolding for the identity. In turn, the condition holds if and only if $\mathfrak{Q}$ is S-dense and consists of (up to isomorphism all) S-injective objects.

(6.10). In the applications we will often consider the case when there is given a functor,

$$F \colon \mathfrak{C} \to \mathfrak{D},$$

and, in addition to the given left left denominator system $S \subseteq \mathfrak{C}$, a given left denominator system T in $\mathfrak{D}$. In this situation the preceding definitions will be applied to the composite functor,

$$(\ )_T F \colon \mathfrak{C} \to T^{-1}\mathfrak{D}. \tag{6.10.1}$$

The functor F is called local with respect to S and T if the functor $(\ )_T F$ is S-local. If T is saturated, then F is local, if

$$s \in S \implies F(s) \in T.$$

We say that RF *exists* if $R\big((\ )_T F\big)$ exists with respect to S, and we use

$$RF \colon S^{-1}\mathfrak{C} \to T^{-1}\mathfrak{D}$$

to denote the extension to $S^{-1}\mathfrak{C}$ of the S-local functor $R\big((\ )_T F\big) \colon \mathfrak{C} \to T^{-1}\mathfrak{D}$. The transformation $(\ )_T F \to R\big((\ )_T F\big)$ induces a transformation of functors $\mathfrak{C} \to T^{-1}\mathfrak{D}$:

$$(\ )_T F \to RF(\ )_S.$$

We say that F is *derivable at* X, resp. *uniformly derivable*, if $(\ )_T F$ is derivable at X, resp. $(\ )_T F$ is uniformly derivable, with respect to S. Similarly, the notions of an *F-unfolded object Q* and an *F-unfolding class* $\mathfrak{Q}$ refer to the corresponding notions for the functor $(\ )_T F$. The functor $RF \colon S^{-1}\mathfrak{C} \to T^{-1}\mathfrak{D}$ is the *derived functor* of F.

Note that the considerations of (6.8) yield limited information on a composition GF in this generalized setup; they apply only to a functor with source $T^{-1}\mathfrak{D}$.

(6.11) The Chain Rule. *Consider categories $\mathfrak{C}$ and $\mathfrak{D}$ with left denominator systems S and T and functors $F\colon \mathfrak{C} \to \mathfrak{D}$ and $G\colon \mathfrak{D} \to \mathfrak{E}$. Assume that F is derivable everywhere and that RG exists. In addition, assume that there is a class $\mathfrak{Q}$ of objects in $\mathfrak{C}$ having the following two properties;*

(1) *The class $\mathfrak{Q}$ is S-dense in $\mathfrak{C}$.*
(2) *For every object $Q \in \mathfrak{Q}$, the morphism $G(FQ) \to RG(FQ)$ is an isomorphism in $\mathfrak{E}$.*

Then the composition GF is derivable everywhere and the canonical transformation is an isomorphism,

$$R(GF) \xrightarrow{\sim} RG\,RF. \tag{6.11.1}$$

Moreover, if the class $\mathfrak{Q}$ consists of F-unfolded objects, then it is GF-unfolding; in particular, then GF is uniformly derivable.

Proof. Note that the functor $RG\colon T^{-1}\mathfrak{D} \to \mathfrak{E}$ appearing in (6.11.1) is the extension to $T^{-1}\mathfrak{D}$ of the T-local functor RG. To be precise in the proof, we will use RG to denote the extension. So the restriction to $\mathfrak{D}$ is the functor $RG\,(\,)_T$, and the natural transformation is the transformation $G \to RG\,(\,)_T$ of functors $\mathfrak{D} \to \mathfrak{E}$.

Let X be an object of $\mathfrak{C}$. By (1), the inclusion of categories,

$$\Phi\colon X/S/\mathfrak{Q} \hookrightarrow X/S,$$

is final. Hence we have in ind-$\mathfrak{C}$ an isomorphism $X_S\Phi \xrightarrow{\sim} X_S$, and it induces in ind-$\mathfrak{D}$ the isomorphism $F(X_S)\Phi \xrightarrow{\sim} F(X_S)$. Apply the functors G and $RG\,(\,)_T$ and the transformation $G \to RG\,(\,)_T$ to obtain a commutative diagram in ind-$\mathfrak{D}$:

$$\begin{tikzcd}
GF(X_S)\,\Phi \arrow[r, "\sim"] \arrow[d] & GF(X_S) \arrow[d] \\
RG\,(\,)_T F(X_S)\,\Phi \arrow[r, "\sim"] & RG\,()_T F(X_S).
\end{tikzcd}$$

By (2), the left vertical morphism is an isomorphism. Hence, so is the right vertical morphism. Since F is derivable at X, we have in ind $-T^{-1}\mathfrak{D}$ the isomorphism $(\,)_T F(X_S) \xrightarrow{\sim} RF(X)$. So the induced morphisms are isomorphisms in ind-$\mathfrak{E}$:

$$GF(X_S) \xrightarrow{\sim} RG\,(\,)_T F(X_S) \xrightarrow{\sim} RG(RF(X))_1.$$

Thus the first part of the Theorem has been proved.

To prove the last part, consider the following commutative diagram in ind-$\mathfrak{E}$, for $X \in \mathfrak{Q}$:

$$\begin{tikzcd}
GF(X_S) \arrow[d, "\wr"'] & GF(X)_1 \arrow[l] \arrow[d] \\
RG\,(\,)_T F(X_S) & RG\,(\,)_T F(X)_1. \arrow[l]
\end{tikzcd}$$

It follows that X is GF-unfolded(??). ∎

(6.12). The preceding definitions generalize in an obvious way to functors of several variables. For simplicity, consider the case of two variables, that is, a functor,

$$F\colon \mathfrak{C}_1 \times \mathfrak{C}_2 \to \mathfrak{D}$$

from the product $\mathfrak{C} = \mathfrak{C}_1 \times \mathfrak{C}_2$ of two categories $\mathfrak{C}_1$ and $\mathfrak{C}_2$ with left denominator systems S_1 and S_2. Then $S := S_1 \times S_2$ is a left denominator system in $\mathfrak{C}$, and there is an isomorphism,

$$S^{-1}\mathfrak{C} = S_1^{-1}\mathfrak{C}_1 \times S_2^{-1}\mathfrak{C}_2.$$

We will say that R_2F *exists*, it $R\big(F(A_1,\ )\big)$ exists with respect to S_2 for every object A_1 in $\mathfrak{C}_1$. Assuming the existence, we may consider R_2F as a functor,

$$R_2F\colon \mathfrak{C}_1 \times S_2^{-1}\mathfrak{C}_2 \to \mathfrak{D}.$$

Similarly, we say that F is derivable everywhere with respect to the second variable if, for every object $A_1 \in \mathfrak{C}_1$, the $F(A_1,\ )\colon \mathfrak{C}_2 \to \mathfrak{D}$ is derivable everywhere with respect to S_2.

Proposition. *If R_2F exists with respect to S_1, then RF exists with respect to $S = S_1 \times S_2$ if and only R_1R_2F exists with respect to S_1. Assuming the existence, there is a canonical isomorphism of functors,*

$$RF \xrightarrow{\sim} R_1R_2F.$$

Similarly, with respect to the appropriate denominator systems, if F is derivable in the second variable, then F is derivable if and only if R_2F is derivable in the first variable.

Proof. The assertions follow from general results about colimits over a product category. □

Remark. If there are subclasses $\mathfrak{Q}_1 \subseteq \mathfrak{C}_1$ and $\mathfrak{Q}_2 \subseteq \mathfrak{C}_2$ such that $\mathfrak{Q}_1$ is unfolding for all the functors $F(\ , A_2)$ for $A_2 \in \mathfrak{C}_2$ and $\mathfrak{Q}_2$ is unfolding for all the functors $F(A_1,\ )$ for $A_1 \in \mathfrak{C}_1$ then the product class $\mathfrak{Q} := \mathfrak{Q}_1 \times \mathfrak{Q}_2$ is unfolding for F. Moreover, then F is uniformly derivable, and we have canonical isomorphisms of functors,

$$R_1R_2F \xrightarrow{\sim} RF \xleftarrow{\sim} R_2R_1F.$$

(6.13) Remark. Assume that $\mathfrak{C}$ and $\mathfrak{D}$ are additive categories and that $F\colon \mathfrak{C} \to \mathfrak{D}$ is an additive functor. Assume that RF exists with respect to the left denominator system S. It follows easily that RF is an additive functor. If $\mathfrak{C}$ and $\mathfrak{D}$ have shift automorphisms $U \mapsto U(1)$ and F commutes with the shifts, it follows easily that RF commutes with the shifts.

(6.14). Consider the case of a triangulated category $\mathfrak{K}$ with a left denominator system S and a triangular functor,

$$F\colon \mathfrak{K} \to \mathfrak{K}',$$

from $\mathfrak{K}$ to a triangulated category $\mathfrak{K}'$.

The following easily proved proposition is a complement to the Unfolding Theorem (6.7). The conditions are with respect to the system S.

Proposition. *Let $\mathfrak{Q}$ be a triangular S-dense class in $\mathfrak{K}$. Then, with respect to the given denominator system S, the following conditions on $\mathfrak{Q}$ are equivalent:*

(i) *The class $\mathfrak{Q}$ is F-unfolding.*
(ii) *Every object in $\mathfrak{Q}$ is F-unfolded.*
(iii) *If Q in $\mathfrak{Q}$ is acyclic, then $F(Q) = 0$ in $\mathfrak{K}'$.*

Often the conditions are applied when the target category is of the form $(S')^{-1}\mathfrak{K}'$, obtained by localizing $\mathfrak{K}'$ with respect to a triangular denominator system S' (and the functor is the functor $(\ )_{S'}F$). In this case that last condition (iii) takes the following form: If Q in $\mathfrak{Q}$ is acyclic, then $(\ )_{S'}F(Q) = 0$. If the system S' is saturated, the condition is equivalent to the following:

(iii') If Q in $\mathfrak{Q}$ is acyclic, then $F(Q)$ is acyclic.

(6.15) Theorem. *In the setup of* (6.14), *consider an exact triangle in $\mathfrak{K}$,*

$$\begin{tikzcd} & Z \arrow[dl, squiggly] & \\ X \arrow[rr] & & Y. \arrow[ul] \end{tikzcd}$$

Assume that F is derivable at X and at Y. Then F is derivable at Z and at $X(1)$, and the following triangle in $\mathfrak{K}'$ is exact:

$$\begin{tikzcd} & RFZ \arrow[dl, squiggly] & \\ RFX \arrow[rr] & & RFY. \arrow[ul] \end{tikzcd}$$

It follows from the Theorem that if F is derivable everywhere then RF is a triangular functor. In addition, if F is uniformly derivable, then the class of F-unfolded objects is a triangular subclass of $\mathfrak{K}$.

We are not going to make any use of the result in the stated generality, but only the special case considered in the following corollary. Therefore we will give a direct proof of the special case, and postpone the proof of the general case.

Corollary. *Assume in the setup of the Theorem that there is a triangular F-unfolding class $\mathfrak{Q}$. Then the functor RF is a triangular functor.*

Direct proof of the Corollary. Let $X \to X' \to X'' \to X(1)$ be an exact triangle in $\mathfrak{K}$. Since $\mathfrak{Q}$ is S-dense, it follows from the denominator conditions that there is a commutative diagram,

$$\begin{tikzcd} X' \arrow[r, "s'"] & Q' \\ X \arrow[u] \arrow[r, "s"] & Q. \arrow[u] \end{tikzcd} \qquad \text{with } Q, Q' \in \mathfrak{Q} \text{ and } s, s' \in S.$$

Embed $Q \to Q'$ into an exact triangle with vertex Q'' in $\mathfrak{Q}$, By (LOC4), the pair (s, s') may be extended to a morphism (s, s', s'') of triangles, with $s'' \in S$,

$$\begin{tikzcd}
& X' \arrow[rr, "s'"] \arrow[dl] & & Q' \arrow[dl] \\
X'' \arrow[rr, "s''"] \arrow[dr, squiggly] & & Q'' \arrow[dr, squiggly] & \\
& X \arrow[rr, "s"'] \arrow[uu] & & Q. \arrow[uu]
\end{tikzcd}$$

Apply F and RF and the transformation $F \to RF$. The result is a commutative diagram of triangles in $\mathfrak{K}'$:

$$\begin{tikzcd}
& RFX' \arrow[rr, "\sim"] \arrow[dl] & & RFQ' \arrow[dl] & & FQ' \arrow[ll, "\sim"'] \arrow[dl] \\
RFX'' \arrow[rr, "\sim"] \arrow[dr, squiggly] & & RFQ'' \arrow[dr, squiggly] & & FQ'' \arrow[ll, "\sim"] \arrow[dr, squiggly] & \\
& RFX \arrow[rr, "\sim"'] \arrow[uu] & & RFQ \arrow[uu] & & FQ. \arrow[ll, "\sim"] \arrow[uu]
\end{tikzcd}$$

The horizontal morphisms are isomorphisms, and the right triangle is exact. Hence the left triangle is exact. Consequently, RF is a triangular functor. ∎

(6.16) Remark. There is a result similar to Corollary (6.15) for the case of a triangular functor $F : \mathfrak{K} \to \mathfrak{A}$ from the triangulated category $\mathfrak{K}$ to an abelian category $\mathfrak{A}$. For the case $\mathfrak{A} = \mathbf{Ab}$ the following result is more precise.

Proposition. *Let $\mathfrak{K}$ be a triangulated category with a triangular denominator system S. Let $G : \mathfrak{K} \to \mathbf{Ab}$ be a triangular functor. Then RG exists and RG is a triangular functor.*

Proof. For every object X of $\mathfrak{K}$ we have by definition

$$RG(X) = \varinjlim_{s \in X/S} G(X_s).$$

So we have to prove for any exact triangle in $\mathfrak{K}$,

$$\begin{tikzcd}
& Z \arrow[dl, squiggly] & \\
X \arrow[rr, "\alpha"] & & Y, \arrow[ul, "\beta"']
\end{tikzcd}$$

that the induced sequence of abelian groups is exact:

$$\varinjlim_{s \in X/S} G(X_s) \xrightarrow{\alpha_*} \varinjlim_{t \in Y/S} G(Y_t) \xrightarrow{\beta_*} \varinjlim_{u \in Z/S} G(Z_u).$$

Let η be an element in $\operatorname{Ker}\beta_*$, and represent η by an element $y \in G(Y_t)$ for some index $t \in Y/S$. Then there is a commutative diagram,

$$\begin{tikzcd} Z \arrow[r, dashed, "u"] & Z_u \\ Y \arrow[u, "\beta"] \arrow[r, "t"'] & Y_t \arrow[u, dashed, "\beta'"'] \end{tikzcd} \qquad \text{with } u \in Z/S,$$

and then $G(\beta')(y) \in G(Z_u)$ represents $\beta_*(\eta)$. Since $\beta_*(y) = 0$, we may modify u in the diagram and assume that $G(\beta')(y) = 0$. Now embed $\beta'\colon Y_t \to Z_u$ in an exact triangle $X' \to Y_t \to Z_u \to X'(1)$. By (LOC 4), the pair (t, u) may be extended to a morphism of triangles (s, t, u) with $s \in S$,

$$\begin{tikzcd} & Z \arrow[dl, squiggly] \arrow[rr, "u"] & & Z_u \arrow[dl, squiggly] \\ X \arrow[rr, dashed, "s"] \arrow[dr] & & X' \arrow[dr] & \\ & Y \arrow[uu] \arrow[rr, "t"'] & & Y_t . \arrow[uu] \end{tikzcd}$$

Then $s\colon X \to X'$ is an index in X/S with $X_s = X'$. Since G is triangular, the following sequence in **Ab** is exact:

$$G(X_s) \to G(Y_t) \to G(Z_u).$$

Moreover, $G(\beta')(y) = 0$. Hence there is an element $x \in G(X_s)$ such that $G(\alpha')(x) = y$. Clearly, then x represents an element $\xi \in \varinjlim_{s \in X/S} G(X_s)$ such that $\alpha_*(\xi) = \eta$. ∎

(6.17). *Proof of Theorem 6.15* Consider an exact triangle $(X, Y, Z, \alpha, \beta, \gamma)$ in $\mathfrak{K}$. Set $U := RF(X)$ and $V = RF(Y)$ and let $a = RF(\alpha)\colon U \to V$. Embed $a\colon U \to V$ in an exact triangle (U, V, W, a, b, c) of $\mathfrak{K}'$. The inductive systems $F(X_S)$ and $F(Y_S)$ are essentially constant. Hence, in the ind-category ind-$\mathfrak{K}'$ there is a commutative diagram with horizontal isomorphisms:

$$\begin{tikzcd} U_1 \arrow[r, "f", "\sim"'] \arrow[d, "a_1"'] & F(X_S) \arrow[d] \\ V_1 \arrow[r, "\sim", "g"'] & F(Y_S). \end{tikzcd} \tag{6.17.1}$$

The shifts in $\mathfrak{K}'$ define natural shifts in the ind-category ind-$\mathfrak{K}'$. Let us first show (with respect to these shifts) that the pair f, g extends to a morphism of triangles in ind-$\mathfrak{K}'$:

$$\begin{tikzcd} & U_1 \arrow[rr, "f", "\sim"'] \arrow[dd, "a_1"] & & F(X_S) \arrow[dd] \\ W_1 \arrow[ur, squiggly, "c_1"] \arrow[rr, dashed, "h"] & & F(Z_S) \arrow[ur, squiggly] & \\ & V_1 \arrow[ul, "b_1"] \arrow[rr, "\sim", "g"'] & & F(Y_S). \arrow[ul] \end{tikzcd} \tag{6.17.2}$$

The morphism $f_1\colon U_1 \to F(X_S)$ in the ind-category is represented by a morphism $\varphi\colon U \to F(X_s)$ for some $s \in X/S$. Similarly, g is represented by a morphism $\psi\colon V \to F(Y_t)$ for $t \in Y/S$. The square (6.17.1) is commutative. Hence, replacing (t, ψ) by an other representative if necessary, we may assume that there are commutative squares, in $\mathfrak{K}$ and in $\mathfrak{K}'$: commutative.

$$\begin{tikzcd}
X \arrow[r, "s"] \arrow[d, "\alpha"'] & X_s \arrow[d, "\alpha'"] & U \arrow[r, "\varphi"] \arrow[d, "a"'] & F(X_s) \arrow[d] \\
Y \arrow[r, "t"'] & Y_t, & V \arrow[r, "\psi"'] & F(Y_t).
\end{tikzcd}$$

Now, let Z_u be the cone of $\alpha'\colon X_s \to Y_t$, and consider the morphisms s, t in S. By (LOC 4), the pair (s, t) may be extended to a morphism of triangles (s, t, u) with $u \in S$:

$$\begin{tikzcd}
& X \arrow[rr, "s"] \arrow[dd] & & X_s \arrow[dd] \\
Z \arrow[ur, rightsquigarrow] \arrow[rr, dashed, "u"] & & Z_u \arrow[ur, rightsquigarrow] & \\
& Y \arrow[ul] \arrow[rr, "t"'] & & Y_t. \arrow[ul]
\end{tikzcd}$$

Since F is triangular, we may extend the pair φ, ψ of morphisms in $\mathfrak{K}'$ to a morphism (φ, ψ, χ) of exact triangles in $\mathfrak{K}'$:

$$\begin{tikzcd}
& U \arrow[rr, "\varphi"] \arrow[dd, "a"] & & F(X_s) \arrow[dd] \\
W \arrow[ur, rightsquigarrow, "c"] \arrow[rr, dashed, "\chi"] & & F(Z_u) \arrow[ur, rightsquigarrow] & \\
& V \arrow[ul, "b"] \arrow[rr, "\psi"'] & & F(Y_t). \arrow[ul]
\end{tikzcd} \tag{6.17.3}$$

Then χ represents a morphism $W_1 \to F(Z_S)$ in ind-$\mathfrak{K}'$ which is the morphism required in (6.17.2).

To finish the proof we show that h is an isomorphism in ind-$\mathfrak{K}'$: It is enough to show for any object A of $\mathfrak{K}'$ that the morphism induced by h is an isomorphism of abelian groups:

$$\mathrm{Hom}\,\mathfrak{K}'(A, W) \xrightarrow{\sim} \varinjlim_{u\in Z/S} \mathrm{Hom}_{\mathfrak{K}'}(A, F(Z_u)). \tag{6.17.4}$$

Consider the triangular functor $G\colon \mathfrak{K} \to \mathbf{Ab}$ defined by $G(X) = \mathrm{Hom}_{\mathfrak{K}'}(A, F(X))$. By the previous result, RG is a triangular functor. Now, the inductive limit on the right side of (6.17.4) is the value $G(Z)$. Let $H_A\colon \mathfrak{K}' \to \mathbf{Ab}$ denote the functor $H_A(\,) = \mathrm{Hom}_{\mathfrak{K}}(A,\ )$. Then $G = H_A F$, and the commutative diagram (6.17.3) with exact triangles induces a commutative diagram in $\mathbf{Ab}$ with exact rows,

$$\begin{tikzcd}
H_A(U) \arrow[r] \arrow[d, "f"', "\wr"] & H_A(V) \arrow[r] \arrow[d, "g"', "\wr"] & H_A(W) \arrow[r] \arrow[d, "h"'] & H_A(U(1)) \arrow[r] \arrow[d, "f"', "\wr"] & H_A(V(1)) \arrow[d, "g"', "\wr"] \\
RG(U) \arrow[r] & RG(V) \arrow[r] & RG(W) \arrow[r] & RG(U(1)) \arrow[r] & RG(V(1)).
\end{tikzcd}$$

The four maps induced by f and g are isomorphisms. So, by the 5-Lemma, the map induced by h is bijective. Therefore, h is an isomorphism, and the proof is complete. $\square$

The homotopy categories

Througout this chapter $\mathfrak{A}$ denotes a fixed abelian category.

1. Complexes.

(1.1) Definition. A complex X in $\mathfrak{A}$ is an infinite sequence of objects and morphisms of $\mathfrak{A}$,

$$X: \quad \cdots \longrightarrow X^{n-1} \xrightarrow{\partial^{n-1}} X^n \xrightarrow{\partial^n} X^{n+1} \longrightarrow \cdots, \tag{1.1.1}$$

such that $\partial^n \partial^{n-1} = 0$ for all integers n.

The object X^n is the *degree-n object* or *degree-n component* of X, the morphism $\partial^n = \partial^n_X$ is the degree-n *differential* or *boundary operator* of X. In addition we define:

$B^n(X) := \operatorname{Im} \partial^{n-1}$, the degree-$n$ *boundary object* of X;
$\widetilde{B}^n(X) := \operatorname{Coim} \partial^n$, the degree-$n$ *coboundary object* of X;
$Z^n(X) := \operatorname{Ker} \partial^n$, the degree-$n$ *cycle object* of X;
$\widetilde{Z}^n(X) := \operatorname{Cok} \partial^{n-1}$, the degree-$n$ *cocycle object* of X.

As $\partial^n \partial^{n-1} = 0$, we have $B^n \subseteq Z^n$, and we may form the quotient,

$H^n(X) := Z^n/B^n$, the degree-n *cohomology object* of X.

Note that H^n is the cokernel of the inclusion $B^n \to Z^n$ and, symmetrically, the kernel of the projection $\tilde{Z}^n \to \tilde{B}^n$: We have an exact commutative diagram,

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B^n \arrow[r] \arrow[d, equal] & Z^n \arrow[r] \arrow[d] & H^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B^n \arrow[r] & X^n \arrow[r] \arrow[d] & \tilde{Z}^n \arrow[r] \arrow[d] & 0 \\
 & & \tilde{B}^n \arrow[r, equal] \arrow[d] & \tilde{B}^n \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{1.1.2}$$

So H^n is the image (or the coimage) of the composition $Z^n \to X^n \to \tilde{Z}^n$. In addition, since $\partial^n \partial^{n-1} = 0$, there is an induced morphism $\delta\colon \tilde{Z}^n \to Z^{n+1}$, and an exact sequence,

$$0 \longrightarrow H^n \longrightarrow \tilde{Z}^n \xrightarrow{\delta} Z^{n+1} \longrightarrow H^{n+1} \longrightarrow 0. \tag{1.1.3}$$

An exact complex is often said to be *acyclic*. A complex is *left bounded*, or a *right complex*, if $X^n = 0$ when $n \ll 0$, and it is *right bounded*, or a *left complex*, if $X^n = 0$ when $n \gg 0$. Left and right bounded complexes are simply *bounded*. A complex is *positive* if $X^n = 0$ when $n < 0$, and *negative* if $X^n = 0$ when $n > 0$. (Some mathematicians doing mostly homology will not agree to the last definitions; maybe the proper phrasing should be co-positive and co-negative.)

(1.2) Note. The components of a complex may be indexed in two natural ways starting from the 0'th component. The complex in (1.1.1) is *increasing*: the index of the target of a differential is 1 bigger than the index of its source. Alternatively, a complex may come with a *decreasing* indexation,

$$Y: \quad \cdots \longrightarrow Y_{m+1} \xrightarrow{\partial_{m+1}} Y_m \xrightarrow{\partial_m} Y_{m-1} \longrightarrow \cdots . \tag{1.2.2}$$

A complex of the form (1.2.2) may be called a *chain complex*, in contrast to the *cochain complex* of (1.1.1). The m'th *homology* of the complex (1.2.2), denoted $H_m(Y)$, is the quotient $\operatorname{Ker} \partial_m / \operatorname{Im} \partial_{m+1}$.

A complex with increasing indexation as in (1.1.1) is turned into a decreasing complex with the definitions $X_m := X^{-m}$ and $\partial_m := \partial^{-m}$. In this notation, the m'th homomology of X is the $(-m)$'th cohomology, $H_m(X) = H^{-m}(X)$.

Let us emphasize the common convention that the differentials of a complex are indexed using the index of their *source*.

(1.3) Definition. A diagram X of the type (1.1.1) with $X^n \in \mathfrak{A}$ corresponds formally to a functor from the graph (quiver) with the integers as vertices,

$$\mathbb{Z}^{\rightarrow} = \quad \cdots \longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow +1 \longrightarrow +2 \longrightarrow \cdots , \tag{1.3.1}$$

to the category $\mathfrak{A}$. As such it is an object in the abelian category $\mathfrak{A}^{\mathbb{Z}^{\rightarrow}}$ of all such diagrams. Recall that a morphism of diagrams $f: X \to Y$ is a $\mathbb{Z}$-indexed family of morphisms $f^n: X^n \to Y^n$ commuting with the differentials. Kernels and cokernels are obtained "component for component" (e.g., $(\operatorname{Ker} f)^n := \operatorname{Ker} f^n$), with induced differentials.

The *category of complexes*, denoted $\mathfrak{C}^\bullet(\mathfrak{A})$ or simply $\mathfrak{A}^\bullet$, is the full subcategory of $\mathfrak{A}^{\mathbb{Z}^{\rightarrow}}$ consisting of complexes. Clearly, if $f: X \to Y$ is a morphism of complexes, then $\operatorname{Ker} f$ and $\operatorname{Cok} f$ are complexes. Hence $\mathfrak{A}^\bullet$ is an abelian category.

From a slightly different point of view, a complex is a $\mathbb{Z}$-indexed family $X = (X^n)$ of objects with a family of morphisms $\partial = \partial_X : X \to X(1)$ such that $\partial^2 = 0$ is the zero morphism $X \to X(2)$; we use $X(p)$ to denote the *p-shifted family*,

$$X(p)^n := X^{p+n}.$$

A morphism of families $f: X \to Y$ is a morphism of complexes if and only if $f\partial_X = \partial_Y f$.

Clearly, there is a full abelian subcategory of *right complexes*, determined by the condition $X^n = 0$ for $n \ll 0$, denoted $\mathfrak{A}^+ = C^+(\mathfrak{A})$. With a similar notation, there are subcategories of *left complexes*, $\mathfrak{A}^- = C^-(\mathfrak{A})$, of *positive* and of *negative complexes*, $\mathfrak{A}^{\geqslant 0} = C^{\geqslant 0}(\mathfrak{A})$ and $\mathfrak{A}^{\leqslant 0} = C^{\leqslant 0}(\mathfrak{A})$, and of *bounded complexes*, $\mathfrak{A}^{\mathrm{b}} = C^{\mathrm{b}}(\mathfrak{A})$. Using the reindexation in (1.2), we may identify:

$$(\mathfrak{A}^{\mathrm{op}})^\bullet = (\mathfrak{A}^\bullet)^{\mathrm{op}}, \qquad (\mathfrak{A}^{\mathrm{op}})^+ = (\mathfrak{A}^-)^{\mathrm{op}}, \qquad (\mathfrak{A}^{\mathrm{op}})^{\geqslant 0} = (\mathfrak{A}^{\leqslant 0})^{\mathrm{op}}, \qquad (\mathfrak{A}^{\mathrm{op}})^{\mathrm{b}} = (\mathfrak{A}^{\mathrm{b}})^{\mathrm{op}}.$$

(1.4) The connecting morhpism. The objects $B^n, Z^n, \dots, H^n$ associated with a complex X of $\mathfrak{A}$ may obviously be considered as additive functors $\mathfrak{A}^\bullet \to \mathfrak{A}$. Clearly, Z^n is left exact and $\tilde{Z}^n$ is right exact. The morphism $\delta\colon \tilde{Z}^n(X) \to Z^n(X)$ of (1.1.3) is a transformation of functors. Hence, for a short exact sequence of complexes,

$$0 \to X' \to X \to X'' \to 0, \tag{1.4.1}$$

there is associated, for every n, an exact commutative diagram,

$$\begin{tikzcd}
 & \tilde{Z}^n(X') \arrow[r] \arrow[d, "\delta'"'] & \tilde{Z}^n(X) \arrow[r] \arrow[d, "\delta"'] & \tilde{Z}^n(X'') \arrow[r] \arrow[d, "\delta''"'] & 0 \\
0 \arrow[r] & Z^{n+1}(X') \arrow[r] & Z^{n+1}(X) \arrow[r] & Z^{n+1}(X''). &
\end{tikzcd}$$

By the exact sequence (1.1.3), the snake morphism $\Delta\colon \operatorname{Ker}\delta'' \to \operatorname{Cok}\delta'$ induced by this diagram is a morphism,

$$\Delta^n\colon H^n(X'') \to H^{n+1}(X');$$

it is called the *connecting morphism* for the given short exact sequence of complexes. It is easy to see that the connecting morphism is functorial with respect to morphisms of short exact sequences.

(1.5) The long exact cohomology sequence. *For a given short exact sequence of complexes* (1.4.1), *the connecting morphisms* Δ^n *fit into a long exact sequence of cohomology objects:*

$$\begin{tikzcd}
\cdots \arrow[r] & H^n(X') \arrow[r] & H^n(X) \arrow[r] & H^n(X'') \arrow[dll, rounded corners] \\
 & H^{n+1}(X') \arrow[r] & H^{n+1}(X) \arrow[r] & H^{n+1}(X'') \arrow[r] & \cdots.
\end{tikzcd}$$

Proof. The assertion is an immediate consequence of The Snake Lemma. ∎

(1.6) The shifts. If X is a complex in $\mathfrak{A}$ we define for each $p \in \mathbb{Z}$ a complex $X(p)$, the *p-shift*, or the p'th *suspension*, of X, as follows:

$$X(p)^n := X^{n+p}, \quad \partial^n_{X(p)} := (-1)^p \partial^{n+p}_X\colon X^{n+p} \to X^{n+p+1}.$$

The p-shift is a functor: For a morphism $f\colon X \to Y$ of complexes, the *shifted* morphism $f(p)\colon X(p) \to Y(p)$ is in degree n equal to $f^{n+p}\colon X^{n+p} \to Y^{n+p}$. Often we write simply

$f\colon X(p) \to Y(p)$ for the p-shift of the morphism f. Note that if the differential is viewed as a family of morphisms $\partial_X\colon X \to X(1)$, then

$$\partial_X(1) = -\partial_{X(1)}\colon X(1) \to X(2).$$

The sign imposed on the differential does not change the cohomology objects. So there is an identification $H^n(X(p)) = H^{n+p}(X)$.

Note that for the "picture" of complexes, say on paper (or in your mind) with the degree-0 object centered, the shift $X(1)$ is a left shifted version of X: all objects of X are translated one step to the left.

Clearly $X \mapsto X(1)$ is an "autofunctor" of $\mathfrak{A}^\bullet$, sometimes denoted Σ, and $X(p) = \Sigma^p X$ is its p'th power.

If A is an object of $\mathfrak{A}$, we write $A(0)$ (or sometimes simply A) for the complex having A as the degree-0 object and the zero object in all other degrees. The functor $A \mapsto A(0)$ identifies $\mathfrak{A}$ with the full subcategory of complexes "*concentrated in degree* 0". We write $A(n)$ for the n-shift of $A(0)$; it has the object A in degree $-n$. Note that morphisms of complexes from $A(0)$ to a complex X correspond to morphisms $A \to Z^0(X)$. In particular, if X is a positive complex, then morphisms $\varepsilon\colon A \to X$ correspond to complexes

$$\cdots \to 0 \to A \xrightarrow{\varepsilon} X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots;$$

the complex X is said to be a *co-augmented complex over* A, with *co-augmentation* ε.

For complexes X, Y we indicate by the notation,

$$f\colon X \rightsquigarrow Y,$$

that f is a morphism of complexes $f\colon X \to Y(1)$. A morphism $f\colon X \rightsquigarrow Y$ induces morphisms of cohomology $H^n(X) \to H^{p+1}(Y)$.

(1.7) The mapping cone of a morphism. Let $f\colon X \to Y$ be a morphism of complexes in $\mathfrak{A}$. The *cone* (or the *mapping cone*) of f is the complex $Z = \operatorname{Con} f$ with

$$Z^n := \oplus\begin{matrix} X^{n+1} \\ Y^n \end{matrix}, \quad \partial_Z^n := \begin{pmatrix} -\partial_X^{n+1} & 0 \\ f^n & \partial_Y^n \end{pmatrix}, \text{ or, shorter, } Z := \oplus\begin{matrix} X(1) \\ Y \end{matrix}, \quad \partial_Z := \begin{pmatrix} -\partial & 0 \\ f & \partial \end{pmatrix},$$

together with the following *triangle*,

$$\begin{tikzcd} & Z \arrow[dl, squiggly, "k=(1\,0)"'] & \\ X \arrow[rr, "f"] & & Y. \arrow[ul, "h=\binom{0}{1}"'] \end{tikzcd}$$

In the matrix defining the family $\partial_Z\colon Z \to Z(1)$, the lower right ∂ is ∂_Y and the upper left ∂ is ∂_X (or more correctly, it is $\partial_X(1)$); so the morphism $-\partial$ is $\partial_{X(1)}$.

Note that the whole triangle is part of the cone Con f. However, when it is unambigous, it may be convenient to use Con f as a notation for the top vertex Z.

Note that the three morphisms of the triangle and their shifts form an infinite sequence of morphisms,

$$\cdots \to X(-1) \to Y(-1) \to Z(-1) \to X \to Y \to Z \to X(1) \to Y(1) \to Z(1) \to \cdots .$$

It is easy to see that the cone and the infinite sequence are functorial with respect to the category of morphisms f (of complexes).

Consider an additive functor $T\colon \mathfrak{A} \to \mathfrak{B}$, where $\mathfrak{B}$ is an abelian category. Clearly, T extends to additive functors on families and on complexes:

$$T\colon \mathfrak{A}^{\mathbb{Z}} \to \mathfrak{B}^{\mathbb{Z}} \text{ and } T^\bullet\colon \mathfrak{A}^\bullet \to \mathfrak{B}^\bullet.$$

It is easy to see that the extension $T^\bullet$ commutes with the formation of cones.

Note. A related, dual, notation is the *cocone* $\mathring{\mathrm{C}}\mathrm{on}\, f$. It consists of the following complex $W = \mathring{\mathrm{C}}\mathrm{on}\, f$,

$$W := \oplus \begin{matrix} X \\ Y(-1) \end{matrix}, \quad \partial_W = \begin{pmatrix} \partial_X & 0 \\ f & -\partial_Y \end{pmatrix},$$

and the following triangle

$$\begin{tikzcd} & W \arrow[dl, "k=(1\,0)"'] & \\ X \arrow[rr, "f"] & & Y. \arrow[ul, squiggly, "h=\binom{0}{1}"'] \end{tikzcd}$$

Obviously, $\mathring{\mathrm{C}}\mathrm{on}(f) = (\mathrm{Con}(-f))(-1)$.

(1.8) The long exact cohomology sequence of a cone. *For a given map $f\colon X \to Y$ of complexes with mapping cone Z, the n'th cohomology H^n applied the morphisms of the triangle and their shifts is a long exact sequence:*

$$\cdots \to H^n(X) \to H^n(Y) \to H^n(Z) \to H^{n+1}(X) \to H^{n+1}(Y) \to H^{n+1}(Z) \to \cdots$$

Proof. Clearly, there is a short exact, and degree-wise split, sequence,

$$0 \longrightarrow Y \xrightarrow{\binom{0}{1}} Z \xrightarrow{(1\,0)} X(1) \longrightarrow 0. \tag{1.8.1}$$

It is easy to see that the connecting morphisms of this short exact sequence, $\delta\colon H^p(X(1)) \to H^{p+1}(Y)$, are the maps induced by f on cohomology. So the cohomology sequence of the triangle is the cohomology sequence of the short exact sequence (1.8.1). By (1.5), it is exact. $\square$

(1.9) Truncations. Let X be a complex. For an integer p, the degree-p *left truncation* $X^{\geqslant p}$ is the complex,

$$X^{\geqslant p}: \quad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X^p \xrightarrow{\partial} X^{p+1} \xrightarrow{\partial} X^{p+2} \longrightarrow \cdots,$$

with X^p in degree p. It may also be denoted $X^{>p-1}$. Similarly, there is a right truncation $X^{\leqslant p} = X^{<p+1}$. The left truncations are subcomplexes of X and they form an decreasing filtration of X:

$$\cdots \hookrightarrow X^{\geqslant p+1} \hookrightarrow X^{\geqslant p} \hookrightarrow X^{\geqslant p-1} \hookrightarrow \cdots \hookrightarrow X. \tag{1.9.1}$$

The right truncations are quotient complexes of X. In fact, there is a natural exact sequence of complexes,

$$0 \longrightarrow X^{>p} \longrightarrow X \longrightarrow X^{\leqslant p} \longrightarrow 0. \tag{1.9.2}$$

Applied to the complex $X := X^{\geqslant p}$, the quotient becomes the complex with X^p concentrated in degree p. So there is an exact sequence of complexes,

$$0 \longrightarrow X^{>p} \longrightarrow X^{\geqslant p} \longrightarrow X^p(-p) \longrightarrow 0. \tag{1.9.3}$$

Consider the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "-\partial"] & X^{p-1} \arrow[r, "-\partial"] \arrow[d] & X^p \arrow[r] \arrow[d, "\partial^p"'] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & X^{p+1} \arrow[r, "\partial"] & X^{p+2} \arrow[r, "\partial"] & \cdots,
\end{tikzcd}$$

where the nontrivial vertical morphism is in degree $p+1$. The top row is $X^{\leqslant p}(-1)$ and the bottom row is $X^{>p}$. The diagram is commutative, since $\partial^2 = 0$. So ∂^p defines a morphism of complexes $X^{\leqslant p}(-1) \to X^{>p}$. Clearly, the cone of this morphism is the given complex X:

$$\begin{tikzcd}
& X \arrow[dl, rightsquigarrow] & \\
X^{\leqslant p}(-1) \arrow[rr, "\partial^p"] & & X^{>p}. \arrow[ul]
\end{tikzcd} \tag{1.9.4}$$

Applied to the truncated complex, $X := X^{\geqslant p}$, the lower left vertex is the complex with the object X^p concentrated in degree $p+1$. So there is a cone of complexes,

$$\begin{tikzcd}
& X^{\geqslant p} \arrow[dl, rightsquigarrow] & \\
X^p(-p-1) \arrow[rr, "\partial"] & & X^{>p}. \arrow[ul]
\end{tikzcd} \tag{1.9.5}$$

(1.10) Quasi isomorphisms. A morphism of complexes $f\colon X \to Y$ is called a *quasi-isomorphism* if for all n the induced morphism on cohomology $H^n(X) \to H^n(Y)$ is an isomorphism. The following result is an immediate consequence of (1.8).

Corollary. *A morphism of complexes $f\colon X \to Y$ is a quasi-isomorphism if and only if the mapping cone of f is acyclic.* ▯

(1.11). Let $f\colon X \to Y$ be a morphism of complexes, and consider the cone $Z := \operatorname{Con} f$ of f. If $g\colon Y \to W$ is a morphism of complexes such that $gf = 0$ then $\tilde{g} := (0\, g)$ is a morphism of complexes $Z \to W$; it is said to be *induced* by g:

$$\begin{tikzcd}
 & Z \arrow[dl, rightsquigarrow] \arrow[drr, dashed, "\tilde{g}"] & & \\
X \arrow[rr, "f"'] & & Y \arrow[ul] \arrow[r, "g"'] & W.
\end{tikzcd}$$

Proposition. *Let $f\colon X \to Y$ be a monomorphism, and consider its cone $Z := \operatorname{Con} f$ and its cokernel $W := \operatorname{Cok} f$. Then the induced morphism $Z \to W$ is a quasi-isomorphism.*

Proof. Use the two long exact sequences, of the cone $X \to Y \to Z$ of f and of the short exact sequence $X \to Y \to W$ defined by f, to obtain the following diagram with exact rows:

$$\begin{tikzcd}
H^n(X) \arrow[r, "f"] \arrow[d, equal] & H^n(Y) \arrow[r, "h"] \arrow[d, equal] & H^n(Z) \arrow[r, "k"] \arrow[d, "\tilde{g}"] & H^{n+1}(X) \arrow[r, "f"] \arrow[d, equal] & H^{n+1}(Y) \arrow[d, equal] \\
H^n(X) \arrow[r, "f"] & H^n(Y) \arrow[r, "g"] & H^n(W) \arrow[r, "\Delta"] & H^{n+1}(X) \arrow[r, "f"] & H^{n+1}(Y).
\end{tikzcd}$$

Except for the square involving k and Δ the squares are obviously commutative. Check that the exceptional square is anticommutative. Conclude by the 5-lemma that the middle vertical morphism is an isomorphism. ▯

Similarly, if $X \to Y$ is an epimorphism of complexes, with kernel V and cone Z, there is an induced quasi-isomorphism $V(1) \to Z$.

(1.12) Cohomology truncations. Let X be a complex. Consider for an integer p the following inclusion of subcomplexes,

$$\begin{tikzcd}
\tau_{\leqslant p}X & \cdots \arrow[r] & X^{p-2} \arrow[r] \arrow[d, equal] & X^{p-1} \arrow[r] \arrow[d, equal] & Z^p \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\tilde{\tau}_{\leqslant p}X & \cdots \arrow[r] & X^{p-2} \arrow[r] & X^{p-1} \arrow[r] & X^p \arrow[r] & B^{p+1} \arrow[r] & 0 \arrow[r] & \cdots,
\end{tikzcd}$$

The inclusion is a quasi-isomorphism and both complexes are degree-p *right cohomology truncations* of X: Their degree-n cohomology is equal to H^nX when $n \leqslant p$ and equal to 0 otherwise. The truncations form an increasing filtration of X, for instance for the $\tau_{\leqslant p}X$:

$$\cdots \hookrightarrow \tau_{\leqslant p-1}X \hookrightarrow \tau_{\leqslant p}X \hookrightarrow \tau_{\leqslant p+1} \hookrightarrow \cdots \hookrightarrow X. \tag{1.12.1}$$

The quotient complex $\tau_{\leqslant p}X/\tau_{<p}X$ is the complex

$$\cdots \longrightarrow 0 \longrightarrow B^p \longrightarrow Z^p \longrightarrow 0 \longrightarrow \cdots,$$

whith Z^p in degree p. It has $Z^p/B^p = H^pX$ as it only nonvanishing cohomology; in fact, the natural morphism is a quasi-isomorphism:

$$\tau_{\leqslant p}X/\tau_{<p}X \to H^p(X)(-p).$$

Similarly, there are two left cohomology filtrations $\tau_{\geqslant p}X$ and $\tilde{\tau}_{\geqslant p}X$ defined as quotients of X. In fact, there are exact sequences,

$$0 \to \tau_{\leqslant p}X \to X \to \tilde{\tau}_{>p}X \to 0,$$
$$0 \to \tilde{\tau}_{\leqslant p}X \to X \to \tau_{>p}X \to 0.$$

(1.13) Definition. Let X and Y be complexes of $\mathfrak{A}$. Define for every integer n an abelian group,

$$\operatorname{Hom}^n(X, Y) := \prod_{p\in\mathbb{Z}} \operatorname{Hom}_{\mathfrak{A}}(X^{p-n}, Y^p).$$

So an element of $f \in \operatorname{Hom}^n(X, Y)$ is an infinite family of morphisms $f_p\colon X^p \to Y^{p+n}$, illustrated by a diagram, not necessarily commutative,

$$\begin{tikzcd}
\cdots \arrow[r] & X^{-n} \arrow[r] \arrow[drr] & \cdots \arrow[r] & X^{-1} \arrow[r] \arrow[drr] & X^0 \arrow[r] \arrow[drr] & X^1 \arrow[r] \arrow[drr] & \cdots & & \\
 & & \cdots \arrow[r] & Y^0 \arrow[r] & \cdots \arrow[r] & Y^{n-1} \arrow[r] & Y^n \arrow[r] & Y^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Equivalently, $\operatorname{Hom}^n(X, Y)$ is the set of all families of morphisms $X(-n) \to Y$ (or families of morphisms $X \to Y(n)$).

Consider the homomorphism of abelian groups,

$$d^n\colon \operatorname{Hom}^n(X, Y) \to \operatorname{Hom}^{n+1}(X, Y),$$

given by $d^n := \partial_Y - (-1)^n\partial_X$. The image of a family $f = (f_p)$ in $\operatorname{Hom}^n(X, Y)$ is the sequence $d^n(f)$ in $\operatorname{Hom}^{n+1}(X, Y)$ with $(d^nf)_p\colon X^{p-n-1} \to Y^p$ given by the equation, for $p \in \mathbb{Z}$,

$$(d^nf)_p = \partial_Y^{p-1}f^{p-1} - (-1)^nf_p\partial_X^{p-n-1}.$$

In terms of the differential $\partial_{X(-n)} = (-1)^n\partial_X$, if $f \in \operatorname{Hom}(X, Y(n))$ then

$$d^n(f) = \partial_Y f - f\partial_{X(-n)}.$$

It is easy to see that the composition $d^{n+1}d^n$ is the zero map. So the groups Hom^n with the maps d^n form a complex of abelian groups, denoted $\operatorname{Hom}^\bullet_{\mathfrak{A}}(X, Y)$. Note that a family $f \in \operatorname{Hom}(X(-n), Y)$ is a cycle in the complex $\operatorname{Hom}^\bullet_{\mathfrak{A}}(X, Y)$ if and only if $\partial_Y f - f\partial_{X(-n)} = 0$, that is, if and only if f is a morphism of complexes $f\colon X(-n) \to Y$. As a formula:

$$Z^n(\operatorname{Hom}^\bullet_{\mathfrak{A}}(X, Y)) = \operatorname{Hom}_{\mathfrak{A}^\bullet}(X(-n), Y) = \operatorname{Hom}_{\mathfrak{A}^\bullet}(X, Y(n)).$$

Note that the definition when Y is a single object B, identified with the complex $B(0)$ having B is degree 0, gives a complex $\mathrm{Hom}^\bullet(X, B)$ which differs by signs in the differentials from the complex obtain by evaluating the functor $\mathrm{Hom}_{\mathfrak{A}}(-, B)$ at the complex X: If the latter functor is denoted $(-)^*$ for simplicity, then

$$\mathrm{Hom}^\bullet(X, B) = \quad \cdots\cdots \longrightarrow X^{2*} \xrightarrow{-\partial^*} X^{1*} \xrightarrow{\partial^*} X^{0*} \xrightarrow{-\partial^*} X^{-1*} \xrightarrow{\partial^*} \cdots;$$

this sign convention is sometimes used when a contravariant functor is applied to a complex.

The formation of the complex $\mathrm{Hom}^\bullet$ is clearly functorial, that is, it defines a functor

$$\mathrm{Hom}^\bullet\colon \big(\mathfrak{A}^\bullet\big)^{\mathrm{op}} \times \mathfrak{A}^\bullet \to \mathbf{Ab}^\bullet.$$

It respects the shifts:

$$\mathrm{Hom}^\bullet(X(-n), Y) = \mathrm{Hom}^\bullet(X, Y(n)) = \mathrm{Hom}^\bullet(X, Y)(n).$$

If $X \in \mathfrak{A}^-$ and $Y \in \mathfrak{A}^+$, then the product defining the group $\mathrm{Hom}^n(X, Y)$ is finite; moreover $\mathrm{Hom}^n(X, Y)$ vanishes when $n \ll 0$, that is, the complex $\mathrm{Hom}^\bullet(X, Y)$ belongs to subcategory $\mathbf{Ab}^+$.

(1.13) Exercises.

1. Let $\varepsilon\colon \mathbb{Z} \to \{\pm 1\}$ be an arbitrary map. For any complex X in $\mathfrak{A}$ let ${}_\varepsilon X$ denote the complex with ${}_\varepsilon X^n := X^n$ and ${}_\varepsilon\partial^n = \varepsilon(n)\partial^n$. Define a canonical functorial isomorphism ${}_\varepsilon X \xrightarrow{\sim} X$.

2. Define, for a morphism of complexes $f\colon X \to Y$ an isomorphism of cones $\mathrm{Con}\,(f) \to \mathrm{Con}\,(-f)$

3. Does the functor $A \mapsto A(0)$, from $\mathfrak{A}$ to $\mathfrak{A}^\bullet$, have a right adjoint? – and a left adjoint?

4. Establish an exact sequence $0 \to \tilde\tau_{<p}X \to \tau_{\leqslant p}X \to (H^pX)(-p) \to 0$.

2. Homotopy.

(2.1) Definition. Recall that we work over a fixed abelian category $\mathfrak{A}$. Let X, Y be complexes of $\mathfrak{A}$. A family of morphisms $f : X \to Y$ is said to be *null homotopic* or *homotopic to zero* if there exists a family of morphisms $s^p : X^p \to Y^{p-1}$ (called a *homotopy*) such that $f^p = \partial^{p-1}s^p + s^{p+1}\partial^p$. Equivalently, in terms of families, $s = (s^p)$ is a family of morphisms $s : X \to Y(-1)$ and

$$f = \partial_Y(-1)s + s(1)\partial_X, \tag{2.1.1}$$

or shorter: $f = \partial s + s\partial$. The family s will be called a homotopy from f to 0, and we will indicate it by writing $s : f \simeq 0$. A null homotopic family of morphisms $f : X \to Y$ commutes with the differentials; hence f is a morphism of complexes $f : X \to Y$.

Two morphisms $f_0, f_1 : X \to Y$ of complexes are *homotopic*, written

$$f_0 \simeq f_1 : X \to Y,$$

if $f_0 - f_1$ is homotopic to zero. Clearly, homotopy is an equivalence relation in the group $\mathrm{Hom}(X, Y)$ of morphisms of complexes, corresponding to the subgroup of null homotopic morphisms. The equivalence classes are called *homotopy classes* of morphisms, and the homotopy class determined by a morphism $f : X \to Y$ is denoted $[f]$ (or simply by the same symbol f).

Assume that $f : X \to Y$ is homotopic to zero. Then, for any morphism $g : Y \to W$ and $h : V \to X$, the compositions gf and fh are homotopic to zero. It follows that we may define a category $\mathrm{Hot}(\mathfrak{A})$ as follows: The objects of $\mathrm{Hot}(\mathfrak{A})$ are complexes of $\mathfrak{A}$ and if X and Y are objects of $\mathrm{Hot}(\mathfrak{A})$ then the morphisms $X \to Y$ in $\mathrm{Hot}(\mathfrak{A})$ are the homotopy classes of morphisms of complexes $X \to Y$. The category $\mathrm{Hot}(\mathfrak{A})$ is the *homotopy category* of $\mathfrak{A}$. It has an additive Hom-structure: the set of morphisms from X to Y in the homotopy category is the quotient group,

$$\mathrm{Hom}_{\,\mathrm{Hot}(\mathfrak{A})}(X, Y) := \mathrm{Hom}_{\mathfrak{A}^\bullet}(X, Y)/\simeq,$$

of morphisms of complexes modulo null homotopic morphisms. The natural functor,

$$\mathfrak{A}^\bullet \to \mathrm{Hot}(\mathfrak{A})$$

(denoted $X \mapsto [X]$) respects finite direct sums. It follows easily the $\mathrm{Hot}(\mathfrak{A})$ is an additive category. In addition, the shifts $X \mapsto X(p)$ are well-defined in the homotopy category.

Restricting to right complexes, left complexes, or bounded complexes, we obtain full subcategories $\mathrm{Hot}^+(\mathfrak{A})$, $\mathrm{Hot}^-(\mathfrak{A})$, and $\mathrm{Hot}^{\mathrm{b}}(\mathfrak{A})$ of $\mathrm{Hot}(\mathfrak{A})$; for every additive subclass $\mathfrak{Q}$ of $\mathfrak{A}$ we get an additive subcategory $\mathrm{Hot}(\mathfrak{Q})$ of complexes of objects from $\mathfrak{Q}$.

It follows from the description in Hot(1.13) that the subgroup of null homotopic morphisms $X \to Y$ is equal to the image of the map $d^{-1} : \mathrm{Hom}^{-1}(X, Y) \to \mathrm{Hom}^0(X, Y)$, that is, equal to the degree-0 boundary $B^0(\mathrm{Hom}^\bullet(X, Y))$. The subgroup of morphisms of families $X \to Y$

that are morphisms of complexes is the group of degree-0 cycles in the complex $\mathrm{Hom}^\bullet(X, Y)$. Hence

$$\mathrm{Hom}_{\mathfrak{A}^\bullet}(X, Y) = Z^0(\mathrm{Hom}^\bullet(X, Y)),$$
$$\mathrm{Hom}_{\mathrm{Hot}(\mathfrak{A})}(X, Y) = H^0(\mathrm{Hom}^\bullet(X, Y)).$$

More generally, see Hot(1.13),

$$\mathrm{Hom}_{\mathrm{Hot}(\mathfrak{A})}(X(-n), Y) = \mathrm{Hom}_{\mathrm{Hot}(\mathfrak{A})}(X, Y(n)) = H^n(\mathrm{Hom}^\bullet(X, Y)).$$

(2.2) Proposition. *Two homotopic morphisms of complexes $f_0 \simeq f_1 \colon X \to Y$ induce the same morphism on cohomology: $H^n(f_0) = H^n(f_1) \colon H^n(X) \to H^n(Y)$.*

Proof. It suffices to prove that a null homotopic $f \colon X \to Y$ induces the zero morphism on cohomology. So assume that $f = \partial s + s\partial$ for a family of morphisms $s \colon X \to Y(-1)$. Consider the following diagram,

$$\begin{tikzcd} & Z^n(X) \arrow[r] \arrow[dl, "\tilde{s}^n"'] \arrow[d, "\tilde{f}^n"] & H^n(X) \arrow[d] \\ Y^{n-1} \arrow[r, "\tilde{\partial}^{n-1}"'] & Z^n(Y) \arrow[r] & H^n(Y). \end{tikzcd}$$

The square, induced by the morphism f, is commutative. In the triangle, the morphism $\tilde{\partial}^{n-1}$ is induced by the differential of Y, and $\tilde{s}^n$ is the restriction of s^n to $Z^n(X)$. The triangle is commutative, since $f^n = \partial^{n-1}s^n + s^{n+1}\partial^n$ and the last term $s^{n+1}\partial^n$ vanishes when restricted to $Z^n(X)$. So the composition $Z^n(X) \to Z^n(Y) \to H^n(Y)$ is the zero morphism. Hence, so is the induced morphism $H^n(X) \to H^n(Y)$. ∎

(2.3). The functors $X \mapsto X^n$, $X \mapsto B^n(X)$, $X \mapsto Z^n(X)$, from $\mathfrak{A}^\bullet$ to $\mathfrak{A}$, are not well-defined on the homotopy category. However, it follows easily from the proposition above that the n'th cohomology, for $n \in \mathbb{Z}$, is a well-defined functor,

$$H^n \colon \mathrm{Hot}(\mathfrak{A}) \to \mathfrak{A}.$$

It is also easy to verify that the bi-functor $\mathrm{Hom}^\bullet$ of Hot(1.3) defines a bi-functor,

$$\mathrm{Hom}^\bullet \colon \mathrm{Hot}(\mathfrak{A})^{\mathrm{op}} \times \mathrm{Hot}(\mathfrak{A}) \to \mathrm{Hot}(\mathfrak{A}).$$

Restriction yields a bi-functor

$$\mathrm{Hom}^\bullet \colon \mathrm{Hot}^-(\mathfrak{A})^{\mathrm{op}} \times \mathrm{Hot}^+(\mathfrak{A}) \to \mathrm{Hot}^+(\mathfrak{A}).$$

The formation of the cone is not a functor on the category of morphisms of the homotopy category. It has, however, the following property:

(2.4) Lemma. *Consider a square diagram of complexes in $\mathfrak{A}^\bullet$,*

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "f"'] & X' \arrow[d, "f'"'] \\ Y \arrow[r, "y"] & Y', \end{tikzcd}$$

and let $Z = \operatorname{Con} f$ and $Z' := \operatorname{Con} f'$ denote the cones of the vertical morphisms. Assume that the square is homotopy commutative, and consider a homotopy $s\colon yf - f'x \simeq 0$. Then the family of morphisms,

$$z := \begin{pmatrix} x & 0 \\ s & y \end{pmatrix} : \begin{matrix} X(1) \\ \oplus \\ Y \end{matrix} \longrightarrow \begin{matrix} X'(1) \\ \oplus \\ Y' \end{matrix},$$

is a morphism of complexes $z\colon Z \to Z'$, and, in the following diagram, the two "squares" involving z are commutative in $\mathfrak{A}^\bullet$:

$$\begin{tikzcd} X \arrow[rr, "x"] \arrow[dd, "f"'] & & X' \arrow[dd, "f'"'] & \\ & Z \arrow[ul, squiggly, "k"'] \arrow[rr, "z"] & & Z' \arrow[ul, squiggly, "k'"'] \\ Y \arrow[rr, "y"] \arrow[ur, "h"'] & & Y' \arrow[ur, "h'"'] & \end{tikzcd}$$

Proof. The first assertion is the equation of families of morphisms, $\partial_{Z'} z = z\partial_Z$ or, in matrix form,

$$\begin{pmatrix} -\partial & 0 \\ f' & \partial \end{pmatrix}\begin{pmatrix} x & 0 \\ s & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ s & y \end{pmatrix}\begin{pmatrix} -\partial & 0 \\ f & \partial \end{pmatrix}.$$

The equations follow from the assumption $yf - f'x = \partial s + s\partial$. Commutativity of the squares, corresponding to the equations $zh = h'y$ and $k'z = xk$, is obvious. ∎

(2.5) Lemma. *Consider in the setup of Lemma* (2.4) *the cones of the horizontal morphisms, $X'' := \operatorname{Con} x$ and $Y'' := \operatorname{Con} y$ and the morphism of complexes, from* $\operatorname{Con} x$ *to* $\operatorname{Con} y$*:*

$$f'' := \begin{pmatrix} f & 0 \\ -s & f' \end{pmatrix} : \begin{matrix} X(1) \\ \oplus \\ X' \end{matrix} \longrightarrow \begin{matrix} Y(1) \\ \oplus \\ Y' \end{matrix}.$$

Then there is a natural isomorphism of complexes, $\operatorname{Con} f'' \xrightarrow{\sim} \operatorname{Con} z$. In fact, the isomorphism on the families,

$$\operatorname{Con} f'' = X(2) \oplus X'(1) \oplus Y(1) \oplus Y' \xrightarrow{\sim} \operatorname{Con} z = X(2) \oplus Y(1) \oplus X'(1) \oplus Y'$$

is given by multiplication by the following matrix,

$$\sigma = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof. The family f'' of morphisms defined by the matrix is a morphism of complexes by Lemma (2.4) applied to the reflected square. Note the sign change in the homotopy s: interchanging the vertical and and the horizontal morphisms changes the sign in the difference $yf - f'x$.

The rest of the assertion is a simple computation. You have to identify the 4×4 matrices defining $\partial_{\mathrm{Con}\, f''}$ and $\partial_{\mathrm{Con}\, z}$ and prove their commutation with σ. ∎

(2.6) Warning. In the setup Lemma (2.4), let Z'' be the mapping cone of the morphism $z\colon Z \to Z'$. Then, by Lemma (2.5), we have to triangles,

$$\begin{tikzcd} & Z'' \arrow[dl, rightsquigarrow, "(1\,0)"'] & \\ Z \arrow[rr, "z"] & & Z' \arrow[ul, "\binom{0}{1}"'] \end{tikzcd}, \qquad \begin{tikzcd} & Z'' \arrow[dl, rightsquigarrow, "(1\,0)\sigma"'] & \\ X'' \arrow[rr, "f''"] & & Y'' \arrow[ul, "\sigma\binom{0}{1}"'] \end{tikzcd}.$$

The first is the cone of z. The second is obtained from the cone of f'' by replacing the top vertex with the isomorphic complex Z'' using the isomorphism σ of (2.5). It should be emphasized, that of the two squares deduced from the morphism in these triangles,

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & Z' \arrow[d] \\ Y'' \arrow[r] & Z'' \end{tikzcd}, \qquad \begin{tikzcd} Z'' \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & Z' \arrow[d, rightsquigarrow] \\ X'' \arrow[r, rightsquigarrow] & X \end{tikzcd},$$

the first is commutative, the second is anti-commutative. The assertion follows from a simple computation.

(2.7) Definition. Two complexes X, Y in $\mathfrak{A}$ are said to be (*homotopy*) *equivalent* if they are isomorphic in $\mathrm{Hot}(\mathfrak{A})$. Equivalently, X and Y are homotopy equivalent if there are morphisms of complexes $f\colon X \to Y$ and $g\colon Y \to X$ such that $gf \simeq 1_X$ and $fg \simeq 1_Y$ (in which case f is said to be a *homotopy equivalence*).

A complex homotopy equivalent to the zero complex is said to be *contractible*. As cohomology is a functor on the homotopy category, it follows for a contractible complex Z that $H^n(Z) = 0$ for all n; in other words, a contractible complex is acyclic.

Consider an additive functor $T\colon \mathfrak{A} \to \mathfrak{B}$, where $\mathfrak{B}$ is an abelian category, and its extensions to additive functors on families and on complexes:

$$T\colon \mathfrak{A}^{\mathbb{Z}} \to \mathfrak{B}^{\mathbb{Z}} \text{ and } T^\bullet\colon \mathfrak{A}^\bullet \to \mathfrak{B}^\bullet.$$

It follows from the definition that T preserves homotopies. As a consequence, $T^\bullet$ defines an additive functor,

$$\mathrm{Hot}(T)\colon \mathrm{Hot}(\mathfrak{A}) \to \mathrm{Hot}(\mathfrak{B}).$$

In particular, if Z is contractible, then so is $T(Z)$.

Characterization. *The following conditions on a complex Z of $\mathfrak{A}$ are equivalent:*

(ii) *The identity of Z is homotopic to zero, $1_Z \simeq 0_Z$, that is, there exists a family of morphisms $s\colon Z \to Z(-1)$ such that $f = \partial(-1)s + s(1)\partial$.*

(ii) *Z is contractible.*

(iii) *The complex* $\mathrm{Hom}_{\mathfrak{A}}(A, Z)$ *is acyclic for every object A of $\mathfrak{A}$.*

(iii)$^{\mathrm{op}}$ *The complex* $\mathrm{Hom}_{\mathfrak{A}}(Z, B)$ *is acyclic for every object B of $\mathfrak{A}$.*

(iv) *Z is isomorphic in $\mathfrak{A}^\bullet$ to a complex of the following form, for some family (U^n):*

$$\begin{tikzcd}[row sep=tiny]
{} \arrow[ddr] & U^n \arrow[ddr] & U^{n+1} \arrow[ddr] & U^{n+2} \arrow[ddr] & \\
\cdots & \oplus & \oplus & \oplus & \cdots \\
 & U^{n-1} & U^n & U^{n+1} &
\end{tikzcd}$$

(v) *Z is isomorphic in $\mathfrak{A}^\bullet$ to the cone of the identity of some complex U.*

Proof. (i) $\Leftrightarrow$ (ii): Indeed, the zero object of an additive category is characterized by the property that the identity is the zero morphism.

(ii) $\Rightarrow$ (iii): If Z is contractible, then, for any additive functor $T\colon \mathfrak{A} \to \mathfrak{B}$, the complex $T(Z)$ is contractible. In particular, $\mathrm{Hom}_{\mathfrak{A}}(U, Z)$ is contractible, and hence acyclic.

(iii) $\Rightarrow$ (iv): Let $U^n := \mathrm{Ker}\,\partial_Z^n$ be the n'th cycle object of Z, with the injection $i : U^{n+1} \to Z^{n+1}$. Then ∂^n factors over U^{n+1} as a product $\partial^n = i\tilde{\partial}$.

The composition $\partial_Z^{n+1} i$ is zero. Hence i is an $(n+1)$-cycle in the complex $\mathrm{Hom}(U^{n+1}, Z)$. By exactness, i is a boundary, that is, there is a morphism $t\colon U^{n+1} \to Z^n$ such that $i = \partial^n t$. This equation implies that $U^{n+1} \subseteq \mathrm{Im}\,\partial^n$, and hence that $U^{n+1} = \mathrm{Im}\,\partial^n$. Moreover, from $i = \partial^n t$ it follows that $\tilde{\partial} t = 1_{U^{n+1}}$. Therefore, the morphism $\tilde{\partial}\colon Z^n \to U^{n+1}$ is a split epimorphism, with t as section. Clearly, $\mathrm{Ker}\,\tilde{\partial} = \mathrm{Ker}\,\partial^n = U_n$. Consequently, we obtain the decomposition $Z^n = U^{n+1} \oplus U^n$, and the decomposition of ∂_Z^n, as asserted.

(iv) $\Rightarrow$ (v): The complex described in (iv) is the cone of the identity of a complex U with zero differentials.

(v) $\Rightarrow$ (i): Assume that U is a complex and that Z is isomorphic to the cone of 1_U. Of the two matrices,

$$\partial_Z := \begin{pmatrix} -\partial_U & 0 \\ 1 & \partial_U \end{pmatrix} \text{ and } s := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

the first is the differential of Z. The second, as a family of morphisms $Z \to Z(1)$, is easily seen to be a homotopy $s\colon 1_Z \simeq 0_Z$.

Thus the equivalence of the conditions (i), ... ,(v) has been established. Their equivalence to (iii)$^{\mathrm{op}}$ hold by a dual argument. ▯

(2.8) Definition. A triangle in the homotopy category $\mathrm{Hot}(\mathfrak{A})$,

$$\begin{tikzcd}
 & Z \arrow[dl, squiggly] & \\
X \arrow[rr] & & Y, \arrow[ul]
\end{tikzcd} \qquad (2.8.1)$$

is called a *homotopy cone* if is isomorphic in $\mathrm{Hot}(\mathfrak{A})$ to the cone of some morphism of complexes, that is, if there exists a morphism of complexes $f: X' \to Y'$ and an isomorphism of triangles in $\mathrm{Hot}(\mathfrak{A})$,

$$\begin{tikzcd}
X' \arrow[rr, "\sim"] \arrow[dd, "f"'] & & X \arrow[dd] & \\
& \mathrm{Con}\, f \arrow[ul, squiggly] \arrow[rr, "\sim"] & & Z \arrow[ul, squiggly] \\
Y' \arrow[rr, "\sim"] \arrow[ur] & & Y \arrow[ur] &
\end{tikzcd}$$

(2.9) Theorem. *The homotopy category* $\mathrm{Hot}(\mathfrak{A})$ *is a triangulated category with the functor* $X \mapsto X(1)$ *as suspension and the class of homotopy cones as the distinguished class.*

Proof. Let us walk through some of the conditions of Cat(5.2):

(1)(a). A triangle isomorphic to a homotopy cone is a homotopy cone. This is obvious from the definition.

(1)(b). Every morphism $\varphi: X \to Y$ of $\mathrm{Hot}(\mathfrak{A})$ embeds into a homotopy cone. Indeed, take any morphism of complexes $f: X \to Y$ representing φ. Then φ embeds into the cone of f.

(1)(c) If X is any complex, then triangle $(X, X, 0; 1_X, 0, 0)$ is a homotopy cone. Indeed, by the Characterization in (2.7) the triangle is isomorphic in $\mathrm{Hot}(\mathfrak{A})$ to the cone of the identity of the complex X.

(2) The rotation axiom is a consequence of the following: Consider a morphism of complexes $f: X \to Y$ and its cone

$$\begin{tikzcd}
& Z \arrow[dl, squiggly, "k"'] & \\
X \arrow[rr, "f"] & & Y, \arrow[ul, "h"']
\end{tikzcd}$$

Then the following two triangles,

$$\begin{tikzcd}
& Z \arrow[dl, "k"'] & \\
X(1) \arrow[rr, squiggly, "-f(1)"] & & Y, \arrow[ul, "h"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& Z(-1) \arrow[dl, "-k(-1)"'] & \\
X \arrow[rr, "f"] & & Y, \arrow[ul, squiggly, "h"']
\end{tikzcd}$$

are, respectively, homotopy equivalent to the cone of h and the cone of $-k(-1)$.

(3) The prism axiom is a consequence of Lemma (2.4).

(4) We leave the verification of the octahedron axiom, and the verification of (2), as an exercise. ▯

(2.10). Clearly, with notations corresponding to the notations used for subcategories of complexes, there are natural triangulated subcategories of $\mathrm{Hot}(\mathfrak{A})$:

$$\mathrm{Hot}^{+}(\mathfrak{A}), \quad \mathrm{Hot}^{-}(\mathfrak{A}), \quad \mathrm{Hot}^{\mathrm{b}}(\mathfrak{A}), \quad \mathrm{Hot}_{\mathfrak{C}}(\mathfrak{A}),$$

where $\mathfrak{C}$ in the last notation is a given thick subcategory of $\mathfrak{A}$.

Of the general properties of triangulated categories we mention here the following:

Corollary. *For any fixed complex A, the functor*

$$\operatorname{Hom}_{\operatorname{Hot}(\mathfrak{A})}(A, -)\colon \operatorname{Hot}(\mathfrak{A}) \to \mathbf{Ab}$$

is exact, that is, any exact triangle (2.8.1) *induces a long exact sequence of groups,*

$$\cdots \to \operatorname{Hom}_{\operatorname{Hot}}(A, Y) \to \operatorname{Hom}_{\operatorname{Hot}}(A, Z) \to \operatorname{Hom}_{\operatorname{Hot}}(A, X(1)) \to \operatorname{Hom}_{\operatorname{Hot}}(A, Y(1)) \to \cdots .$$

In particular, for a morphism of complexes $f\colon X \to Y$ it follows that f is a homotopy equivalence if and only if the cone of f is contractible.

By the characterization in (2.4), a morphism of complexes $f\colon X \to Y$ is a homotopy equivalence, if and only if for every object A of $\mathfrak{A}$ the induced morphism of complexes of abelian groups $\operatorname{Hom}_{\mathfrak{A}}(A, X) \to \operatorname{Hom}_{\mathfrak{A}}(A, Y)$ is a quasi-isomorphism.

A special property of the triangulated category $\operatorname{Hot}(\mathfrak{A})$ is that the p'th cohomology $H^p\colon \operatorname{Hot}(\mathfrak{A}) \to \mathbf{Ab}$ (for any fixed p) is an exact functor on $\operatorname{Hot}(\mathfrak{A})$. Indeed, $H^p(X(n)) = H^{p+n}(X)$, and so the assertion follows from (1.7): For any exact triangle (2.8.1) there is an induced long exact sequence in $\mathfrak{A}$:

$$\cdots \to H^p(Y) \to H^p(Z) \to H^{p+1}(X) \to H^{p+1}(Y) \to \cdots .$$

Note also that the functor,

$$\operatorname{Hot}(T)\colon \operatorname{Hot}(\mathfrak{A}) \to \operatorname{Hot}(\mathfrak{B}),$$

for any given additive functor $T\colon \mathfrak{A} \to \mathfrak{B}$, is exact, that is, it takes exact triangles to exact triangles.

(2.11). Let $f\colon X \to Y$ be a morphism of complexes, and let Z be its cone. It follows from the long exact sequence of cohomology that f is a quasi-isomorphism if and only if the cone Z is acyclic. It follows easily that the class of acyclic complexes, as a class in the homotopy category, is a triangular subclass, and that the system of quasi-isomorphisms is the corresponding system of morphisms. As a consequence we obtain the following result.

Proposition. *The system of quasi-isomorphism is a saturated denominator system in the homotopy category* $\operatorname{Hot}(\mathfrak{A})$.

We emphasize in particular the *denominator property* and the *equalizer property*:

(LOC 1) Any pair of morphisms of complexes $s\colon X \to X'$ and $f\colon X \to Y$ where s is a quasi-isomorphism may be completed to a homotopy commutative diagram,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[d, dashed, "s'"] \\ X' \arrow[r, dashed, "f'"] & Y', \end{tikzcd}$$

where s' is a quasi-isomorphism. An conversely, if f' and s' are the given morphisms with s' a quasi-isomorphism, then they may be completed with s, f to a homotopy commutative diagram with a quasi-isomorphism s.

(LOC 2) If two morphisms of complexes $f, g\colon X \to Y$ are equalized up to homotopy by a quasi-isomorphism $s\colon X' \to X$ then they may be coequalized up to homotopy by a quasi-isomorphism $t\colon Y \to Y'$.

(2.12) The homotopy theorem of injectives. *Let $X \in \mathfrak{A}^\bullet$ be an acyclic complex and $Q \in \mathfrak{A}^+$ a right complex of injective objects. Then any morphism of complexes $f\colon X \to Q$ is homotopic to zero.*

Proof. We have to construct a family of morphisms $s\colon X \to Q(-1)$ such that $f = s(1)\partial + \partial(-1)s$. The morphisms $s^p\colon X^p \to Q^{p-1}$ will be constructed inductively. Consider the equations in degrees at most n,

$$f^{p-1} = s^p\partial^{p-1} + \partial^p s^{p-1} \quad \text{for } p \leqslant n. \tag{2.12.1}$$

Since $Q \in \mathfrak{A}^+$, we have $Q^p = 0$ (and $f^p = 0$) when $p \ll 0$. So, with $s^p := 0$ for $p \ll 0$, we may assume that (2.12.1) holds when $n \ll 0$. Proceed by induction: Assume the s^p are defined for $p \leqslant n$ such that (2.12.1) holds. We have to chose s^{n+1} such that the equation in (2.12.1) holds for $p = n+1$, that is, such that $f^n = s^{n+1}\partial^n + \partial^{n-1}s^n$. Now, for the morphism $h := f^n - \partial^{n-1}s^n\colon X^n \to Q^n$, we see that

$$h\partial_X^{n-1} = (f - \partial s)\partial = f\partial - \partial s\partial = \partial(f - s\partial) = \partial\partial s = 0.$$

Hence h extends to a morphism $h'\colon \operatorname{Cok}\partial_X^{n-1} \to Q^n$. Now, as X is acyclic in degree n, $\operatorname{Cok}\partial_X^{n-1} = \operatorname{Im}\partial_X^n$ injects into X^{n+1} and Q is injective. Therefore, h' extends to a morphism $h''\colon X^{n+1} \to Q^n$. By construction,

$$h''\partial^n = h = f^n - \partial^{n-1}s^n.$$

Hence $s^{n+1} := h''$ is the proper choice. ∎

(2.13) Corollary. *Every acyclic right complex Q of injectives is contractible. Every quasi-isomorphism $Q \to Q'$ between right complexes of injectives is a homotopy equivalence.*

Proof. To prove the first part, note that, by the Theorem, the identity 1_Q is homotopic to zero; hence Q is contractible by Characterization (2.4). If $f\colon Q \to Q'$ is a quasi-isomorphism, then the cone of f is a acyclic, since cohomology is an exact functor. Hence the second part is a consequence of the first part. ∎

(2.14) Corollary. *Let A, B be objects and let X and Q be positive complexes over A and B respectively, say with co-augmentations $\varepsilon\colon A \to X$ and $\eta\colon B \to Q$. Assume that $A \to X$ is a resolution, that is, the mapping cone,*

$$\overline{X}: \quad \cdots \to 0 \to A \xrightarrow{\varepsilon} X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots$$

is acyclic, and that Q is a complex of injectives. Then any morphism $f\colon A \to B$ extends to a morphism of complexes $\tilde{f}\colon X \to Q$,

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tilde{f}} \\ B & \xrightarrow{\eta} & Q, \end{array}$$

and $\tilde{f}$ is unique up to homotopy.

Proof. The two compositions in the square are morphisms of complexes from A, concentrated in degree 0, to the positive complex Q. Therefore the two compositions are equal if and only if they are homotopy equivalent. Thus the we are looking for morphisms $\tilde{f}$ making the diagram commutative up to homotopy, that is, morphisms $\tilde{f}$ in the homotopy category such that $\tilde{f}\varepsilon$ is the given composition ηf.

By the preceding corollary, $\operatorname{Hom}_{\mathrm{Hot}}(\overline{X}, Q) = 0$. Hence, by the dual of the long exact sequence of Corollary (2.10), the morphism ε induces an isomorphism,

$$\operatorname{Hom}_{\mathrm{Hot}}(X, Q) \xrightarrow{\sim} \operatorname{Hom}_{\mathrm{Hot}}(A, Q),$$

So, the morphism εf on the right hand side is hit by a unique morphism on the left hand sides. ∎

(2.15) Exercises.

1. Let $f\colon X \to Y$ be a monomorphism of complexes, split monic in every degree. Let $W := \operatorname{Cok} f$ and $Z := \operatorname{Con} f$ be the cokernel and the cone of f. Prove that the induced morphism $Z \to W$ is a homotopy equivalence. [Hint: Use (1.6?) on the complexes obtained by applying the functor $\operatorname{Hom}(A, -)$.]

2. Let $f\colon X \to Y$ be a map of complexes. Then with $Z := \operatorname{Con} f$ there is a natural short exact sequence, split in every degree,

$$0 \to Y \xrightarrow{h} Z \xrightarrow{k} X(1) \to 0.$$

Consider the cone $\operatorname{Con} h$ of h. By the previous excercise, the induced morphism $\operatorname{Con} h \to X(1)$ is a homotopy equivalence, that is, an isomorphism in the homotopy category. So, replacing $\operatorname{Con} h$ by $X(1)$ we have obtained a homotopy cone,

$$\begin{tikzcd} & Z \arrow[dl, "k"'] & \\ X(1) \arrow[rr, rightsquigarrow, "f(1)"] & & Y. \arrow[ul, "h"'] \end{tikzcd}$$

The conclusion is incorrect, cf. the homotypy cones in Theorem (2.9)/2). Where is the error in the argument?

3. In the second square of (2.6) the counter clocwise composition $Z'' \rightsquigarrow X'' \rightsquigarrow X$ is determined by the following computations:

$$(1\,0)(1\,0)\sigma = (1,\,0)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\sigma = (1\,0\,0\,0)\sigma = (-1\,0\,0\,0).$$

Explain the computations, especially the first equation.

3. Bicomplexes.

(3.1) Definition. A *bicomplex* in $\mathfrak{A}$ is a diagram,

$$
\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & X^{p,q+1} \arrow[u] \arrow[r, "\partial_1"] & X^{p+1,q+1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & X^{p,q} \arrow[u, "\partial_2"] \arrow[r, "\partial_1"] & X^{p+1,q} \arrow[u, "\partial_2"] \arrow[r] & \cdots , \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}
\tag{3.1.1}
$$

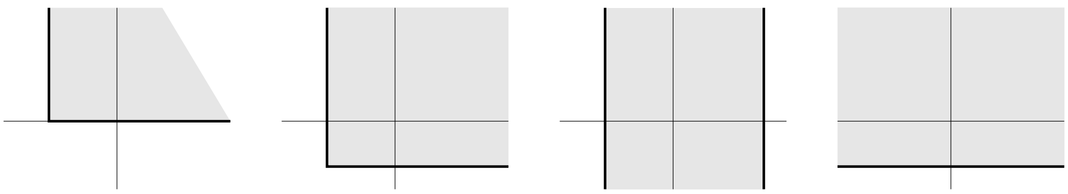
of objects $X^{p,q}$ and morphisms $\partial_1^{pq} : X^{p,q} \to X^{p+1,q}$ and $\partial_2^{pq} : X^{p,q} \to X^{p,q+1}$ for p, q in $\mathbb{Z}$, such that

$$\partial_1\partial_1 = 0, \quad \partial_2\partial_2 = 0, \quad \partial_1\partial_2 + \partial_2\partial_1 = 0. \tag{3.1.2}$$

The first two equations mean that each *row* $X^{\bullet,q}$ and each *column* $X^{p,\bullet}$ is a complex. The last equation means that each small square in the diagram is anticommutative; in particular, the morphisms $\partial_1^{p,\bullet}$ do not define a morphism of complexes $X^{p,\bullet} \to X^{p+1,\bullet}$.

There is an obvious abelian category of bicomplexes in $\mathfrak{A}$, denoted $\mathfrak{A}^{\bullet,\bullet}$.

(3.2). The *support* of a bicomplex is the set of pairs $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ such that $X^{p,q} \neq 0$. With restrictions on the support we obtain several natural subcategories of bicomplexes. For instance, the category $\mathfrak{A}^{+,\geqslant 0}$ consists of all bicomplexes having support in a region of the form $[N, \infty) \times [0, \infty)$ for some integer N. With similar notations we obtain subcategories with support in regions as indicated on the figure:



$\mathfrak{A}^{+,\geqslant 0}$ $\mathfrak{A}^{+,+}$ $\mathfrak{A}^{\mathrm{b},\bullet}$ $\mathfrak{A}^{\bullet,+}$

(3.3) The total complex. We denote by $\mathfrak{A}^{\bullet,\bullet}_{\divideontimes}$ the full subcategory of $\mathfrak{A}^{\bullet,\bullet}$ consisting of bicomplexes X such that on every the line $p+q = n$ there is only a finite number of nonzero $X^{p,q}$. Note that $\mathfrak{A}^{\bullet,\bullet}_{\divideontimes}$ includes any of the three first subcategories indicated on the figure in (3.2), but not the fourth.

For every X in $\mathfrak{A}^{\bullet,\bullet}_{\divideontimes}$ there is an associated *total complex* $\mathrm{Tot}(X)$ defined as follows: In degree n,

$$\mathrm{Tot}^n(X) = \prod_{p+q=n} X^{p,q}, \tag{3.3.1}$$

and the differential $\partial\colon \mathrm{Tot}^n \to \mathrm{Tot}^{n+1}$ is the sum $\partial = \partial_1 + \partial_2$ of the two (diagonal) morphisms determined by the two differentials ∂_1 and ∂_2. It follows from the equations (3.1.2) that $\partial\partial = 0$. So the total complex determines a functor, obviously exact,

$$\mathrm{Tot}\colon \mathfrak{A}^{\bullet;\bullet}_{\divideontimes} \to \mathfrak{A}^{\bullet}.$$

Clearly, $\mathfrak{A}^{+,+}$ and $\mathfrak{A}^{\mathrm{b},\bullet}$ are contained in $\mathfrak{A}^{\bullet;\bullet}_{\divideontimes}$, and restriction of the total complex functor defines functors $\mathrm{Tot}\colon \mathfrak{A}^{+,+} \to \mathfrak{A}^{+}$ and $\mathfrak{A}^{\mathrm{b},\bullet} \to \mathfrak{A}^{\bullet}$.

A bicomplex has shifts in two directions: The primary shifted bicomplex $X(1,0)$ has $X(1,0)^{p,q} = X^{p+1,q}$ and both differentials are multiplied by -1; the secondary shift $X(0,1)$ is defined similarly. Clearly, if $X \in \mathfrak{A}^{\bullet;\bullet}_{\divideontimes}$ then

$$\mathrm{Tot}(X(1,0)) = \mathrm{Tot}(X(0,1)) = \mathrm{Tot}(X)(1). \tag{3.3.1}$$

Clearly, if $\mathfrak{A}$ has $\prod_{\mathbb{N}}$'s, the total complex may be defined by (3.3.1) for any bicomplex; alternatively, the finite direct sum could be replaced by an infinite direct sum.

(3.4) Definition. It is often useful to view a bicomplex as *bifamily*, that is, a $(\mathbb{Z}\times\mathbb{Z})$-indexed family $X = (X^{p,q})$ of objects, with two given morphisms of families $\partial_1\colon X \to X(1,0)$ and $\partial_2\colon X \to X(0,1)$ satisfying the equations in (3.1.2). Note that the functor Tot is defined on the category of families corresponding to $\mathfrak{A}^{\bullet;\bullet}_{\divideontimes}$. The differential in $\mathrm{Tot}(X)$, for a bicomplex $X \in \mathfrak{A}^{\bullet;\bullet}_{\divideontimes}$, is in fact the sum $\partial_{\mathrm{Tot}} = \mathrm{Tot}(\partial_1) + \mathrm{Tot}(\partial_2)$.

(3.5) Commuting a bicomplex. In a bicomplex X, each column $X^{p,\bullet}$ is a complex, but the differential $\partial_1\colon X^{p,\bullet} \to X^{p+1,\bullet}$ is not a morphism of complexes since the small squares in Diagram (3.1.1) are not commutative. In other words, a complex of complexes corresponds to a *commutative* diagram (3.1.1) (with $\partial_1^2 = \partial_2^2 = 0$), and not to a bicomplex.

Since each small square in the diagram of a bicomplex is anticommutative, a sign change in 1 or 3 of its arrows will make the small square commutative. There are several conventions for chosing sign changes that make the whole diagram of a bicomplex commutative, and hence turn a bicomplex into a complex of complexes. The same choice will then turn a complex of complexes into a bicomplex.

A *sign function* for the Diagram (3.1.1) is a ±1-valued function ε defined on the underlying graph. So, for the two edges beginning at p, q the function has a value ε_1^{pq} in the primary direction and a value ε_2^{pq} in the other direction. The sign function is *odd*, resp. *even*, if for any small square of the diagram the product of the four signs corresponding to the four edges is equal to -1, resp. equal to 1.

Let ε be sign function. For any diagram X of the form (3.1.1), denote by ${}_{\varepsilon}X$ the diagram obtained from X by multiplying the morphism $\partial_i^{p,q}$ with the sign determined by ε, that is,

$${}_{\varepsilon}\partial_i^{p,q} := \varepsilon_i^{p,q}\partial_i^{p,q}, \quad i = 1, 2.$$

Observation 1. *If ε is an even sign function, then there is a canonical choice of signs $\alpha^{p,q} = \pm 1$ for all $p, q \in \mathbb{Z}$ such that $\alpha^{0,0} = 1$ and such that for any diagram X of the form* (3.1.1) *multiplication by $\alpha^{p,q}$ in $X^{p,q}$ defines an isomorphism $X \xrightarrow{\sim} {}_\varepsilon X$.*

Proof. Define $\alpha^{p,q}$ as follows: Choose a path from $(0, 0)$ to (p, q) in the graph underlying the diagram, and let $\alpha^{p,q}$ be the product of the signs given by ε on the edges in the path. The product is independent of the choice, since ε is even. As a consequence, $\alpha^{p+1,q} = \varepsilon_1^{p,q}\alpha^{p,q}$, and it follows that $\alpha\partial_1 = {}_\varepsilon\partial_1\alpha$ and, similarly, $\alpha\partial_2 = {}_\varepsilon\partial_2\alpha$. Hence $\alpha\colon X \to {}_\varepsilon X$ is a morphism of diagrams. ∎

Observation 2. *Let ε be an odd sign function. If X is a complex of complexes, then ${}_\varepsilon X$ is bicomplex. If η is a second odd sign function, then the bicomplexes ${}_\varepsilon X$ and ${}_\eta X$ are canonically isomorphic.*

Proof. The first assertion is immediate, the second follows from Observation 1, since η/ε is even and transforms ${}_\varepsilon X$ into ${}_\eta X$. ∎

An odd sign function ε transforms a complex of complexes X into a bicomplex ${}_\varepsilon X$, and $X \mapsto {}_\varepsilon X$ is obviously a functor isomorphism $(\mathfrak{A}^\bullet)^\bullet \xrightarrow{\sim} \mathfrak{A}^{\bullet,\bullet}$. We denote by $(\mathfrak{A}^\bullet)^\bullet_{\divideontimes}$ the subcategory of $\mathfrak{A}^\bullet)^\bullet$ corresponding to $\mathfrak{A}^{\bullet,\bullet}_{\divideontimes}$. For a complex of complexes X in $(\mathfrak{A}^\bullet)^\bullet_{\divideontimes}$, the total complex $\mathrm{Tot}_\varepsilon(X) := \mathrm{Tot}({}_\varepsilon X)$ is defined. Clearly, $X \mapsto \mathrm{Tot}_\varepsilon X$ is a functor,

$$\mathrm{Tot}_\varepsilon\colon (\mathfrak{A}^\bullet)^\bullet_{\divideontimes} \to \mathfrak{A}^\bullet.$$

A different odd sign function defines a functor Tot_η which, by Observation 2, is canonically isomorphic to Tot_ε. In fact, the two complexes $\mathrm{Tot}_\varepsilon X$ and $Tot_\eta X$ have the same degree n component $\bigoplus_{p+q=n} X^{p,q}$, and the isomorphism is given by a diagonal automorphism with the signs ± 1.

In the sequel we use the following sign function,

$$\varepsilon_i^{p,q} = \begin{cases} 1 & \text{if } i = 1, \\ (-1)^p & \text{if } i = 2. \end{cases} \tag{3.5.1}$$

Its effect on a diagram (3.1.1) is to multiply all the differentials in the odd columns by -1.

(3.6) Lemma. *The functor* $\mathrm{Tot}_\varepsilon\colon (\mathfrak{A}^\bullet)^\bullet_{\divideontimes} \to \mathfrak{A}^\bullet$, *where ε is the sign function* (3.5.1), *commutes with shifts, commutes with the formation of the cone, and respects homotopy.*

Proof. The first assertion is the equality, for a complex X in $(\mathfrak{A}^\bullet)^\bullet_{\divideontimes}$,

$$\mathrm{Tot}_\varepsilon(X(1)) = (\mathrm{Tot}_\varepsilon(X))(1); \tag{3.6.1}$$

the shift on the left side is the *primary shift*: X is a complex $\cdots \to X^p \xrightarrow{\partial_1} X^{p+1} \to \cdots$ with a differential (called the primary differential). In turn each X^p, being a complex, is viewed as column, with a differential ∂_2, called the secondary differential. The primary shift moves the X^p and multiplies the differential ∂_1 by -1. The columns in X of even index are placed in odd degrees in $X(1)$, and so, when forming ${}_\varepsilon(X(1))$, their differentials are multiplied by -1.

So, compared with the differentials of ${}_\varepsilon X$, all differentials of ${}_\varepsilon(X(1))$ have been multiplied by -1. In other words, ${}_\varepsilon(X(1)) = {}_\varepsilon X(1,0)$. Hence (3.6.1) follows from (3.3.1).

Consider next a morphism $f\colon X \to Y$ of complexes $(\mathfrak{A}^\bullet)^\bullet_\circledast$ and its mapping cone $Z = X(1) \oplus Y$. As a bifamily we have $Z = X(1,0) \oplus Y$; the primary and secondary differentials of Z are:

$$\partial_1 = \begin{pmatrix} -\partial_{X,1} & 0 \\ f & \partial_{Y,1} \end{pmatrix} \text{ and } \partial_2 = \begin{pmatrix} \partial_{X,2} & 0 \\ 0 & \partial_{Y,2} \end{pmatrix}$$

Passing to the bicomplex ${}_\varepsilon Z$, the primary differential is unchanged: ${}_\varepsilon\partial_1 = \partial_1$; in the secondary differential, for a column of degree q, the differential $\partial_{X,2}$ is in a column of degree $q+1$ and $\partial_{Y,2}$ is in a column of degree q. So the secondary differential is changed to the following,

$${}_\varepsilon\partial_{Z,2} = \begin{pmatrix} {}_\varepsilon\partial_{X,2} & 0 \\ 0 & {}_\varepsilon\partial_{Y,2} \end{pmatrix}$$

Apply the functor Tot, and add the Tot of the two differentials: It follows that $\operatorname{Tot}_\varepsilon Z$ is the family $(\operatorname{Tot} X)(1) \oplus \operatorname{Tot} Y$ with the differential,

$${}_\varepsilon\partial_{\operatorname{Tot} Z} = \begin{pmatrix} -\operatorname{Tot}{}_\varepsilon\partial_{X,1} - \operatorname{Tot}{}_\varepsilon\partial_{X,2} & 0 \\ \operatorname{Tot} f & \operatorname{Tot}{}_\epsilon\partial_{Y,1} + \operatorname{Tot}{}_\varepsilon\partial_{Y,2} \end{pmatrix},$$

which is the differential of the cone of $\operatorname{Tot}(f)\colon \operatorname{Tot}_\varepsilon(X) \to \operatorname{Tot}_\varepsilon(Y)$.

Finally, assume that $f\colon X \to Y$ is null homotopic as a morphism of complexes, say $f = \partial_Y s + s\partial_X$ where s is a family of morphisms $s\colon X \to Y(-1)$ that is, a family of morphisms $s\colon X^p \to Y^{p-1}$. Note that each of X^p and Y^{p-1} is a complex, and so s is a morphism of complexes. In other words, s commutes with the secondary differentials,

$$s\partial_{X,2} = \partial_{Y,2}s.$$

Now, passing from X and Y to the bicomplexes ${}_\varepsilon X$ and ${}_\varepsilon Y$, the differentials of X^p and Y^{p-1} are multiplied with opposite signs. Hence commutation becomes anti-commutation, $s\,{}_\varepsilon\partial_{X,2} + {}_\varepsilon\partial_{Y,2}s = 0.$, and it follows that

$$\operatorname{Tot} s\ \operatorname{Tot}{}_\varepsilon\partial_{X,2} + \operatorname{Tot}{}_\varepsilon\partial_{Y,2}\ \operatorname{Tot} s = 0.$$

Hence

$$\operatorname{Tot}(f) = \operatorname{Tot}(\partial_1 s + s\partial_1) = \operatorname{Tot}(\partial_1 s + s\partial_1 + s\partial_2 + \partial_2 s) = \partial_{\operatorname{Tot}}\operatorname{Tot} s + \operatorname{Tot} s\,\partial_{\operatorname{Tot}},$$

and $\operatorname{Tot} f$ is null homotopic. ▯

(3.7) The Column Theorem. *If a bicomplex X in $\mathfrak{A}^{\bullet;\bullet}_\circledast$ has acyclic columns, then the total complex* $\operatorname{Tot}(X)$ *is acyclic. If, in a complex of complexes $X \in (\mathfrak{A}^\bullet)^\bullet_\circledast$, each component X^p is an acyclic complex, the* $\operatorname{Tot}_\varepsilon X$ *is acyclic.*

Proof. Clearly, the two assertions are equivalent. We prove the second. Consider, as in Hot(1.9) the truncated complexes $X^{\geqslant p}$ and the mapping cone,

$$\begin{tikzcd} & X^{\geqslant p} \arrow[dl, squiggly] & \\ X^p(-p-1) \arrow[rr, "\partial"] & & X^{>p}. \arrow[ul] \end{tikzcd} \tag{3.7.2}$$

Each vertex is a complex of complexes. Apply the functor Tot_ε. The lower left vertex is the complex having X^p concentrated in degree $p+1$. It follows from the Lemma that $\mathrm{Tot}_\varepsilon\, X^p(-p-1)$ is the internal shift $X^p(-p-1)$ of the complex X^p. So the Lemma yields this mapping cone:

$$\begin{tikzcd} & \mathrm{Tot}_\varepsilon\, X^{\geqslant p} \arrow[dl, squiggly] & \\ X^p(-p-1) \arrow[rr, "\partial"] & & \mathrm{Tot}_\varepsilon\, X^{>p}. \arrow[ul] \end{tikzcd} \tag{3.7.2}$$

By hypothesis, the lower left vertex is acyclic. Therefore, by the long exact cohomology sequence of the cone, we obtain for all integers n an isomorphism,

$$H^n(\mathrm{Tot}_\varepsilon\, X^{>p}) \xrightarrow{\sim} H^n(\mathrm{Tot}_\varepsilon\, X^{\geqslant p}).$$

Now X is in $(\mathfrak{A}^\bullet)^\bullet_\divideontimes$. So, for a fixed n there are only finitely many nonzero components $X^{p,q}$ on the lines $p+q=n-1$, $p+q=n$, and $p+q=n+1$. It follows, when $p \ll 0$ that the complexes $\mathrm{Tot}_\varepsilon\, X$ and $\mathrm{Tot}_\varepsilon\, X^{\geqslant p}$ have the same components of degrees $n-1$, n, and $n+1$; in particular, they have the same degree-n cohomology. Similarly, it follows for $p \gg 0$ that $\mathrm{Tot}_\varepsilon\, X^{\geqslant p}$ vanishes in degree n; in particular, its degree-n cohomology vanishes. Hence, with $p \ll 0$ and $p' \gg 0$,

$$H^n(\mathrm{Tot}_\varepsilon(X)) = H^n(\mathrm{Tot}_\varepsilon(X^{\geqslant p})) = \cdots = H^n(\mathrm{Tot}_\varepsilon(X^{\geqslant p'})) = 0.$$

Therefore $\mathrm{Tot}_\varepsilon(X)$ is acyclic. □

Naturally there is a corresponding *Row Theorem*; it may be obtained from the bicomplex version of the Column Theorem by interchanging rows and columns.

(3.7) The Row Theorem. *If a bicomplex X in $\mathfrak{A}^{\bullet,\bullet}_\divideontimes$ has acyclic rows, then the total complex* $\mathrm{Tot}(X)$ *is acyclic. If X is an acyclic complex of complexes $X \in (\mathfrak{A}^\bullet)^\bullet_\divideontimes$, then* $\mathrm{Tot}_\varepsilon\, X$ *is acyclic.*

(3.8). Assume there is given an additive subclass $\mathfrak{Q} \subseteq \mathfrak{A}$ such that every object A of $\mathfrak{A}$ admits a monomorphism $A \hookrightarrow Q$ into an object Q of $\mathfrak{Q}$. Then every object A has a resolution with objects from $\mathfrak{Q}$, that is, an exact sequence,

$$0 \to A \xrightarrow{\partial} Q^0 \xrightarrow{\partial^0} Q^1 \xrightarrow{\partial^1} Q^2 \longrightarrow \cdots, \tag{3.8.1}$$

with Q^i in $\mathfrak{Q}$. Indeed, the construction is inductive: Take a monomorphism $\partial\colon A \hookrightarrow Q^0$ into an object Q^0 of $\mathfrak{Q}$. Let A^1 be the cokernel of ∂, and take a monomorphism $A^1 \hookrightarrow Q^1$ into an object of $\mathfrak{Q}$. Let ∂^1 be the composition $Q^0 \to A^1 \to Q^1$, and let A^2 be the cokernel of ∂^1. Continue by induction.

An additive class $\mathfrak{R} \subseteq \mathfrak{A}$ is said to have (*right*) *dimension* at most N if, for every exact sequence,

$$R^0 \xrightarrow{\partial_0} \cdots \longrightarrow R^{N-1} \longrightarrow R \longrightarrow 0,$$

such that the first N objects $R^0, \dots, R^{N-1}$ belong to $\mathfrak{R}$, also the last object R belongs to $\mathfrak{R}$. Clearly, if the additive class $\mathfrak{Q}$ is of dimension at most N, then the resolution (3.8.1) may be taken to be of length at most N, that is, with $Q^i = 0$ for $i > N$.

The condition for $N = 0$ means that $\mathfrak{Q} = \mathfrak{A}$; for $N = 1$ it means that any quotient of an object in $\mathfrak{Q}$ belongs to $\mathfrak{Q}$.

(3.9) Lemma. *Assume the conditions of* (3.8) *for the class* $\mathfrak{Q}$. *Then, every complex* X *in* $\mathfrak{A}^\bullet$ *admits a monomorphism* $X \to Y$ *into a contractible complex* Y *of objects from* $\mathfrak{Q}$, *and every positive complex admits a monomorphism into a positive complex of objects of* $\mathfrak{Q}$.

Proof. Chose for every n a monomorphism $\iota_n \colon X^n \hookrightarrow Q^n$ into an object Q^n of $\mathfrak{Q}$. Let Q be the family of objects Q^n, and view Q as a complex with zero differentials. Let Y be the mapping cone of the identity of Q. Then

$$Y^n = \begin{matrix} Q^{n+1} \\ \oplus \\ Q^n \end{matrix}, \quad \partial_Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and Y is contractible. Moreover, the family $\binom{\iota\partial}{\iota}$ is a morphism of complexes $X \to Y$, and obviously a monomorphism.

If $X \in \mathfrak{A}^{\geqslant 0}$, we may take $Q^n = 0$ for $n \geqslant 0$. Then $Q \in \mathfrak{A}^{\geqslant 0}$, but Y has Q^0 as a component in degree -1. However, truncation of the negative components, that is, the redefinition $Y^n := 0$ for $n < 0$, yields a positive complex Y, as desired. ∎

(3.10) The Density Theorem. *Assume the conditions of* (3.8) *for the class* $\mathfrak{Q}$. *Then every positive complex* X *in* $\mathfrak{A}^{\geqslant 0}$ *admits a quasi-isomorphism* $X \xrightarrow{\sim} Y$ *into a positive complex* Y *of objects of* $\mathfrak{Q}$. *Moreover, if* $\mathfrak{Q}$ *is of finite dimension, then every complex in* $\mathfrak{A}^\bullet$ *admits a quasi-isomorphism into a complex of objects from* $\mathfrak{Q}$.

Proof. It follows from the Lemma that X admits a monomorphism into a positive complex of objects of $\mathfrak{Q}$. Therefore, by the observation at the beginning of (???), applied to the abelian category $\mathfrak{A}^{\geqslant 0}$ and the class $\mathfrak{Q}^{\geqslant 0}$, there is a resolution (3.8.1) of X by objects $Y^n \in \mathfrak{Q}^{\geqslant 0}$. The resolution is the cone of the morphism $X(0) \to Y$, from X as a complex concentrated in degree 0 to the positive complex $Y \in (\mathfrak{A}^{\geqslant 0})^{\geqslant 0}$. This cone belongs to $(\mathfrak{A}^{\geqslant 0})^{+} \subseteq \mathfrak{A}^{\bullet;\bullet}_{\divideontimes}$, and, by construction, its rows are exact. Therefore, the total complex of the cone is exact. So the morphism $X \to \operatorname{Tot}(Y)$ has exact cone. Consequently, $X \to \operatorname{Tot}(Y)$ is a quasi-isomorphism. Since $\operatorname{Tot}(Y)$ is a positive complex with objects in $\mathfrak{Q}$, the first assertion has been verified.

The proof of the second assertion is similar: If $\mathfrak{Q}$ has dimension at most N, then the class of complexes $\mathfrak{Q}^\bullet$ in $\mathfrak{A}^\bullet$ has dimension of most N. So the resolution Y may be taken to be a finite resolution, of length at most N. So the cone of $X \to Y$ belongs to $(\mathfrak{A}^\bullet)^{\mathrm{b}} \subseteq \mathfrak{A}^{\bullet;\bullet}_{\divideontimes}$. The rest of the argument is identical to the argument of the first assertion. ∎

(3.11) Example. Fix a complex $X \in \mathfrak{A}^\bullet$. Recall that for an object B of $\mathfrak{A}$ we denote by $\mathrm{Hom}^\bullet(X, B)$ the complex defined in Hot(1.13) (with the sign conventions on the differential). The construction is clearly functorial in B, so it defines a functor $\mathfrak{A} \to \mathbf{Ab}^\bullet$. Extending it to complexes, we obtain a functor

$$\mathfrak{A}^\bullet \to (\mathbf{Ab}^\bullet)^\bullet.$$

It associates with complex of complexes $Y = \cdots \to Y^{p-1} \to Y^p \to Y^{p+1} \to \cdots$ the following complex of complexes,

$$\cdots \to \mathrm{Hom}^\bullet(X, Y^{p-1}) \to \mathrm{Hom}^\bullet(X, Y^p) \to \mathrm{Hom}^\bullet(X, Y^{p+1}) \to \cdots .$$

Its degree-(p, q) term is the abelian group $\mathrm{Hom}_{\mathfrak{A}}(X^{-q}, Y^p)$. The category $\mathbf{Ab}$ has infinite products, so the total complex is defined for an arbitrary bicomplex. It is easy to that the total complex of this bicomplex is the complex $\mathrm{Hom}^\bullet(X, Y)$ defined in (1.10).

4. Multicomplexes.

5. Additive functors.

6. Standard filtrations.

Derivable functors in abelian categories

Let $\mathfrak{A}$ and $\mathfrak{B}$ be abelian categories.

1. The classical sequence of derived functors.

(1.0) Setup. An additive functor $T: \mathfrak{A} \to \mathfrak{B}$ has obvious extensions to functors of complexes,

$$T: \mathfrak{A}^\bullet \to \mathfrak{B}^\bullet, \qquad T: \mathfrak{A}^+ \to \mathfrak{B}^+, \quad \text{etc.},$$

and to triangular functors on the homotopy categories,

$$T\colon \operatorname{Hot}(\mathfrak{A}) \to \operatorname{Hot}(\mathfrak{B}), \qquad T\colon \operatorname{Hot}^+(\mathfrak{A}) \to \operatorname{Hot}^+(\mathfrak{B}), \quad \text{etc.},$$

(1.1) Definition. An additive functor $T: \mathfrak{A} \to \mathfrak{B}$ is called *right uniformly derivable*, or simply *derivable*, if there exists an additive class $\mathfrak{Q}$ of objects $\mathfrak{A}$ satisfying the following two conditions:

(i) Every object A of $\mathfrak{A}$ admits a monomorphism $A \hookrightarrow Q$ into an object of $\mathfrak{Q}$.
(ii) If U is an exact right complex of objects in $\mathfrak{Q}$, then the complex TU is exact.

A class $\mathfrak{Q}$ with the two properties is said to be T*-unfolding*.

Note that the first condition is independent of the functor T; we shall refer til the condition by saying that the class $\mathfrak{Q}$ is a (right) *dense subclass* of $\mathfrak{A}$.

If T is derivable, then the n'th *derived functor* R^nT is defined on a right complex X as follows: Chose a quasi-isomorphism $s: X \to U$ into a right complex U of objects from the T-unfolding class $\mathfrak{Q}$; this is possible by The Density Theorem Hot(3.10). Define $R^nT(X)$ as the n'th cohomology,

$$R^nT(X) := H^nTU. \tag{1.1.1}$$

Note that the morphism s induces a morphism $TX \to TU$ and hence a morphism of cohomology

$$H^n(TX) \to R^nT(X). \tag{1.1.2}$$

It is important to notice the following consequence of condition (2): If $s: U \to U'$ is a quasi-isomorphism between right complexes of objects from $\mathfrak{Q}$, then $Ts: TU \to TU'$ is a quasi-isomorphism. Indeed, the cone of $U \to U'$ is acyclic with objects in $\mathfrak{Q}$. Hence the cone of $TU \to TU'$ is acyclic by (2). Therefore, $TU \to TU'$ is a quasi-isomorphism.

(1.2) Proposition. *Assume that $T: \mathfrak{A} \to \mathfrak{B}$ is derivable. Then the formation of the object $R^nT(X)$, for right complexes X, determines a well-defined, triangular functor from the triangulated homotopy category to the abelian category $\mathfrak{B}$,*

$$R^nT\colon \operatorname{Hot}^+(\mathfrak{A}) \to \mathfrak{B}.$$

Moreover, a quasi-isomorphism $X \to Y$ induces an isomorphism $R^nT(X) \xrightarrow{\sim} R^nT(Y)$. Finally, with respect to shifts, we have the equality $R^nT(X) = R^0T(X(n))$.

Proof. Consider, in the first part of the proof, a morphism $f: X \to X'$ of complexes in $\mathfrak{A}^+$. Chose quasi-isomorphisms $s: X \to U$ and $s'\colon X' \to U'$ with $U, U' \in \mathfrak{Q}^+$. We have to prove that there is a natural morphism $H^n(TU) \to H^n(TU')$, depending only on the homotopy class of f. Apply the left denominator property Hot(2.8)(LOC 1) to the morphisms s and $s'f$ to obtain a homotopy commutative diagram in $\mathfrak{A}^+$, with a quasi-isomorphism t,

$$\begin{array}{ccccc} X & \xrightarrow{f} & X' & \xrightarrow{s'} & U' \\ {\scriptstyle s}\downarrow & & & & \downarrow{\scriptstyle t} \\ U & & \xrightarrow{\quad h \quad} & & V. \end{array} \qquad (1.2.1)$$

There is a quasi-isomorphism from V to a right complex in $\mathfrak{Q}^+$; Replacing V by the target, we may assume that $V \in \mathfrak{Q}^+$. Then Tt is a quasi-isomorphism $TU' \to TV$, and we define $H^n(TU) \to H^n(TU')$ as the composition, denoted h_* for simplicity,

$$h_*\colon H^n(TU) \xrightarrow{H^n(Th)} H^n(TV) \xrightarrow{H^n(Tt)^{-1}} H^n(TU').$$

We claim that the morphism $h_*\colon H^n(TU) \to H^n(TU')$ is independent of the diagram (1.2.1). Indeed, consider a second homotopy commutative diagram, say with morphisms $\hat{h}\colon U \to \hat{V}$ and $\hat{t}\colon U' \to \hat{V}$, and the corresponding morphism $\hat{h}_*$. The equality $h_* = \hat{h}_*$ is obvious, if the second diagram is obtained from the first by replacing V with the target of a quasi-isomorphism $V \to \hat{V}$. In general, apply the left denominator property Hot(2.8)(LOC 1) to the morphisms $t\colon U' \to V$ and $\hat{t}\colon U' \to \hat{V}$. It follows, replacing if necessary V and $\hat{V}$ by targets under quasi-isomorphisms, that we may assume in the homotopy category that $V = \hat{V}$ and $t = \hat{t}$. So it remains to prove that $H^n(Th) = H^n(T\hat{h})$ if $h, \hat{h}\colon U \to V$ both make the diagram (1.2.1) homotopy commutative. If this is the case, then h and $\hat{h}$ are equalized in the homotopy category by the quasi-isomorphism s. So, by equalizer property Hot(2.8)(LOC 1), h and $\hat{h}$ are coequalized by a quasi-isomorphism $u\colon V \to \tilde{V}$. Replacing $\tilde{V}$ by the target of a quasi-isomorphism, we may assume that $\tilde{V}$ is in $\mathfrak{Q}^+$. Then Tu is a quasi-isomorphism coequalizing Th and $T\hat{h}$. Therefore $H^n(Th)$ and $H^n(T\hat{h})$ are coequalized by the isomorphism $H^n(Tu)$. Hence $H^n(Th) = H^n(T\hat{h})$.

Take $X' = X$ and $f = 1_X$ in this result. In particular, then the top morphism in diagram (1.2.1) is a quasi-isomorphism. Hence the bottom morphism h is a quasi-isomorphism. Consequently, Th is a quasi-isomorphism, and the induced morphism is a canonical isomorphism

$H^n(TU) \xrightarrow{\sim} H^n(TU')$. In other words, the object $R^nT(X) := H^n(TU)$ is well-defined. Again, by the result in the first part, $X \mapsto R^nT(X)$ is functor on the homotopy category, since the diagram (1.2.1) is only assumed to be commutative up to homotopy.

To prove that R^nT is triangular, that is, takes exact triangles to exact sequences, consider an exact triangle in the homotopy category $\mathrm{Hot}^+(\mathfrak{A})$. We may assume that the third vertex X'' is the cone of a morphism $f: X \to X'$ of complexes in $\mathfrak{A}^+$. Chose a quasi-isomorphism$X \to U$ into a complex $U \in \mathfrak{Q}^+$. By the left denominator property Hot(2.11)(LOC 1) there is a commutative square in the homotopy category, with a quasi-isomorphism s',

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & X' \arrow[d, "s'"] \\ U \arrow[r, "h"] & U'. \end{tikzcd}$$

We may assume that $U' \in \mathfrak{Q}^+$. Let U'' be the cone of h. Then $U'' \in \mathfrak{Q}^+$, and the diagram extends with a morphism $s'': X'' \to U''$ to a morphism of triangles. As the two morphisms s and s' are quasi-isomorphisms, so is the third. So, the sequence $R^nT(X) \to R^nT(X') \to R^nT(X'')$ is, by definition, the sequence, $H^n(TU) \to H^n(TU') \to H^n(TU'')$ which is part of the long exact cohomology sequence of the cone of h, and hence exact.

Finally, it results immediately from the definition, that if $X \to Y$ is a quasi-isomorphism, then $R^nT(X) \to R^nT(Y)$ is an isomorphism. □

(1.3) Notes. (1) If the category $\mathfrak{A}$ has enough injectives, then the class of injectives is unfolding for any additive functor T. Indeed, take as $\mathfrak{Q}$ the class of injective objects of $\mathfrak{A}$. Then condition (i) is exactly the condition of having enough injectives. And (ii) is automatic, because an acyclic right complex U of injectives is contractible; hence TU is contractible, and hence acyclic.

(2) The derived functors R^nT are, in particular, defined on complexes concentrated in degree 0, that is, they define functors,

$$R^nT: \mathfrak{A} \to \mathfrak{B}.$$

To obtain the value $R^nT(A)$ for an object $A \in \mathfrak{A}$, choose a resolution,

$$0 \to A \to Q^0 \to Q^1 \to \cdots,$$

with $Q^i \in \mathfrak{Q}$ (this is possible by (i)). It defines a quasi-isomorphism $A \to Q$, and $R^nT(A) = H^n(TQ)$. Note that the morphism (1.1.1) for $n = 0$ is a transformation of functors,

$$TA \to R^0T(A). \tag{1.3.1}$$

The sequence $0 \to A \to Q^0 \to Q^1$ is left exact. Hence (1.3.1) is an isomorphism if T is left exact.

(1.4) Properties of derived functors. Let us emphasize that the exactness part of Proposition (1.2), given the commutation with respect to shifts, is the following assertion: Assume that $T\colon \mathfrak{A} \to \mathfrak{B}$ is derivable. Then every exact triangle in the homotopy category $\mathrm{Hot}^+(\mathfrak{A})$,

$$\begin{tikzcd} & Z \arrow[dl, squiggly, "w"'] & \\ X \arrow[rr, "u"] & & Y, \arrow[ul, "v"'] \end{tikzcd}$$

induces a long exact sequence connecting the functors R^nT:

$$\begin{array}{l} \cdots \to \\ R^nT(X) \to R^nT(Y) \to R^nT(Z) \to \\ R^{n+1}T(X) \to R^{n+1}T(Y) \to R^{n+1}T(Z) \to \\ \cdots \end{array}$$

Several important properties are straight forward consequences of the definition. For instance: If X is a complex in $\mathfrak{A}^{\geqslant N}$, then $R^iT(X) = 0$ for $i < N$. In particular, if A is an object of $\mathfrak{A}$, then $R^nT(A) = 0$ for $n < 0$.

The most important is the following property, immediate from the definition:

Theorem. *A quasi-isomorphism $X \to Y$ of complexes in $\mathfrak{A}^+$ induces an isomorphism of the n'th derived functor $R^nT(X) \xrightarrow{\sim} R^nT(Y)$.*

For a short exact sequence $0 \to X' \to X \to X'' \to 0$ of complexes in $\mathfrak{A}^+$, there is a long exact sequence similar to the one above,

$$\begin{array}{l} \cdots \to R^nT(X') \to R^nT(X) \to R^nT(X'') \to \\ R^{n+1}T(X') \to R^{n+1}T(X) \to R^{n+1}T(X'') \to \cdots \end{array}$$

Indeed, let Z be the cone of the morphism $X' \to X$. Then the induced morphism $Z \to X''$ is a quasi-isomorphism. By the theorem, the induced morphism $R^nT(Z) \to R^nT(X'')$ are isomorphisms. Hence the second long exact sequence is obtained from the first by replacing, for each n, $R^nT(Z)$ by $R^nT(X'')$.

In particular, when applied to a short exact sequence $0 \to A' \to A \to A'' \to 0$ of objects of $\mathfrak{A}$, it follows that the functor $R^0T\colon \mathfrak{A} \to \mathfrak{B}$ is left exact. Hence, the morphism $T \to R^0T$ is an isomorphism if (as noted in (1.3)(2)) and only if T is left exact.

(1.5) Acyclic objects. If $T\colon \mathfrak{A} \to \mathfrak{B}$ is derivable, then an object Q of $\mathfrak{A}$ is called T-*acyclic* if $TQ \to R^0T(Q)$ is an isomorphism and $R^nT(Q) = 0$ for $n > 0$. Clearly, any object from the given T-unfolding class is T-acyclic. Conversely, we have the following assertion. It is part of the assertion that the derived functor R^nT is independent of the unfolding class that is part of its definition.

Observation. *If $T : \mathfrak{A} \to \mathfrak{B}$ is derivable, then the class of T-acyclic objects is T-unfolding, and it may be used in the computation of the derived functors R^nT.*

Proof. Let $\mathfrak{Q}$ be a T-unfolding class.

We prove first for any right complex U consisting of T-acyclic objects U^i that the morphism of (1.1.2) is an isomorphism:

$$H^n(TU) \to R^nT(U). \tag{1.5.1}$$

Assume for simplicity that U is positive: $U^i = 0$ for $i < 0$. Recall that a quasi-isomorphism from U to a positive complex of objects from $\mathfrak{Q}$ may be obtained as follows: There exists an exact sequence of positive complexes with $U^i \in \mathfrak{Q}^{\geqslant 0}$:

$$\cdots \to 0 \to U \to U^0 \to U^1 \to U^2 \to \cdots, \tag{1.5.1}$$

and then $U \to \operatorname{Tot} U^\bullet$ is the required quasi-isomorphism; in particular, then $R^nT(U) = H^n(T \operatorname{Tot} U^\bullet)$. The i'th row in (1.5.1) is a resolution of U^i. Apply the functor T to (1.5.1). Since each U^j is T-acyclic, the resulting i'th row is still exact. Consequently, $TU \to T \operatorname{Tot} U^\bullet$ is a quasi-isomorphism. Whence, (1.5.1) is an isomorphism.

Assume now that U is an exact right complex of T-acyclic objects. Then the zero-morphism $U \to 0$ is a quasi-isomorphism. So, by (1.4), we have $R^nT(U) = 0$. Therefore, by the isomorphism (1.5.1), TU is exact. So condition (ii) holds for the class of T-acyclic objects. Moreover, condition (i) holds, because it holds for the class $\mathfrak{Q}$.

So the class of T-acyclic objects is T-unfolding. If $s : X \to U$ is quasi-isomorphism of right complexes and U is a complex of T-acyclic objects, then $R^nT(X) \xrightarrow{\sim} R^nT(U)$ since s is a quasi-isomorphism, and $R^nT(U) = H^n(TU)$ by (1.5.1). So $R^nT(X)$ could have been defined as the cohomology of TU. ▯

(1.6) Definition. Let $T : \mathfrak{A} \to \mathfrak{B}$ be an additive functor. Let $T^\bullet$ be a positive complex of additive functors from $\mathfrak{A}$ to $\mathfrak{B}$,

$$\Pi^\bullet : \quad \cdots \to 0 \to \Pi^0 \to \Pi^1 \to \Pi^2 \to \cdots,$$

with a given *coaugmentation* $\epsilon : T \to \Pi^\bullet$. So each Π^i is an additive functor $\mathfrak{A} \to \mathfrak{B}$, and the composition $\Pi^n \to \Pi^{n+1} \to \Pi^{n+2}$ is zero. In addition, the coaugmentation is a morphism of functors $\epsilon : T \to \Pi^0$ such that the composition $T \to \Pi^0 \to \Pi^1$ is zero. Note that $\Pi^\bullet$ extends to complexes: If $X \in \mathfrak{A}^\bullet$ is a complex, then each $\Pi^i(X)$ is a complex in $\mathfrak{B}^\bullet$, and

$$\Pi^\bullet(X) : \quad \cdots \to 0 \to \Pi^0(X) \to \Pi^1(X) \to \Pi^2(X) \to \cdots$$

is a complex of complexes. If $X \in \mathfrak{A}^+$, say $X \in \mathfrak{A}^{\geqslant N}$, then each $\Pi^i(X)$ belongs to $\mathfrak{A}^{\geqslant N}$; hence $\Pi^\bullet(X) \in (\mathfrak{A}^\bullet)^\bullet_\divideontimes$ and we may form the associated total complex. Viewing TX as a complex of complexes concentrated in degree 0, the coaugmentation is a morphism of complexes (of complexes) $\epsilon_X : TX \to \Pi^\bullet(X)$, and it induces a morphism of total complexes:

$$\epsilon_X : TX \to \operatorname{Tot} \Pi^\bullet(X). \tag{1.6.1}$$

The complex of functors Π with the coaugmentation $\epsilon\colon T \to \Pi^\bullet$ is called an *exact resolvent complex* for T, if the following two conditions hold:

(i) Every object A of $\mathfrak{A}$ embeds into an object Q such that the following sequence is exact:

$$0 \to T(Q) \to \Pi^0(Q) \to \Pi^1(Q) \to \cdots . \tag{1.6.2}$$

(ii) Each functor Π^i is exact.

(1.7) Theorem. *Assume that the additive functor $T\colon \mathfrak{A} \to \mathfrak{B}$ has an exact resolvent complex $\epsilon\colon T \to \Pi^\bullet$. Then T is derivable, and, for any complex X in $\mathfrak{A}^+$ and any integer n, there is a canonical isomorphism in $\mathfrak{B}$,*

$$R^nT(X) \simeq H^n(\operatorname{Tot}\Pi^\bullet(X)). \tag{1.7.1}$$

Moreover, the class $\mathfrak{Q}$ of objects Q such that the sequence (1.6.2) *is exact is equal to the class of T-acyclic objects.*

Proof. First, for any complex $X \in \mathfrak{A}^+$ consider the complex of complexes,

$$\overline{\Pi}^\bullet(X): \qquad \cdots \to 0 \to TX \to \Pi^0(X) \to \Pi^1(X) \to \cdots ,$$

with TX in degree -1. The coaugmentation $\epsilon\colon TX \to \Pi^\bullet(X)$ is a morphism of complexes (of complexes), and $\overline{\Pi}^\bullet(X)$ is its mapping cone. Hence $\operatorname{Tot}\overline{\Pi}^\bullet(X)$ is the mapping cone of the morphism ε_X in (1.6.1).

Now we make the following two observations:

(a) If $Z \in \mathfrak{A}^+$ is an exact complex then, by condition (ii), the complex $\Pi^\bullet(Z)$ has exact columns. Hence, by the Column Theorem, the complex $\operatorname{Tot}\Pi(Z)$ is exact. As a consequence, since $\operatorname{Tot}\Pi^\bullet$ preserves cones, if $X \to Y$ is a quasi-isomorphism of complexes in $\mathfrak{A}^+$, then $\operatorname{Tot}\Pi^\bullet(X) \to \operatorname{Tot}\Pi^\bullet(Y)$ is a quasi-isomorphism.

(b) Let U be a right complex of objects from $\mathfrak{Q}$. Then the complex $\overline{\Pi}^\bullet(U)$ has exact rows. Hence, by the Row Theorem, $\operatorname{Tot}\overline{\Pi}^\bullet(U)$ is exact. The latter complex is the cone of the morphism ε_U. Therefore, the morphism ε_U of (1.6.1) is a quasi-isomorphism.

Now, consider an exact right complex U of objects from $\mathfrak{Q}$. It follows from (b) that $\varepsilon_U\colon TU \to \operatorname{Tot}\Pi^\bullet(U)$ is a quasi-isomorphism, and it follows from (a) that $\operatorname{Tot}\Pi(U)$ is exact. Therefore, TU is exact. Hence condition (1.2)(ii) is satisfied. Since condition (1.2)(ii) is part of the hypothesis, it follows that the class $\mathfrak{Q}$ is T-unfolded.

Let X be a right complex and choose a quasi-isomorphism $s\colon X \to U$ into a right complex U of objects from $\mathfrak{Q}$. Then we have the following commutative diagram of degree-n cohomology,

$$\begin{tikzcd} H^n(TX) \arrow[r, "\epsilon_X"] \arrow[d, "s"', "\wr"] & H^n(\operatorname{Tot}\Pi^\bullet(X)) \arrow[d, "s"'] \\ H^n(TU) \arrow[r, "\epsilon_U", "\sim"'] & H^n(\operatorname{Tot}\Pi^\bullet(U)). \end{tikzcd}$$

The left vertical morphism s is an isomorphism since $TX \to TU$ is a quasi-isomorphism by (a). The lower horizontal morphism ε_U is an isomorphism, since $TU \to \operatorname{Tot}\Pi^\bullet(U)$ is a quasi-isomorphism by (b). Composition of the isomorphisms yields the isomorphism (1.7.1):

$$R^nT(X) = H^n(TU) \xrightarrow{\sim} H^n(\operatorname{Tot}\Pi^\bullet(U)) \xleftarrow{\sim} H^n(\operatorname{Tot}\Pi^\bullet(X)).$$

When the complex X is an object $Q \in \mathfrak{A}$, concentrated in degree 0, the total complex $\operatorname{Tot}\Pi^\bullet(X)$ is simply the complex $\Pi^\bullet(A)$. So, by the isomorphism above, we have that $R^nT(Q) = H^n(\Pi^\bullet(Q))$. Hence Q is T-acyclic if and only if the sequence (1.6.2) is exact, that is, if and only if $Q \in \mathfrak{Q}$. ∎

(1.8) Confusing example. Let $\mathfrak{A} := \mathfrak{B}^\bullet$ be the abelian category of complexes in $\mathfrak{B}$. Consider the following two functors:

$$I, K\colon \mathfrak{A} \to \mathfrak{B},$$

given by $I(A) = \operatorname{Im}(A^{-1} \to A^0)$ and $K(A) := \operatorname{Ker}(A^0 \to A^1)$. Let $\Pi^\bullet(A)$ be the truncated complex $\Pi^\bullet(A) = A^{\geqslant 0}$, and view $\Pi^\bullet$ as the complex of functors $\Pi^n\colon \mathfrak{A} \to \mathfrak{B}$, given by $\Pi^n(A) = A^n$ for $n = 0, 1, \dots$. So each functor Π^n is exact. With the obvious coaugmentations given by the inclusions $I(A) \hookrightarrow K(A) \hookrightarrow A^0$, the complex $\Pi^\bullet$ is a resolvent complex for both functors I and K. Indeed, every object A of $\mathfrak{A}$ embeds into an acyclic object Q (for instance A embeds into the mapping cone of the identity of A, and the mapping cone is even contractible), and it suffices to note that the sequence (1.6.2), for $T = K$ is exact if Q is acyclic in positive degrees, and for $T = I$ is exact if Q is acyclic in nonnegative degrees.

In particular, it follows that the n'the cohomology of an object A in $\mathfrak{B}^\bullet$, for $n \geqslant 1$, is the n'th derived functor,

$$R^nI(A) = R^nK(A) = H^n(A).$$

For $n = 0$, we have that $R^0I(A) = R^0K(A) = K(A)$.

Note that the derived functors R^nK are defined on right complexes in $X \in \mathfrak{A}^+$, that is, on the category $((\mathfrak{B})^\bullet)^+$ of right complexes of complexes in $\mathfrak{B}$. It follows from the description i (1.7) that if $X \in (\mathfrak{B}^{\geqslant 0})^+$, then the value $R^nK(X)$ is equal $H^n(\operatorname{Tot} X)$ for all n.

(1.9) Remark. An exact resolvent complex for the identity functor 1 of $\mathfrak{A}$,

$$\cdots \to 0 \to 1 \to \Pi^0 \to \Pi^1 \to \cdots \tag{1.9.1}$$

is also called an (exact) *resolution* of the identity, since, for instance by the theorem, the sequence (1.9.1) is necessarily exact. Consider the following condition: Every object A of $\mathfrak{A}$ embeds into an object Q such that the sequence,

$$\cdots \to 0 \to Q \to \Pi^0Q \to \Pi^1Q \to \dots,$$

is contractible (in the terminology of relative abelian categories the sequence (1.9.1) is a *relative* resolvent complex for the identity). Clearly, under this condition, if $T\colon \mathfrak{A} \to \mathfrak{B}$ is any additive functor such that all the functors $T\Pi^n$ are exact, then $T \to T\Pi^\bullet$ is an exact resolvent complex for T (in fact, a relatively exact resolvent complex).

(1.10) Example. Let B be an object of $\mathfrak{A}$ with a projective resolution $P \to B$,

$$\cdots \to P_2 \to P_1 \to P_0 \to B \to 0 \to \cdots .$$

Consider the functor $T = \mathrm{Hom}(B, -)$. Then there is a T-augmented complex of functors $\mathfrak{A} \to \mathbf{Ab}$,

$$\cdots \to 0 \to \mathrm{Hom}(B, -) \to \mathrm{Hom}(P_0, -) \to \mathrm{Hom}(P_1, -) \to \cdots ,$$

and each functor $\mathrm{Hom}(P_i, -)$ is exact. The complex is a resolvent complex for $\mathrm{Hom}(B, -)$, if and only if $\mathrm{Hom}(B, -)$ is derivable (for instance, if $\mathfrak{A}$ has enough injectives).

Note that in any case, the cohomology $H^n \operatorname{Tot} \mathrm{Hom}(P, X)$, for any complex $X \in \mathfrak{A}^+$ (in fact, for any complex $X \in \mathfrak{A}^\bullet$ if Tot of a bicomplex of abelian groups is determined by products), is the ext-group,

$$H^n \operatorname{Tot} \mathrm{Hom}(P, X) = \mathrm{Ext}^n(B, X).$$

Indeed, we have the equalities,

$$\begin{aligned} \mathrm{Ext}^n(B, X) = \mathrm{Hom}_D(B, X(n)) &\xrightarrow{\sim} \mathrm{Hom}_D(P, X(n)) \\ = \mathrm{Hom}_{\mathrm{Hot}}(P, X(n)) = H^n \mathrm{Hom}^\bullet_{\mathfrak{A}}(P, X) &= H^n \operatorname{Tot} \mathrm{Hom}_{\mathfrak{A}}(P, X). \end{aligned}$$

The first is the definition of the ext-group as the hom-group in the derived category $D = D(\mathfrak{A})$, the second holds because $P \to B$ is a quasi-isomorphism, the third holds because P is a left complex of projectives, the fourth holds by definition of the homotopy category $\mathrm{Hot} = \mathrm{Hot}(\mathfrak{A})$, and the last holds by the definition of $\mathrm{Hom}^\bullet$ as Tot of a bicomplex.

2. Derived categories of complexes.

(2.1) Setup. Recall that the homotopy categories $\mathrm{Hot}(\mathfrak{A})$ and $\mathrm{Hot}^+(\mathfrak{A})$ of complexes and of right complexes are triangulated categories. A morphism is a quasi-isomorphism if it induces an isomorphism in cohomology, or, equivalently, if its cone is acyclic.

The class of quasi-isomorphisms is a saturated denomonator system in the homotopy category. The *full derived category* of $\mathfrak{A}$ is the category obtained by localizing the homotopy category $\mathrm{Hot}(\mathfrak{A})$ at the class of quasi-isomorphisms; it is denoted $\mathrm{D}(\mathfrak{A})$. The objects of $\mathrm{D}(\mathfrak{A})$ are the complexes X of objects of $\mathfrak{A}$. The morphisms in $\mathrm{D}(\mathfrak{A})$ from X to Y are represented by pairs (s, f), where $s\colon Y \to Y_t X \to$ is a quasi-isomorphism, and $f\colon X \to Y_t$ is a morphism of complexes. The well-known equivalence relation of pairs takes into account that the localization is obtained from the homotopy category where the morphisms are homotopy classes of morphisms of complexes.

A similar derived category $\mathrm{D}^+(\mathfrak{A})$ is obtained by localizing $\mathrm{Hot}^+(\mathfrak{A})$ at the class of quasi-isomorphisms. The subcategory $\mathrm{Hot}^+(\mathfrak{A})$ of $\mathrm{Hot}(\mathfrak{A})$ is localizing with respect to quasi-isomorphisms: If X is a right complex and $s\colon X \to X'$ is a quasi-isomorphism into an arbitrary complex X', then there exists a right complex X'' and a quasi-isomorphism $t\colon X' \to X''$. As a consequence, the derived from $\mathrm{Hot}^+(\mathfrak{A})$ is a full subcategory,

$$\mathrm{D}^+(\mathfrak{A}) \subseteq \mathrm{D}(\mathfrak{A}).$$

3. The functors Ext and RHom.

Simplicial cohomology

1. The basic simplicial categories.

The categories (**s**), (**ss**), and (**sss**). Simplicial objects. Simplicial sets. Augmentation. Simplicial categories. The associated complex, and its cohomology. Simple retraction.

2. Basic examples.

The functor $[\ ]_{\mathbf{s}}\colon (\bar{\mathbf{s}}) \to (\mathrm{GrCat})$, $[\ ]_{\mathbf{ss}}\colon (\overline{\mathbf{ss}}) \to (\mathbf{Cat})$, $[\ ]\mathbf{sss}\colon (\overline{\mathbf{sss}}) \to (\mathrm{PreCat})$.

Powers: The co-$\bar{\mathbf{s}}$-object $n \mapsto A^{[n]}$, the $\mathbf{s}$-object $n \mapsto A^{\oplus[n]}$; the notation $A^{\oplus I}$ indicates the direct sum (the co-product) of an I-indexed family of objects all equal to A).

The functors $n \mapsto \mathbb{R}^{[n]}$ and $n \mapsto \mathbb{R}^{\oplus[n]}$. Then notation $\mathbb{R}^{\oplus I}$ is rather ambigous, since it depends on the category in which $\mathbb{R}$ is considered as an object. In the context here we will think of the abelian category of $\mathbb{R}$-modules. In particular, then for finite sets we may identify $\mathbb{R}^I$ and $\mathbb{R}^{\oplus I}$, but the dependencies on I are different: the formation of $\mathbb{R}^I$ is contravariant in I, that of $\mathbb{R}^{\oplus I}$ is covariant in I.

For instance, for a morphism $\varphi\colon n \to p$ in $\mathbf{s}$, that is, a map of sets $\varphi\colon [n] \to [p]$, there are induced morphism morphims $\varphi\colon \mathbb{R}^{\oplus[n]} \to \mathbb{R}^{\oplus[p]}$ and $\varphi^*\colon \mathbb{R}^{[p]} \to \mathbb{R}^{[n]}$; they are given, respectively, by

$$\varphi(t_0, \dots, t_n) = (u_0, \dots, u_p), \qquad u_i = \sum_{\varphi(j)=i} t_{\varphi j};$$
$$\varphi^*(u_0, \dots, u_p) = (t_0, \dots, t_n), \qquad t_j = u_{\varphi j}.$$

$$n \mapsto \Delta^n_{\mathrm{sing}} = \{\, t \in \mathbb{R}^{[n]} \mid t_j \geqslant 0, \textstyle\sum t_j = 1 \,\}$$

subfunctor of $n \mapsto \mathbb{R}^{\oplus[n]}$.

The functor

$$[\ ]\colon (\overline{\mathbf{ss}}) \to (\mathbf{Cat}).$$

For any small category I, let $\mathfrak{C}^I = \mathrm{Funct}(I, \mathfrak{C})$ denote the category of functors $I \to \mathfrak{C}$. Its objects are the functors $I \to \mathfrak{C}$, and its morphisms are the transformation of functors. The formation is contravariant in I: for any functor $\varphi\colon J \to I$, there is an induced functor $\mathfrak{C}^I \to \mathfrak{C}^J$. If K and J are small categories, then $\mathfrak{K}^I = \mathrm{Funct}(I, K)$ is a set; so we obtain at category $\mathbf{Cat}$ of small categories such that

$$\mathfrak{K}^I = \mathrm{Funct}(I, K) = \mathrm{Hom}_{\mathbf{Cat}}(I, K);$$

in particular, each Hom-set in ($\mathbf{Cat}$) is a category.

For any small category I, we obtain an $\overline{\mathbf{ss}}$-object in ($\mathbf{Cat}$):

$$n \mapsto I^{[n]} = \mathrm{Hom}_{\mathbf{Cat}}([n], I).$$

The category $I^{[n]}$ has as objects the set of all *n-strings* of I:

$$i_0 \to \cdots \to i_n$$

For any small category I, let I^* be the category,

$$I^* = \mathrm{Hom}_{\mathbf{Cat}}(I, [1]_{\mathbf{ss}}).$$

Note that an object in I^* is a division of the objects of I into two disjoint classes, $I = I_0 \cup I_1$, with the property that there are no arrows from an object in I_1 to an object in I_0.

Let $(\mathbf{Cat}_{00})$ be the category of small categories with chosen extremal objects (an initial object and a terminal object); the morphisms in $(\mathbf{Cat}_{00})$ are the functors preserving the extremal objects. Clearly, for two objects I, J in $(\mathbf{Cat}_{00})$, we may view the Hom-set,

$$\mathrm{Hom}_{\mathbf{Cat}_{00}}(I, J) \subseteq \mathrm{Hom}_{\mathbf{Cat}}(I, J),$$

as a full subcategory.

Note that I^* is in $(\mathbf{Cat}_{00})$: its two extremal objects are the two constant functors. So we may view $I \mapsto I^*$ as a contravariant functor $(\mathbf{Cat}) \to (\mathbf{Cat}_{00})$. By composition, we obtain a contravariant functor,

$$(\mathbf{ss}) \xrightarrow{[\,]} (\mathbf{Cat}) \xrightarrow{(\,)^*} (\mathbf{Cat}_{00}),$$

and hence for every J in $(\mathbf{Cat}_{00})$ a co-**ss**-category,

$$n \mapsto \mathrm{Hom}_{\mathbf{Cat}}([n]^*, J).$$

Note that the elements of $[n]^*$ are the increasing maps $[n] \to [1]$. For $0 \leqslant i \leqslant n+1$, let $i^* \colon [n] \to [1]$ be the increasing map such that $i^*(x) = 0$ for $x < i$ and $i^*(x) = 1$ otherwise (in particular, 0^* is the constant map 1, and $(n+1)^*$ is the constant map 0). Then $[n]^*$ is the ordered set,

$$[n]^* = \{\, (n+1)^* < n^* < \cdots < 0^* \,\}$$

In other words, the objects of the category $\mathrm{Hom}_{\mathbf{Cat}_{00}}([n]^*, J)$ is the category of *terminated n-strings* of j,

$$a \to j_n \to \cdots j_1 \to b,$$

where a and b are the terminal objects of J.

When $J = [0, 1]$ is the unit interval as an ordered set we obtain the description,

$$\mathrm{Hom}_{\mathbf{Cat}_{00}}([n]^*, [0, 1]) = \{(t_n, \ldots, t_1) \mid 0 \leqslant t_n \leqslant \cdots \leqslant t_1 \leqslant 1\},$$

4. 2.20 (1) Prove that the two contravariant functors, $(\mathbf{Cat}) \to (\mathbf{Cat}_{00})$ and $(\mathbf{Cat}_{00}) \to (\mathbf{Cat})$, are adjoint:

$$\mathrm{Hom}_{\mathbf{Cat}}(I, J^*) = \mathrm{Hom}_{\mathbf{Cat}_{00}}(J, I^*).$$

(2) Prove that $[n]^{**} = [n]$ for $n = -1, 0, \ldots$.

3. Simplicial sets.

4. Homology of simplicial sets. Acyclic simplicial sets.

5. Standard resolutions and canonical resolutions.

(5.1) The star product. Consider functors $F, G\colon \mathfrak{K} \to \mathfrak{K}'$ and $F', G'\colon \mathfrak{K}' \to \mathfrak{K}''$. Let $\varphi\colon F \to G$ and $\varphi'\colon F' \to G'$ be transformations of functors. Then there is a *star product* of transformations,

$$\varphi' * \varphi\colon F'F \to G'G,$$

of functors $\mathfrak{K} \to \mathfrak{K}''$ defined as follows: For any object X in $\mathfrak{K}$, we have the morphism $\varphi(X)\colon FX \to GX$ in $\mathfrak{K}'$ and hence, since φ' is a transformation of functors, a commutative diagram in $\mathfrak{K}''$,

$$\begin{tikzcd} F'FX \arrow[r, "F'(\varphi(X))"] \arrow[d, "\varphi'(FX)"'] & F'GX \arrow[d, "\varphi'(GX)"] \\ G'FX \arrow[r, "G'(\varphi(X))"'] & G'GX. \end{tikzcd}$$

Define $(\varphi' * \varphi)(X)$ as the composition in the diagram, $F'FX \to G'GX$. The arrows in the diagram are given by transformations, denoted respectively $F'\varphi$, $\varphi'G$, $G'\varphi$, and $\varphi'F$. In particular, the star product is a transformation of functors $\mathfrak{K} \to \mathfrak{K}''$.

The formation of the star product is functorial in the following sense: Assume there are further functors $H\colon \mathfrak{K} \to \mathfrak{K}'$ and $H'\colon \mathfrak{K}' \to \mathfrak{K}''$, and transformations $\psi\colon G \to H$ and $\psi'\colon G' \to H'$. Then we have the equality,

$$(\psi' * \psi)(\varphi' * \varphi) = (\psi'\varphi') * (\psi\varphi), \tag{5.1.1}$$

of transformations $F'F \to H'H$.

It is easy to extend the definition to a star product of more that two transformations: For composable functors and transformations $\varphi_i\colon F_i \to G_i$ for $i = 0, \dots, n$ the star product is a transformation,

$$\varphi_0 * \cdots * \varphi_n\colon F_0 \cdots F_n \to G_0 \cdots G_n,$$

with functorial properties extending (5.1.1).

(5.2) Setup. Fix in the following a category $\mathfrak{K}$, a functor $F\colon \mathfrak{K} \to \mathfrak{K}$, and a transformation,

$$\delta\colon 1 \to F,$$

from the identity functor 1 to the functor F. Consider for $n \geqslant -1$ the composition,

$$F^{[n]} = \overbrace{F \cdots F}^{n+1};$$

for $n = -1$, the composition is the identity functor. Let $f\colon p \to n$ be a morphism in $(\overline{\mathbf{sss}})$, that is, a strictly increasing map $f\colon [p] \to [n]$. Associate with f a transformation (denoted by the same symbol) $f\colon F^{[p]} \to F^{[n]}$ as follows: For $j = 0, \dots, n$, if j is in the image of f, let $F_j := F$ and let $\varphi_j = 1\colon F_j \to F$ be the identity transformation; otherwise, let $F_j := 1$

be the identity functor and let $\varphi_j = \delta\colon F_j \to F$ be the given transformation (beware that the symbol 1 is used both for the identity functor of $\mathfrak{K}$ and for the identity transformation of a given functor). Define the transformation f as the star product,

$$f = \varphi_0 * \cdots * \varphi_n\colon F^{[p]} = \overbrace{F_0 \cdots F_n}^{n+1} \to \overbrace{F \cdots F}^{n+1} = F^{[n]}.$$

Clearly, with this definition, $n \to F^{[n]}$ is a functor from the category $(\overline{\mathbf{sss}})$ to the category of endomorphisms of $\mathfrak{K}$. In other words, it is a co-$\overline{\mathbf{sss}}$-object of endomorphisms of $\mathfrak{K}$,

$$F^{[\,]}\colon \qquad 1 \to F \rightrightarrows F^{[1]} \mathrel{\substack{\to\\\to\\\to}} F^{[2]} \mathrel{\substack{\to\\\to\\\to\\\to}} \cdots. \tag{5.2.1}$$

It is called the (co-augmented) *standard object* associated to the transformation $\delta\colon 1 \to F$. Evaluation at any object X of $\mathfrak{K}$ yields a co-$\overline{\mathbf{sss}}$-object $F^{[\,]}X$ of $\mathfrak{K}$,

Note that face morphisms corresponding to the maps $\delta_i^n\colon n \to n+1$ have a simple inductive description: under the identification $F^{[n+1]} = F^{[n]}F$,

$$\delta_{n+1}^n = 1 * \delta\colon F^{[n]} \to F^{[n]}F, \qquad \delta_i^n = \delta_i^{n-1} * 1\colon F^{[n-1]}F \to F^{[n]}F,$$

where the factor 1 in the first star product is the identity transformation of $F^{[n]}$ and in the second product (for $0 \leqslant i \leqslant n$) is the identity transformation of F.

(5.3) Examples. (1) For $\mathfrak{K} = (\mathbf{Cat})$, consider the functor $I \mapsto I^+$ that adds a final object to the category I. The obvious inclusion,

$$I \to I^+,$$

is transformation of functors $\delta\colon 1 \to (\ )^+$. The corresponding standard object, evaluated on any category I is a functor $(\overline{\mathbf{sss}}) \to (\mathbf{Cat})$. In particular, evaluation at the empty category $\emptyset$ yields the standard functor, $n \mapsto [n]$, from $(\overline{\mathbf{sss}})$ to $(\mathbf{Cat})$.

(2) For $\mathfrak{K} = (\mathrm{Top})$, consider the functor $X \mapsto C(X)$, where $C(X)$ is the cone over X (the mapping cone of the identity of X). The obvious inclusion,

$$X \hookrightarrow C(X),$$

is a transformation of functors $1 \to C$. The corresponding standard object, evaluated on $X = \emptyset$, is the topological realization, $n \mapsto \Delta^n_{\mathrm{top}}$, restricted to $(\overline{\mathbf{sss}})$.

(3) Consider a pair of adjoint functors, $\mathfrak{K} \underset{\rho}{\overset{\lambda}{\rightleftarrows}} \mathfrak{L}$, with ρ right adjoint to λ. Let $F := \rho\lambda$. The adjunction isomorphism,

$$\mathrm{Hom}_{\mathfrak{L}}(\lambda X, Y) = \mathrm{Hom}_{\mathfrak{K}}(X, \rho Y),$$

is, like any functorial map from the left side set to the right side set, given by a functorial morphism $X \to \rho\lambda X$, that is, by a transformation $1 \to \rho\lambda = F$.

(5.4) The Standard Contraction Lemma. (a) *Let* $S\colon \mathfrak{K}' \to \mathfrak{K}$ *be a functor such that the transformation* $\delta * 1\colon S \to FS$ *has a retraction. Then the co-*$\overline{\text{sss}}$*-object of functors* $\mathfrak{K}' \to \mathfrak{K}$*,*

$$F^{[\,]}S: \qquad S \to FS \rightrightarrows F^{[1]}S \;\substack{\to\\\to\\\to}\; F^{[2]} \;\substack{\to\\\to\\\to\\\to}\; \cdots,$$

has a simple contraction.

(b) *Let* $T\colon \mathfrak{K} \to \mathfrak{K}'$ *be a functor such the transformation* $1 * \delta\colon T \to TF$ *has a retraction. Then the co-*$\overline{\text{sss}}$*-object of functors* $\mathfrak{K} \to \mathfrak{K}'$*,*

$$TF^{[\,]}: \qquad T \to TF \rightrightarrows TF^{[1]} \;\substack{\to\\\to\\\to}\; TF^{[2]} \;\substack{\to\\\to\\\to\\\to}\; \cdots,$$

has a simple contraction.

Proof. For (a), let $r = r^0\colon FS \to S$ be a retraction for the transformation $\delta * 1\colon S \to FS$. For $n \geqslant 1$, define $r^n\colon F^{[n]}S \to F^{[n-1]}S$ as the star product,

$$r^n = 1^n * r\colon F^{[n]}S = F^n(FS) \to F^nS = F^{[n-1]}S,$$

where 1^n is the identity transformation of F^n. From the description of the face morphisms δ_i^n, it follows easily that $r^{n+1}\delta_{n+1}^n = 1$ and that $r^{n+1}\delta_i^n = \delta_i^{n-1}r^n$ for $0 \leqslant i \leqslant n$. Hence the r^n form a simple retraction of the co-$\overline{\text{sss}}$-object.

The proof of (b) is similar. □

(5.5) Example. (1) Fix a category J. Consider the transformation $I \to I^+$ of (5.3)(1), and the corresponding co-$\overline{\text{sss}}$-object of endomorphisms of (**Cat**). Composition with the (contravariant) functor $T = \mathrm{Hom}_{\mathbf{Cat}}(\ ,J)$ gives an $\overline{\text{sss}}$-object of contravariant functors from (**Cat**) to (**Sets**). Evaluation at I is the following $\overline{\text{sss}}$-set:

$$\mathrm{Hom}(I,J) \leftarrow \mathrm{Hom}(I^+,J) \leftleftarrows \mathrm{Hom}(I^{++},J) \;\substack{\leftarrow\\\leftarrow\\\leftarrow}\; \cdots. \tag{5.5.1}$$

Assume that the category J has a final object e. Then there is an obvious functor $r\colon J^+ \to J$ equal to the identity on $J \subset J^+$. So, for any functor $f\colon I \to J$ we obtain a functor $rf^+\colon I^+ \to J^+ \to J$ extending f. In other words, $f \to rf^+$ is a section of $\mathrm{Hom}(I^+,J) \to \mathrm{Hom}(I,J)$, and in fact a section of the transformation $T(\ )^+ \to T$. Therefore, by the Standard Contraction Lemma, if J has a final object, then, for any category I, the $\overline{\text{sss}}$-set (5.5.1) has a simple contraction. In particular, for $I = \emptyset$, the following $\overline{\text{sss}}$-set has a simple contraction:

$$* \leftarrow \mathrm{Hom}([0],J) \leftleftarrows \mathrm{Hom}([1],J) \;\substack{\leftarrow\\\leftarrow\\\leftarrow}\; \mathrm{Hom}([2],J) \;\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}\; \cdots. \tag{5.5.2}$$

The Hom-sets are for the category (**Cat**); each $[n]$ is considered as a partially ordered set, and hence as a category. Thus $\mathrm{Hom}([n],J)$ is the set of n-strings $j_0 \to j_1 \to \cdots \to j_n$ in J.

A modification of the functor $(\ )^+$ for the category of sets yields a similar result: If M is a set, let M^+ be the set obtained by adding an extra element to M. By the same arguments we obtain the following result: If M is a nonempty set, then the following $\overline{\mathbf{sss}}$-set has a simple contraction:

$$* \leftarrow M^{[0]} \leftleftarrows M^{[1]} \leftleftarrows M^{[2]} \leftleftarrows \cdots . \tag{5.5.3}$$

(2) Fix a topological space Y. Consider the transformation $X \to CX$ of (5.3)(2), and the corresponding co-$\overline{\mathbf{sss}}$-object of endomorphisms of (Top). Composition with the (contravariant) functor $T = \mathrm{Hom}_{\mathrm{Top}}(\ ,Y)$ gives an $\overline{\mathbf{sss}}$-object of contravariant functors from (Top) to (**Sets**). Evaluation at X is the following $\overline{\mathbf{sss}}$-set:

$$\mathrm{Hom}(X,Y) \leftarrow \mathrm{Hom}(CX,Y) \leftleftarrows \mathrm{Hom}(C^2X,Y) \leftleftarrows \cdots . \tag{5.5.4}$$

The Hom-sets are the sets of continuous maps. In particular, with $X = \emptyset$, we obtain the topological n-simplex $\Delta^n = C^n\emptyset$, and the *singular* $\mathbf{sss}$-*set* of Y,

$$* \leftarrow \mathrm{Hom}(\Delta^0,Y) \leftleftarrows \mathrm{Hom}(\Delta^1,Y) \leftleftarrows \mathrm{Hom}(\Delta^2,Y) \leftleftarrows \cdots . \tag{5.5.5}$$

Assume that the topological space Y is *contractible*, that is, the inclusion $Y \to CY$ has a retraction r. As above, we obtain for any map $X \to Y$ an extension $CX \to Y$, wich defines a section of the map $\mathrm{Hom}(CX,Y) \to \mathrm{Hom}(X,Y)$. Therefore, by the Standard Contraction Lemma, if Y is contractible, then, for any topological space X, the $\overline{\mathbf{sss}}$-set (5.5.4) has a simple contraction; in particular, the singular $\overline{\mathbf{sss}}$-set of Y has a simple contraction.

(5.6) The standard complex. If, in the setup of (5.2), the category $\mathfrak{K}$ is additive then there is a co-augmented cochain complex of functors associated to the co-$\overline{\mathbf{sss}}$-object $F^{[\,]}$:

$$\overline{C}_{\mathrm{stand}}: \qquad 0 \to 1 \to F^{[0]} \to F^{[1]} \to F^{[2]} \to \cdots . \tag{5.6.1}$$

It is called the *standard complex* associated to transformation $\delta\colon 1 \to F$. The differential $d^n\colon F^{[n]} \to F^{[n+1]}$ is the alternating sum

$$d^n = \sum\nolimits_{i=0}^{n+1} (-1)^i \delta_i^n .$$

For $n = -1$, the definition reduces to the given transformation $\delta\colon 1 \to F$. More generally, if $\mathfrak{K}$ is an arbitrary category and $T\colon \mathfrak{K} \to \mathfrak{B}$ is a functor into an additive category $\mathfrak{B}$, then there is a standard complex $T\overline{C}_{\mathrm{stand}}$ associated to the co-$\overline{\mathbf{sss}}$-object $TF^{[\,]}$.

(5.7) Proposition. *Let $\mathfrak{A}$ be an abelian category, let $F\colon \mathfrak{A} \to \mathfrak{A}$ be an additive endomorphism, and let $\delta\colon 1 \to F$ be a transformation. Assume the following conditions:* (i) *F is exact,* (ii) *the transformation $\delta\colon 1 \to F$ is monic, and* (iii) *the transformation $\delta * 1\colon F \to FF$ has a retraction. Let $T\colon \mathfrak{A} \to \mathfrak{B}$ be an additive functor into an abelian category such that TF is exact. Then the standard complex,*

$$T\overline{C}_{\mathrm{stand}}: \qquad 0 \to T \to TF^{[0]} \to TF^{[1]} \to TF^{[2]} \to \cdots , \tag{5.7.1}$$

is a resolvent complex for T. In particular, the standard complex is a resolution of the identity of $\mathfrak{A}$:

$$0 \to 1 \to F^{[0]} \to F^{[1]} \to F^{[2]} \to \cdots . \tag{5.7.2}$$

Proof. Since F and TF are exact, it follows that the functors $TF^{[n]}$, for $n \geqslant 0$, in (5.7.1) are exact.

The condition (iii) implies, by (5.4) with $S := F$, that the co-$\overline{\text{sss}}$-object $F^{[\,]}F$ has a contraction. Therefore, the complex (5.7.2) is contractible when evaluated on an object of the form FX. Hence, so is the complex (5.7.1). In particular, the complex (5.7.1) is acyclic when evaluated on an object of the form $Q = FX$. It follows from condition (ii), that every object X has an embedding into an object of this form. Therefore (5.7.1) is a resolvent complex for T. The assertion for the identity is the special case $T = 1$. □

(5.8) The canonical complex. Assume that $\mathfrak{A}$ is an abelian category, that $F\colon \mathfrak{A} \to \mathfrak{A}$ is an additive functor and that $\delta\colon 1 \to F$ is a transformation. In particular, then the standard complex $\overline{C}_{\text{stand}}$ associated to δ is defined. A second complex associated to δ is obtained as follows: For any object X of $\mathfrak{A}$, let $\kappa(X)\colon FX \to NX$ be the cokernel of the morphism $X \to FX$. Then N is an endomorphism of $\mathfrak{A}$, $\kappa\colon F \to N$ is a transformation, and we have a right exact sequence of functors $1 \to F \to N \to 0$. By composing with the powers N^i, we obtain right exact sequences,

$$\begin{gathered} 1 \to F \to N \to 0, \\ N \to FN \to N^2 \to 0, \\ N^2 \to FN^2 \to N^3 \to 0, \\ \cdots \end{gathered}$$

Hence, with the compositions $FN^n \to N^{n+1} \to FN^{n+1}$ as differentials, we obtain a co-augmented cochain complex of functors,

$$\overline{C}_{\text{can}}: \quad 0 \to 1 \to F \to FN \to FN^2 \to FN^2 \to \cdots , \tag{5.8.1}$$

called the *canonical complex* associated to the transformation δ. Its degree-n part is FN^n, for $n \geqslant 0$. From the right exact sequences we obtain for the cohomology, also for $n = -1$,

$$H^n(\overline{C}_{\text{can}}) = \operatorname{Ker}(N^{n+1} \to FN^{n+1}).$$

In particular, the canonical complex $\overline{C}_{\text{can}}$ is acyclic if and only if $1 \to F$ is monic .

It is easy to check that the transformations defined by the star product,

$$1 * \kappa^{*n}\colon F^{[n]} = FF^n \to FN^n,$$

define a morphism of complexes,

$$\begin{tikzcd} 0 \arrow[r] & 1 \arrow[r] \arrow[d, equal] & F^{[0]} \arrow[r] \arrow[d, equal] & F^{[1]} \arrow[r] \arrow[d] & F^{[2]} \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & 1 \arrow[r] & F \arrow[r] & FN \arrow[r] & FN^2 \arrow[r] & \cdots \end{tikzcd}$$

from the standard complex $\overline{C}_{\text{stand}}$ to the canonical complex $\overline{C}_{\text{can}}$.

(5.9) Proposition. *Assume in the setup of* (5.8) *that the transformation* $\delta * 1\colon F \to F^2$ *has a retraction. Then the complex,*

$$\overline{C}_{\mathrm{can}} F,$$

i.e., the canonical complex composed with the functor F*, is contractible.*

Proof. Let us say for the moment that an endomorphism G of $\mathfrak{A}$ is *special* if the transformation $\delta * 1\colon G \to FG$ has a retraction. Clearly, if G is special and H is any endomorphism, then GH is special. Note also that if G is special and $G \to H$ is a transformation with a retraction (or with a section), then H is special. By hypothesis, F is special.

Consider the right exact sequence $1 \to F \to N \to 0$, and compose it with G to obtain the right exact sequence,

$$G \to FG \to NG \to 0.$$

Assume that G is special. Then the sequence is split exact. Hence the epimorphism $FG \to NG$ has a section. Moreover, FG is special because F is special. Therefore, NG is special.

Apply the argument repeatedly. It follows, for $n = 0, 1, \dots$, that $N^n F$ is special. Therefore, the right exact sequences of (5.8) become split exact when composed with F. Consequently, the complex $\overline{C}_{\mathrm{can}} F$, built out of these sequences, is contractible. ▯

(5.10) Corollary. *In the setup of* (5.7)*, the same conclusion holds if the standard complex is replaced by the canonical complex.*

(5.11) Theorem. *Assume in the setup of* (5.8) *that,* (i) *the transformation* $\delta * 1\colon F \to FF$ *has a retraction, and* (ii) *the sequence,*

$$0 \to F1 \xrightarrow{1*\delta} FF \xrightarrow{1*\kappa} FN \to 0,$$

is split exact. Then the morphism of complexes $\overline{C}_{\mathrm{stand}} \to \overline{C}_{\mathrm{can}}$ *is a homotopy equivalence.*

Proof. Consider, for $n \geqslant 0$, the following diagram of functors:

$$\begin{tikzcd}
1 \arrow[r, "\delta"] \arrow[d, equal] & F \arrow[r, "\partial"] \arrow[d, equal] & \cdots \arrow[r, "\partial"] & FN^n \arrow[r, "d^0"] \arrow[d, equal] & F^{[1]}N^n \arrow[r, "d^1"] \arrow[d] & F^{[2]}N^n \arrow[r, "d^2"] \arrow[d] & \cdots \\
1 \arrow[r, "\delta"] & F \arrow[r, "\partial"] & \cdots \arrow[r, "\partial"] & FN^n \arrow[r, "\partial"] & FN^{n+1} \arrow[r, "d^0"] & F^{[1]}N^{n+1} \arrow[r, "d^1"] & \cdots .
\end{tikzcd}$$

The top sequence has FN^q in degree q for $0 \leqslant q \leqslant n$, and $F^{[p]}N^n$ in degree $p+n$. The morphisms labelled ∂ are the differentials of the canonical complex $\overline{C}_{\mathrm{can}}$. The morphisms labelled d are induced from the differentials of $\overline{C}_{\mathrm{stand}}$; in fact, the top sequence from the index n is the complex $C_{\mathrm{stand}} N^n$ obtained from the nonaugmented complex C_{stand} by composition with N^n. Since the last morphism ∂ in the top row it the composition $FN^{n-1} \to N^n \to FN^n$, the composition $d^0\partial$ in the top row is equal to zero. Hence the top row is a complex. Denote it $\overline{C}_n$. Then the bottom row is the complex $\overline{C}_{n+1}$. The nontrivial vertical morphisms in the diagram are the transformations, for $p \geqslant 1$,

$$1 * \kappa * 1 : F^{[p]}N^n = F^p F N^n \to F^p N N^n = F^{[p-1]}N^{n+1}. \tag{5.11.1}$$

It is easily seen that the diagram is commutative. Hence the diagram defines a morphism of complexes,

$$t_n \colon \overline{C}_n \to \overline{C}_{n+1}\,.$$

The complex $\overline{C}_0$ is the standard complex $\overline{C}_{\text{can}}$. Moreover, the morphism $t\colon \overline{C}_{\text{can}} \to \overline{C}_{\text{stand}}$ is the infinite composition,

$$t = \cdots t_2 t_1 t_0,$$

which is finite in every degree. Therefore it suffices to prove that t_n is a homotopy equivalence. In degree $q \leqslant n$, the morphism is the identity. Consider t_n in degree $n + p$, for $p \geqslant 1$. It is the morphism (5.11.1). In particular, it is induced by the morphism $FF \to FN$. Therefore, by condition (ii), it is a split epimorphism, with kernel equal to F^pN^n. In fact, it is not hard to see that the kernel of t_n, as a complex, up to a shift in degree, is the complex,

$$\overline{C}_{\text{can}} F N^n.$$

The condition (i) implies, by (5.4), that the complex $\overline{C}_{\text{can}} F$ is contractible. Hence so is the complex $\overline{C}_{\text{can}} F N^n$. Therefore each t_n, and hence also t, is a homotopy equivalence. □

(5.12) A relativized version. The previous results are even more appealing(?) in the setting of relativized abelian categories. Consider for simplicity the adjoint functors case: There is given a pair of adjoint functors of abelian categories, $\mathfrak{A} \underset{\rho}{\overset{\lambda}{\rightleftarrows}} \mathfrak{A}_0$, with ρ right adjoint to λ, and the transformation δ is the canonical transformation, $\delta\colon 1 \to \rho\lambda$, of endomorphisms of $\mathfrak{A}$. (Similarly, there is a canonical transformation $\varepsilon\colon \lambda\rho \to 1$, of endomorphisms of $\mathfrak{A}_0$.)

For an abelian category $\mathfrak{A}$, indicate with the notation $\mathfrak{A}_{\text{split}}$ that $\mathfrak{A}$ is considered as a relative abelian category: the relative monomorphisms are the split monomorphisms. Now λ is a left adjoint, and hence right exact. Therefore, a second relativization of $\mathfrak{A}$, indicated with the notation $\mathfrak{A}_{\lambda,\text{split}}$, is obtained via λ: a morphism s in $\mathfrak{A}_{\lambda,\text{split}}$ is a relative monomorphism if $\lambda(s)$ is a split monomorphism. So, almost by definition, λ is a relatively exact functor,

$$\lambda\colon \mathfrak{A}_{\lambda,\text{split}} \to (\mathfrak{A}_0)_{\text{split}}. \tag{5.12.1}$$

Any additive functor $T\colon \mathfrak{A} \to \mathfrak{B}$ of abelian categories is a relatively exact functor $\mathfrak{A}_{\text{split}} \to \mathfrak{B}_{\text{split}}$. So the functor $F = \rho\lambda$ is a relatively exact functor,

$$F\colon \mathfrak{A}_{\lambda,\text{split}} \to \mathfrak{A}_{\text{split}}; \tag{5.12.2}$$

in particular, F is a relatively exact endomorphism of $\mathfrak{A}_{\lambda,\text{split}}$.

The category of functors into $\mathfrak{A}$ (from any fixed source) is relativized similarly: A transformation $S \to S'$ is relatively monic if the transformation $\lambda S \to \lambda S'$ is split monic. According to this definition, the natural transformation $\delta\colon 1 \to \rho\lambda$, or

$$\delta\colon 1 \to F, \tag{5.12.3}$$

is a relative monomorphism of functors. Indeed, $1 * \delta\colon \lambda \to \lambda F$ is retracted by $\varepsilon * 1$.

(5.13) Proposition. *In the setup of* (5.12), *the standard complex and the canonical complex,*

$$\begin{aligned}\overline{C}_{\text{stand}}:&\quad \cdots \to 0 \to 1 \to F^{[0]} \to F^{[1]} \to F^{[2]} \to \cdots,\\ \overline{C}_{\text{can}}:&\quad \cdots \to 0 \to 1 \to F \to FN \to FN^2 \to \cdots,\end{aligned}$$

define relatively exact resolvent complexes for the identity as a functor $\mathfrak{A}_{\lambda,\text{split}} \to \mathfrak{A}_{\text{split}}$. *In particular, any morphism between the two complexes which is equal to the identity in degree* -1, *is a homotopy equivalence.*

Proof. Recall that if $T\colon \mathfrak{A} \to \mathfrak{B}$ is an additive functor between relative abelian categories, then a complex of functors $\mathfrak{A} \to \mathfrak{B}$,

$$\overline{C}:\quad \cdots \to 0 \to T \to C^0 \to C^1 \to C^2 \to \cdots,$$

is a relatively exact resolvent complex for T, if the following two conditions hold:

(i) Each C^n is relatively exact $\mathfrak{A} \to \mathfrak{B}$.

(ii) For every object A of $\mathfrak{A}$ there is a relative monomorphism into an object Q for which the complex $\overline{C}(Q)$ is relatively exact in $\mathfrak{B}$.

Consider the standard complex. First, to verify (i), note that $F\colon \mathfrak{A}_{\lambda,\text{split}} \to \mathfrak{A}_{\lambda,\text{split}}$ is relatively exact. Hence $F^n\colon \mathfrak{A}_{\lambda,\text{split}} \to \mathfrak{A}_{\lambda,\text{split}}$ is relatively exact for all $n \geqslant 0$. Therefore, viewing F as the relatively exact functor (5.12.2), it follows that $F^{[n]} = FF^n$ is relatively exact $\mathfrak{A}_{\lambda,\text{split}} \to \mathfrak{A}_{\text{split}}$. To verify (ii), note that $A \to FA$ is a relative monic, and that $\overline{C}_{\text{stand}}F$ is contractible by (5.4)(a); in fact, $\overline{C}_{\text{stand}}\lambda$ is contractible since $\delta * 1\colon \lambda \to F\lambda$ is retracted by $1 * \varepsilon$.

Consider the canonical complex, and the condition (i). The functor N is defined as the cokernel of $1 \to F$, that is, by the right exact sequence,

$$0 \to 1 \to F \to N \to 0. \tag{*}$$

Apply λ. Since $1 \to F$ is relatively monic, the resulting sequence, λ(*), is split exact. Plug in (as columns) a relatively short exact sequence in $\mathfrak{A}_{\lambda,\text{split}}$; the result is 3×3 commutative diagram in $\mathfrak{A}_0$ with split exact rows. The first column is split exact in $\mathfrak{A}_0$. As $\lambda F = F\lambda$, it follows that the second column is split exact. Hence the third column is split exact. Therefore, the functor N is relatively exact $\mathfrak{A}_{\lambda,\text{split}} \to \mathfrak{A}_{\lambda,\text{split}}$. By the argument used above for F, it follows that FN^k is relatively exact $\mathfrak{A}_{\lambda,\text{split}} \to \mathfrak{A}_{\text{split}}$. Condition (ii) follows from (5.9). ▯

(5.14). As a consequence, any additive functor $T\colon \mathfrak{A} \to \mathfrak{B}$, viewed as a functor of relative abelian categories $\mathfrak{A}_{\lambda,\text{split}} \to \mathfrak{B}_{\text{split}}$, has relatively resolvent complexes, for instance TC_{stand} and TC_{can}.

Corollary. *Assume in the setup of* (5.12) *that* $F = \rho\lambda$ *is exact and that* $\delta\colon 1 \to \rho\lambda$ *is monic. If* $T\colon \mathfrak{A} \to \mathfrak{B}$ *is any additive functor such that* TF *is exact, then* TC_{stand} *and* TC_{can} *are resolvent complexes for* T.

6. Cohomology of categories I (Draft).

(6.1) Setup. Fix a small category I and an abelian category $\mathfrak{A}$. Assume that products in $\mathfrak{A}$ indexed by I (that is, indexed by sets K of cardinality at most equal to the cardinality of the union of the Hom-sets of I) exist. These products, as functors from the category of K-indexed families of objects of $\mathfrak{A}$ to the category $\mathfrak{A}$, are left exact. It is essentially Grothendieck's axiom AB4* that they are exact.

Consider the category $\mathfrak{A}^I$ of *I-systems* in $\mathfrak{A}$ (or *$\mathfrak{A}$-valued coefficient systems on I*), that is, the category of functors $\mathfrak{F}\colon I \to \mathfrak{A}$. In this notation, $\mathfrak{A}^{|I|}$ is the category of *I-families* in $\mathfrak{A}$, that is, the category of all I-indexed families $i \mapsto \mathfrak{G}_i$ of objects of $\mathfrak{A}$. Note that a sequence of I-systems,

$$0 \to \mathfrak{F}' \to \mathfrak{F} \to \mathfrak{F}'' \to 0, \tag{6.1.1}$$

is exact in $\mathfrak{A}^I$ if and only if the following sequences in $\mathfrak{A}$, for all $x \in I$, are exact:

$$0 \to \mathfrak{F}'_x \to \mathfrak{F}_x \to \mathfrak{F}''_x \to 0\,. \tag{6.1.2}$$

In other words, if $\Box\colon \mathfrak{A}^I \to \mathfrak{A}^{|I|}$ is the obvious forgetful functor, then a sequence in $\mathfrak{A}^I$ is exact if and only if its image under $\Box$ is exact in $\mathfrak{A}^{|I|}$.

The sequence (6.1.1) is called *relatively short exact* if all the sequences (6.1.2) are split exact; a complex $\mathfrak{F}$ of I-systems is called *relatively exact* if all the complexes $\mathfrak{F}_x$ in $\mathfrak{A}$ are contractible. In the language of relative abelian categories, the category $\mathfrak{A}^I$ is *split relativized* via the functor $\Box$.

By the Kan construction, the forgetful functor $\Box$ has a right adjoint functor ρ,

$$\Box\colon \mathfrak{A}^I \to \mathfrak{A}^{|I|}, \qquad \rho\colon \mathfrak{A}^{|I|} \to \mathfrak{A}^I.$$

It associates with an I-family $\mathfrak{G} = \{\mathfrak{G}_z\}$ the I-system $\rho\mathfrak{G}$ defined by

$$(\rho\mathfrak{G})_x = \prod_{x \to z} \mathfrak{G}_z\,;$$

for an arrow $f\colon x \to y$ in I, the morphism $f\colon (\rho\mathfrak{G})_x \to (\rho\mathfrak{G})_y$ is given by

$$\mathrm{pr}_{y\to z}\, f = \mathrm{pr}_{x \to z}\,,$$

where the right side arrow $x \to z$ is the composition of the left side arrow with f. Note that the functor $\rho\colon \mathfrak{A}^{|I|} \to \mathfrak{A}^I$ is left exact; in the presence of AB4*, it is even exact.

An I-system of the form $\rho\mathfrak{G}$ is called *coinduced*. In the setup of relative abelian categories, an I-system is relatively injective if and only if it is a direct summand of a coinduced I-system.

Let $\pi := \rho\Box$ be the composition, and $\delta\colon 1 \to \pi$ the adjunction morphism; in addition, let $\kappa\colon \pi \to \nu$ be the cokernel of δ. For an I-system $\mathfrak{F}$, the morphism $\delta\colon \mathfrak{F} \to \pi\mathfrak{F}$ is determined by the morphisms,

$$\delta_x\colon \mathfrak{F}_x \to \prod_{x \to y} \mathfrak{F}_y,$$

whose projection, $\mathrm{pr}_{x\to y}\,\delta_x$, on an arrow $x\to y$ is the morphism $\mathfrak{F}_x \to \mathfrak{F}_y$ induced by the I-system $\mathfrak{F}$. Note that the index set of the product contains the identity arrow of x; the corresponding projection of δ_x is the identity of $\mathfrak{F}_x$. So the cokernel $(\nu\mathfrak{F})_x$ may be identified with the product,

$$(\nu\mathfrak{F})_x = {\prod_{x\to y}}' \mathfrak{F}_y,$$

where the product is over all arrows different from the identity of x. More precisely, if $\pi'\mathfrak{F}_x$ is product on the right side, then $\pi\mathfrak{F}_x = \mathfrak{F}_x \oplus \pi'\mathfrak{F}_x$. Accordingly, $\delta_x = (1, \delta'_x)^{\mathrm{tr}}$ with a morphism $\delta'_x\colon \mathfrak{F}_x \to \pi'\mathfrak{F}_x$. So $(\nu\mathfrak{F})_x = \pi'\mathfrak{F}_x$ and $\kappa\colon \pi\mathfrak{F}_x \to \pi'\mathfrak{F}_x$ is the morphism $(-\delta'_x, 1)$.

The transformation $\delta\colon 1 \to \pi$ induces the standard complex C_{stand} with $C^n_{\mathrm{stand}} = \pi^{[n]}$ and the canonical complex C_{can} with $C^n_{\mathrm{can}} = \pi\nu^n$. The functors $\pi^{[n]}$ and $\pi\nu^n$ are determined as follows:

$$(\pi^{[n]}\mathfrak{F})_x = \prod_{x\to x_0\to\cdots\to x_n} \mathfrak{F}_{x_n}, \quad (\pi\nu^n\mathfrak{F})_x = {\prod_{x\to x_0\to\cdots\to x_n}}' \mathfrak{F}_{x_n};$$

the first product is over all $(n+1)$-strings of the category I, the second is over those $(n+1)$-strings where none of the last n arrows are identities. The two complexes define relatively exact resolutions of the identity of $\mathfrak{A}^I$. In particular, since a relative monomorphism is a monomorphism, they are also exact resolutions.

Let $\mathfrak{B}$ be an abelian category and $T\colon \mathfrak{A}^I \to \mathfrak{B}$ an additive functor. Then a *relatively exact resolvent complex* for T is T-augmented complex of functors $\mathfrak{A}^I \to \mathfrak{B}$,

$$\overline{C}:\quad \cdots \to 0 \to T \to C^0 \to C^1 \to C^2 \to \cdots, \tag{6.1.3}$$

such that,

(i) if Q is an induced I-system, then $\overline{C}(Q)$ is a contractible complex of $\mathfrak{B}$.
(ii) each functor C^i takes relatively short exact sequences of $\mathfrak{A}^I$ into split exact sequences of $\mathfrak{B}$; and,

Relatively exact resolvent complexes are unique, up to homotopy. And they exist: TC_{stand} and TC_{can} are examples. Since $\overline{C}_{\mathrm{stand}}$ is an exact resolution of the identity, it follows in particular that its first part, $0 \to 1 \to C^0_{\mathrm{stand}} \to C^1_{\mathrm{stand}}$, is exact. Therefore, if T is left exact, then the sequence $0 \to T \to TC^0_{\mathrm{stand}} \to TC^1_{\mathrm{stand}}$ is exact; hence so is the sequence $0 \to T \to C^0 \to C^1$, for any relatively exact resolvent complex (6.1.3). In other words, T is the kernel of $C^0 \to C^1$.

Note that if, in a relatively resolvent complex (6.1.3), each functor C^n is exact, then the complex is an exact resolvent complex for T; in particular, then T is derivable.

(6.2) The inverse limit. The inverse limit is a functor $\varprojlim_I\colon \mathfrak{A}^I \to \mathfrak{A}$. A relatively exact resolvent complex for this functor,

$$\overline{C}:\quad \cdots \to 0 \to \varprojlim_I \to C^0 \to C^1 \to C^2 \to \cdots, \tag{6.2.1}$$

will simply be called a *resolvent complex for the category I*. The p'th *cohomology of I with coefficients* in $\mathfrak{F}$ is, by definition, the p'th cohomology of the nonaugmented complex $C(\mathfrak{F})$; it is independent of the choice of complex, and it is denoted

$$H^p(I,\mathfrak{F}) \quad \text{or} \quad \varprojlim\nolimits_I^{(p)} \mathfrak{F} \quad \text{or} \quad \varprojlim\nolimits_{x\in I}^{(p)} \mathfrak{F}_x.$$

Since the inverse limit is a left exact functor, it is necessarily equal to the kernel of $C^0 \to C^1$, that is,

$$\varprojlim\nolimits_I \mathfrak{F} = H^0(I,\mathfrak{F}).$$

In the presence of AB4*, the functor $\varprojlim_I^{(p)}$ is the p'th derived functor of $\varprojlim_I$; in general, it is the relatively derived functor.

Clearly, for an induced system $\rho\mathfrak{G}$ we obtain for the inverse limit,

$$\varprojlim_{x\in I}(\rho\mathfrak{G})_x = \varprojlim_{x\in I}\prod_{x\to y}\mathfrak{G}_y = \prod_{y\in I}\mathfrak{G}_y. \tag{6.2.2}$$

Hence, from the standard complex associated with the transformation $1 \to \pi$, we obtain a resolvent complex with

$$C^n\mathfrak{F} = \varprojlim\nolimits_I \pi^{[n]}\mathfrak{F} = \prod_{x_0\to\cdots\to x_n}\mathfrak{F}_{x_n}. \tag{6.2.3}$$

It will be denoted $C(I,\mathfrak{F})$. Similarly, from the canonical complex we obtain a resolvent complex with

$$C^n\mathfrak{F} = \varprojlim\nolimits_I \pi\nu^n\mathfrak{F} = \prod\nolimits'_{x_0\to\cdots\to x_n}\mathfrak{F}_{x_n}, \tag{6.2.4}$$

where the product is over n-strings of I with no identities.

(6.3) Observation. Consider for an index z of I and an object A of $\mathfrak{A}$ the I-family $\mathfrak{G}$ with $\mathfrak{G}_y = 0$ for $y \neq z$ and $\mathfrak{G}_z = A$. The corresponding coinduced I-system $\rho\mathfrak{G}$, denoted $\rho_z A$, is determined by

$$(\rho_z A)_x = \prod_{x\to z} A = A^{\mathrm{Hom}(x,z)}, \quad \text{and} \quad \varprojlim\nolimits_I \rho_z A = A;$$

the last equation follows from (6.2.2), or directly. Clearly any coinduced I-system $\rho\mathfrak{G}$ is a product of I-systems of this special form; in fact, $\rho\mathfrak{G} = \prod_z \rho_z\mathfrak{G}_z$. Hence, if the functors C^n in a complex (6.2.1) are relatively exact and commute with products indexed by I, then the complex is resolvent for I if and only if all the complexes $\overline{C}(\rho_z A)$ are contractible.

(6.4) Example. Consider the "one-point-category" (or better "one-arrow-category") 1. It has one object, denoted 1, one arrow, denoted 1, and hence one endomorphism, denoted 1, one transformation, denoted 1, ... Clearly, $\mathfrak{A}^1 = \mathfrak{A}$ and the inverse limit is the identity. The standard and canonical resolvent complex for the category 1 are the following:

$$\cdots \to 0 \to A = A \xrightarrow{0} A = A \xrightarrow{0} \cdots \quad \text{and} \quad \cdots \to 0 \to A = A \to 0 \to \cdots.$$

Consider similarly the category $I = (\bullet' \to \bullet \leftarrow \bullet'')$, with three objects and two nontrivial arrows. An I-system $\mathfrak{F}$ is a diagram $A' \xrightarrow{f'} A \xleftarrow{f''} A''$ in $\mathfrak{A}$. It is easy to see that a resolvent complex of functors for this category is determined by the following complex:

$$\overline{C}(\mathfrak{F})\colon \quad \cdots \to 0 \to A' \times_A A'' \to A' \oplus A'' \xrightarrow{f'-f''} A \longrightarrow 0 \to \cdots .$$

(6.5) Example. Take as category I the category $(0 \rightrightarrows 1)$. Here an I-system $\mathfrak{F}$ is a pair of morphisms $f, g\colon \mathfrak{F}_0 \to \mathfrak{F}_1$. The following complex is resolvent for I:

$$\overline{C}\colon \quad \cdots \to 0 \to \varprojlim_I \to C^0 \xrightarrow{f-g} C^1 \to 0 \to 0 \to \cdots, \quad \text{with } C^i\mathfrak{F} = \mathfrak{F}_i .$$

Indeed, each functor C^n is relatively exact. The two complexes $\overline{C}(\rho_z A)$ with $z = 0$ and $z = 1$ are:

$$0 \to A \xrightarrow{1} A \xrightarrow{0} 0 \to 0, \qquad 0 \to A \xrightarrow{(1,1)^{\mathrm{tr}}} A^2 \xrightarrow{(1,-1)} A \to 0;$$

clearly, both complexes are contractible.

As a consequence,

$$H^0(\mathfrak{F}_0 \rightrightarrows \mathfrak{F}_1) = \operatorname{Ker}(f-g), \quad \text{and} \quad H^1(\mathfrak{F}_0 \rightrightarrows \mathfrak{F}_1) = \operatorname{Cok}(f-g),$$

and cohomology in degree higher than 1 vanish.

(6.6) Note. An I-system $\mathfrak{F}\colon I \to \mathfrak{A}$ may alternatively be viewed as a (covariant) functor $\mathfrak{F}\colon I^{\mathrm{op}} \to \mathfrak{A}^{\mathrm{op}}$, that is, as an I^{op}-system in $\mathfrak{A}^{\mathrm{op}}$. As such, its p'th cohomology is an object in $\mathfrak{A}^{\mathrm{op}}$. As an object in $\mathfrak{A}$, it is called the p'th *homology* of I with coefficients in $\mathfrak{F}$, and denoted

$$H_p(I, \mathfrak{F}) \quad \text{or} \quad \varinjlim_I^{(p)} \mathfrak{F} \quad \text{or} \quad \varinjlim_{x \in I}^{(p)} \mathfrak{F}_x .$$

A contravariant functor $\mathfrak{G}\colon I \to \mathfrak{A}$ may be viewed as an I^{op}-system in $\mathfrak{A}$. As such, the homology and cohomology are objects of $\mathfrak{A}$,

$$H_p(I^{\mathrm{op}}, \mathfrak{G}) \quad \text{and} \quad H^p(I^{\mathrm{op}}, \mathfrak{G}). \tag{6.6.1}$$

The indication in (6.6.1) that the opposite category of I is considered is hardly necessary since it is only in the case of a constant functor that its variance is not obvious. And even in the case of a constant functor, the indication is not necessary. Indeed, consider more generally a *local system* $\mathfrak{F}$ on I, that is, a functor $\mathfrak{F}\colon I \to \mathfrak{A}$ transforming any arrow of I into an isomorphism of $\mathfrak{A}$. Then there is an associated contravariant functor $\mathfrak{F}^{-1}\colon I \to \mathfrak{A}$. It is given by $\mathfrak{F}^{-1}_x := \mathfrak{F}_x$; if $f\colon x \to y$ is an arrow of I, then $\mathfrak{F}^{-1}(f)\colon \mathfrak{F}_y \to \mathfrak{F}_x$ is the inverse of the isomorphism $\mathfrak{F}(f)\colon \mathfrak{F}_x \to \mathfrak{F}_y$. Consider the standard complexes (6.2.3) for $\mathfrak{F}$ and $\mathfrak{F}^{-1}$:

$$C^n(I, \mathfrak{F}) = \prod_{x_0 \to \cdots \to x_n} \mathfrak{F}_{x_n} \quad \text{and} \quad C^n(I^{\mathrm{op}}, \mathfrak{F}^{-1}) = \prod_{x_0 \leftarrow \cdots \leftarrow x_n} \mathfrak{F}_{x_n} .$$

Since $\mathfrak{F}$ is local system, there is an obvious isomorphism $t^n\colon C^n(\mathfrak{F}) \to C^n(\mathfrak{F}^{-1})$, and it is easy to see that the morphisms $(-1)^{n(n+1)/2}t^n$ form an isomorphism of complexes $\overline{C}(\mathfrak{F}) \to \overline{C}(\mathfrak{F}^{-1})$. In particular, there is an identification,

$$H^p(I, \mathfrak{F}) = H^p(I^{\mathrm{op}}, \mathfrak{F}^{-1}). \tag{6.6.2}$$

(6.7) Example. Let $\mathfrak{s}$ be one of the simplicial categories (**s**), (**ss**), or (**sss**). An $\mathfrak{s}$-system $\mathfrak{F}$ in $\mathfrak{A}$ is a co-$\mathfrak{s}$-object of $\mathfrak{A}$; it may be visualized as a diagram,

$$\mathfrak{F}\colon \quad \mathfrak{F}^0 \rightrightarrows \mathfrak{F}^1 \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathfrak{F}^3 \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots,$$

where only the face morphisms δ_i^n have been indicated. Let $C(\mathfrak{F})$ be the associated complex, that is, $C^n(\mathfrak{F}) = \mathfrak{F}^n$ with the differential $d^n = \sum(-1)^i\delta_i^n$. The following complex is resolvent for the category $\mathfrak{s}$:

$$\overline{C}\colon \quad \cdots \to 0 \to \varprojlim_{\mathfrak{s}} \to C^0 \to C^1 \to C^2 \to \cdots;$$

in particular, the inverse limit $\varprojlim_{\mathfrak{s}} \mathfrak{F}$ is equal to the kernel of $d^0 = \delta_0^0 - \delta_1^0\colon \mathfrak{F}^0 \to \mathfrak{F}^1$.

Indeed, each C^n is exact and relatively exact, and commutes with products. It remains to consider the complex $\overline{C}(\rho_p A)$ of a coinduced system of the form $\rho_p A$. The complex will be denoted $\widetilde{C}(p, A)$. Clearly, $\widetilde{C}^{-1}(p, A) = \varprojlim_{\mathfrak{s}} \rho_p A = A$ and, for $n \geqslant 0$,

$$\widetilde{C}^n(p, A) = \prod_{n \to p} A = A^{\mathrm{Hom}_{\mathfrak{s}}(n,p)}.$$

In fact, the complex $\widetilde{C}(p, A)$ is the associated complex of the co-$\bar{\mathfrak{s}}$-object obtained as the composition of $T\colon n \mapsto \mathrm{Hom}_{\bar{\mathfrak{s}}}(n, p)$ and $S \mapsto A^S$,

$$\widetilde{C}(p, A)\colon \ \cdots \to 0 \to A \to A^{\mathrm{Hom}_{\mathfrak{s}}(0,p)} \to A^{\mathrm{Hom}_{\mathfrak{s}}(1,p)} \to A^{\mathrm{Hom}_{\mathfrak{s}}(2,p)} \to \cdots. \tag{6.7.1}$$

It remains to prove the following:

Lemma. *The complex* (6.7.1), *for a fixed object A of $\mathfrak{A}$ and a fixed $p \geqslant 0$, is contractible.*

Proof. Note that $\mathrm{Hom}_{\bar{\mathfrak{s}}}(n, p)$ is a singleton when $n = -1$. So the n'term in the complex is equal to $A^{\mathrm{Hom}_{\bar{\mathfrak{s}}}(n,p)}$ also when $n = -1$.

Below, we define, especially for $\mathfrak{s} =$ (**sss**), a contraction for the complex. Later we will give a different proof. Let us, just for fun, note that the assertion for the two cases (**ss**) and (**s**) follows from the Standard Contraction Lemma of Section 5. For $\mathfrak{s} =$ (**ss**) note that the set $[p]$ is a partially ordered set. Hence $[p]$ may be viewed as a category, and clearly $\mathrm{Hom}_{\mathbf{Cat}}([n], [p]) = \mathrm{Hom}_{\mathbf{ss}}(n, p)$. Therefore, the $\overline{\mathbf{sss}}$-set of (5.5.2), with $J = [p]$, is the following:

$$* \leftarrow \mathrm{Hom}_{\mathbf{ss}}(0, p) \leftleftarrows \mathrm{Hom}_{\mathbf{ss}}(1, p) \substack{\leftarrow\\\leftarrow\\\leftarrow} \mathrm{Hom}_{\mathbf{ss}}(2, p) \substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow} \cdots. \tag{6.7.2}$$

Since $[p]$ (as a category) has a final object, it follows from Section (5.5) that the $\overline{\mathbf{sss}}$-set has a simple contraction. Therefore, the complex (6.7.1), obtained from (6.7.2) via the functor $S \mapsto A^S$, is contractible. The argument for $\mathfrak{s} = (\mathbf{s})$ is similar, using (5.5.3) with $M := [p]$.

Of course, the contraction of $\widetilde{C}(p, A)$ given by the considerations in (5.5) may be determined explicitly. Set $T_n := \operatorname{Hom}_{\mathfrak{s}}(n, p)$. For any map $f : [n] \to [p]$, let $\tilde{f} : [n+1] \to [p]$ be the map with $\tilde{f}(i) = f(i)$ for $i \leqslant n$ and $\tilde{f}(n+1) = p$. Note for $\mathfrak{s} = (\mathbf{s})$ and $\mathfrak{s} = (\mathbf{ss})$, that if $f \in T_n$, then $\tilde{f} \in T_{n+1}$. Consider the morphisms

$$s^{n+1} : A^{T_{n+1}} \to A^{T_n} \quad \text{given by } \operatorname{pr}(f) s^{n+1} = \operatorname{pr}(\tilde{f}),$$

where $\operatorname{pr}(f)$ is the projection $A^{T_n} \to A$ corresponding to the index $f \in T_n$. The morphisms define the contraction of (6.7.1) since the following equation is easily verified:

$$s^{n+1} d^n - d^{n-1} s^n = (-1)^{n+1}. \tag{6.7.3}$$

In the case $\mathfrak{s} = (\mathbf{sss})$, the definition of the morphisms s have to be modified. In this case, the elements of T_n are the strictly increasing maps $f : [n] \to [p]$, and $\tilde{f}$ is only strictly increasing if f is *frontal*, that is, if $f(n) < p$. Define $s : A^{T_{n+1}} \to A^{T_n}$ by

$$\operatorname{pr}(f) s = \begin{cases} \operatorname{pr}(\tilde{f}) & \text{if } f \text{ is frontal,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $f : n \to p$ be a morphism in $(\overline{\mathbf{sss}})$. Clearly,

$$\tilde{f} \delta_i^n = \begin{cases} f & \text{if } i = n+1; \\ (f \delta_i^{n-1})\tilde{} & \text{if } i \leqslant n. \end{cases}$$

Consequently, if f is frontal, then $\operatorname{pr}(f) s \delta_{n+1}^n = \operatorname{pr}(f)$ and if $i \leqslant n$ then

$$\operatorname{pr}(f) s \delta_i^n = \operatorname{pr}(\tilde{f}) \delta_i^n = \operatorname{pr}(\tilde{f} \delta_i^n) = \operatorname{pr}((f \delta_i^{n-1})\tilde{}) = \operatorname{pr}(f) \delta_i^{n-1} s;$$

Hence,

$$\operatorname{pr}(f) s d - \operatorname{pr}(f) d s = (-1)^{n+1} \operatorname{pr}(f).$$

On the other side, if f is not frontal, then $\operatorname{pr}(f) s d = 0$; moreover, if $i < n$ then $f \delta_i^{n-1}$ is not frontal, and then $\operatorname{pr}(f) \delta_i^{n-1} s = \operatorname{pr}(f \delta_i^{n-1}) s = 0$. Hence,

$$\operatorname{pr}(f) s d - \operatorname{pr}(f) d s = -(-1)^n \operatorname{pr}(f) \delta_n^{n-1} s = (-1)^{n+1} \operatorname{pr}((f \delta_n^{n-1})\tilde{}) = (-1)^{n+1} \operatorname{pr}(f),$$

where the last equation holds because $f(n) = p$. Therefore, with this definition of s, equation (6.7.3) holds also when $\mathfrak{s} = (\mathbf{sss})$. Hence $\widetilde{C}(p, A)$ is contractible in all three cases. □

(6.8) Example. Let S be a simplicial set, that is, an $\mathfrak{s}$-object in the category of sets, visualized by a diagram,

$$S_0 \leftleftarrows S_1 \,\overset{\leftarrow}{\leftleftarrows}\, S_2 \,\overset{\leftleftarrows}{\leftleftarrows}\, \cdots .$$

Then there is category associated to S: The objects of the category are the *simplices* of S (the elements in the disjoint union of the sets S_n), the morphisms from a simplex $y \in S_p$ to a simplex $x \in S_n$ are the morphisms $f\colon p \to n$ in $\mathfrak{s}$ for which $f^*x = y$. With this structure on S as a category, an S-system $\mathfrak{F}$ associates with each simplex $x \in S_n$ an object $\mathfrak{F}_x$ of $\mathfrak{A}$, and with each morphism $f\colon p \to n$ in $\mathfrak{s}$ a morphism $\mathfrak{F}_{f^*x} \to \mathfrak{F}_x$ (with obvious compatibilities). The following complex of functors is resolvent for the category S:

$$\overline{C}\colon \quad \cdots \to 0 \to \varprojlim_S \to C^0 \to C^1 \to C^2 \to \cdots, \quad \text{with } C^n\mathfrak{F} := \prod_{x\in S_n} \mathfrak{F}_x;$$

it is easily seen that the $C^n\mathfrak{F}$, in a natural way, form a co-$\mathfrak{s}$-object of $\mathfrak{A}$ and the complex $C(\mathfrak{F})$ is the associated complex.

Indeed, each functor $C^n\colon \mathfrak{A}^S \to \mathfrak{A}$ is obviously relatively exact, and commutes with products. It remains to prove, for a fixed simplex $z \in S_p$ and an object A of $\mathfrak{A}$, that the complex $\overline{C}(\rho_z A)$ is contractible. Clearly, for a simplex x in S, say $x \in S_n$,

$$(\rho_z A)_x = \prod_{x\to z} A = \prod_{f^*z=x} A,$$

where the product is over all morphisms $f\colon n \to p$ in $\mathfrak{s}$ for which $f^*z = x$. By taking the product over $x \in S_n$, it follows that

$$\overline{C}^n(\rho_z A) = A^{\operatorname{Hom}_{\mathfrak{s}}(n,p)}.$$

Hence, by the lemma in (6.7), the complex $\overline{C}(\rho_z A)$ is contractible. As a consequence, the inverse limit $\varprojlim_S \mathfrak{F}$ is equal to the kernel of the morphism,

$$d^0 = \delta_0^0 - \delta_1^0\colon \prod_{x\in S_0} \mathfrak{F}_x \to \prod_{x\in S_1} \mathfrak{F}_x .$$

(6.9) Example. Let G be a group, considered as category of one object and the elements of G as morphisms. Then a G-system in $\mathfrak{A}$ is an object A of $\mathfrak{A}$ with a given representation $G \to \operatorname{Aut}(A)$; it is also called a (co-)G-object of $\mathfrak{A}$. Its inverse limit is often denoted A^G (which should not be confused with a coinduced G-system, of the form $\prod_{g\in G} A$ for an object A of $\mathfrak{A}$).

The standard complex (6.2.3) or the canonical complex (6.2.4) define the cohomology of a G-object. For special groups, special resolvent complexes are given in the examples section. Let us just note here, for the cyclic group $G = \mathbb{Z}/2$ of order 2 and a constant $\mathbb{Z}/2$-object A, that the canonical complex is the following:

$$0 \to A \xrightarrow{1} A \xrightarrow{0} A \xrightarrow{2} A \xrightarrow{0} A \xrightarrow{2} A \to \cdots;$$

in particular, $H^0(\mathbb{Z}/2, A) = A$, $H^n(\mathbb{Z}/2, A) = {}_2A$ (the kernel of 2_A) when $n > 0$ is odd, and $H^n(\mathbb{Z}/2, A) = A/2$ (the cokernel of 2_A) when $n > 0$ is even.

(6.10) Example. Let I be a *unique factorization category*, that is, there is a subset P of arrows of I such that any morphism f of I factors uniquely as a composition $f = p_1 \cdots p_k$ with arrows p_ν in P. Then there is a resolvent complex for I,

$$\overline{C}\colon \quad \cdots \to 0 \to \varprojlim_I \to C^0 \xrightarrow{d^0} C^1 \to 0 \to 0 \to \cdots , \tag{6.10.1}$$

with

$$C^0\mathfrak{F} = \prod_x \mathfrak{F}_x, \quad C^1\mathfrak{F} = \prod_{u\to x}{}' \mathfrak{F}_x,$$

where the second product is over the arrows in P. The C^0 above is the C^0 of the standard resolvent complex (6.2.3) and the C^1 above is a quotient of the standard C^1; accordingly, the differential d^0 in (6.10.1) is the difference $d^0 = \delta_0' - \delta_1'$ where $\mathrm{pr}_{u\to x}\,\delta_0' = \mathrm{pr}_x$ and $\mathrm{pr}_{u\to x}\,\delta_1'$ is the projection pr_u on $\mathfrak{F}_u$ followed by the morphism $\mathfrak{F}_u \to \mathfrak{F}_x$.

To prove the assertion, consider the evaluation of $\overline{C}$ on a coinduced system $\rho_z A$. It yields the complex,

$$0 \to A \to \prod_{x\to z} A \to \prod_{u\to x\to z}{}' A \to 0, \tag{6.10.2}$$

where the second product is over strings $u{\to}x{\to}z$ with $u{\to}x$ in P. Consider the morphisms,

$$s\colon \prod_{x\to z} A \to A, \quad \text{and} \quad t\colon \prod_{u\to x\to z}{}' A \to \prod_{x\to z} A,$$

where s is the projection pr_1 on the identity of z and

$$\mathrm{pr}_{x\to z}\, t = \sum_{x\to u\to y\to z} \mathrm{pr}_{u\to y\to z},$$

where the sum, for a given arrow $f\colon x \to z$ is over all factorizations $f = (x \to u \to y \to z)$ with an arrow $u \to y$ in P. Of course the sum if finite: if $f = p_1 \ldots p_k$ is the "prime" factorization of f, then the possible $u \to y$ in the sum are the p_ν. It is easy to see that the pair (s, t) defines a splitting of the complex (6.10.1).

Note that unique factorization holds for partially ordered sets with the property that each interval $[x, y]$ is a finite totally ordered set; in particular, it holds for the ordered sets $(\mathbb{Z}, \leqslant)$ and $(\mathbb{N}, \geqslant)$. It also holds for a free monoid (generated by an alphabet P), considered as a category with one object. And it holds for the categories in (6.5).

(6.11) Definition. Consider, for an object A of $\mathfrak{A}$ the constant functor $A\colon I \to \mathfrak{A}$. Clearly, there is a canonical morphism $A \to \varprojlim_I A$, and hence for every resolvent complex $\overline{C}$ for I a *reduced complex*,

$$\widetilde{C}(A)\colon \quad \cdots \to 0 \to A \to C^0(A) \to C^1(A) \to \cdots ,$$

obtained by a change in the coaugmentation. For instance C may be taken as the standard resolvent complex $C(I,\ )$. Note that the two complexes $\overline{C}(A)$ and $\widetilde{C}(A)$ differ only in their term of degree -1. The cohomology of the complex $\widetilde{C}(A)$ is the *reduced cohomology* of I with constant coefficients A, denoted $\widetilde{H}^p(I, A)$. It differs from the cohomology $H^p(I, A)$ only in degree -1 and 0. Moreover, $\widetilde{H}^{-1}(I, A) = \widetilde{H}^0(I, A) = 0$ if and only if $\varprojlim_I A = A$. The *reduced homology* $\widetilde{H}_p(I, A)$ is defined similarly.

The category I is called *acyclic* if one of the following equivalent conditions hold:

(i) $\widetilde{H}_n(I, \mathbb{Z}) = 0$ for all n. Equivalently, $H_n(I, \mathbb{Z}) = 0$ for $n > 0$ and $H_0(I, \mathbb{Z}) = \mathbb{Z}$.

(ii) $\widetilde{H}^n(I, A) = 0$ for all n and all abelian groups A.

(iii) $\widetilde{C}(I, A)$ is contractible for any A in any abelian category $\mathfrak{A}$.

To see that the conditions are equivalent, let $\widetilde{C}_\bullet := \widetilde{C}_\bullet(I, \mathbb{Z})$ be the reduced homology complex corresponding to the constant system $\mathbb{Z}$. Then (i) holds if and only if $\widetilde{C}_\bullet$ is acyclic. Now $\widetilde{C}_\bullet$ is a left complex of free $\mathbb{Z}$-modules; hence it is acyclic if and only if it is contractible. Therefore (i) holds if and only if $\widetilde{C}_\bullet$ is contractible.

On the other hand, for any abelian group A, we have that

$$\widetilde{C}(I, A) = \mathrm{Hom}(\widetilde{C}_\bullet, A).$$

Hence $\widetilde{C}(I, A)$ is acyclic for all abelian groups A if and only if $\widetilde{C}_\bullet$ is contractible. Thus (i) and (ii) are equivalent.

If $\widetilde{C}_\bullet$ is contractible, then a homotopy $s\colon \overline{C}_\bullet \to \overline{C}_\bullet(-1)$ from the identity 1 to 0, yields by transposing, a homotopy

$$s^{\mathrm{tr}}\colon \overline{C}(I, A)(1) \to \overline{C}(I, A),$$

from 1 to 0, for any object A in any abelian category $\mathfrak{A}$. Therefore (iii) is a consequence of (i). Conversely, it is obvious that (i) and (ii) follow from (iii).

(6.12) Lemma. (0) *An acyclic category is nonempty and connected.*

(1) *If I is acyclic, then so is I^{op}.*

(2) *A nonempty filtering (to the left or to the right) category is acyclic.*

(3) *A nonempty directed union of acyclic categories is acyclic.*

Proof. (0) Assume that I is acyclic. Then, in particular, the first part of the reduced homology complex is exact:

$$C_1(I, \mathbb{Z}) \to C_0(I, \mathbb{Z}) \to \mathbb{Z} \to 0.$$

Here

$$C_0 = \bigoplus_{x \in I} \mathbb{Z}, \quad C_1 = \bigoplus_{x_1 \to x_0} \mathbb{Z}.$$

Since $C_0 \to \mathbb{Z}$ is surjective, it follows that I is nonempty. For any pair of indices x_0, x_1 in I, the element $x_0 - x_1$ in C_0 is a cycle, and hence the boundary of an element in C_1. From that element in C_1 it is easy to connect x_0 and x_1 with arrows.

(1) The assertion follows from the considerations in (6.6).

(2) Assume that I is filtering to the right. Then, as is easily seen, the direct limit,

$$\varinjlim_I : (\mathbf{Ab})^I \to (\mathbf{Ab}),$$

is an exact functor, and the direct limit of the constant system $\mathbb{Z}$ is equal to $\mathbb{Z}$. Therefore, since the standard complex is an exact left resolution of the identity of $(\mathrm{Ab})^I$, it follows that left complex $\overline{C}_\bullet(I, \mathbb{Z})$ is exact.

(3) Again, the assertion holds, because the direct limit over a directed set is an exact functor. □

(6.13) The restriction. Let $\varphi\colon J \to I$ be a functor of small categories. Any I-system $\mathfrak{F}\colon I \to \mathfrak{A}$ restricts to a J-system $\mathfrak{F}\varphi\colon J \to \mathfrak{A}$ and restriction is a functor $\mathrm{Res}\colon \mathfrak{A}^I \to \mathfrak{A}^J$, often denoted φ^* or $\square^I_J$; it is obviously exact and relatively exact. Clearly, for the standard resolvent complexes there is an obvious *restriction morphism*,

$$C(I, \mathfrak{F}) \to C(J, \mathrm{Res}\,\mathfrak{F}). \tag{6.13.1}$$

(6.14) Proposition. *In the setup of* (6.13), *the following conditions are equivalent:*

(i) *The restriction morphism* (6.13.1) *is a homotopy equivalence.*
(ii) *All left fibers J/x, for $x \in I$, of the functor $\varphi\colon J \to I$ are acyclic.*

Proof. Recall that the left fiber J/x, over an object x of I, is the category whose objects are pairs (v, f), with v is an index in J and f is a morphism $f\colon \varphi v \to x$; a morphism from (v, f) to (u, g) is a morphism $h\colon u \to v$ in J such that $f\varphi(h) = g$.

Clearly, there is a canonical (functorial) morphism $\varprojlim_I \mathfrak{F} \to \varprojlim_J \mathrm{Res}\,\mathfrak{F}$. The target is the kernel of $C^0(J, \mathrm{Res}\,\mathfrak{F}) \to C^1(J, \mathrm{Res}\,\mathfrak{F})$. So the right side complex in (6.13.1) may be co-augmentated with the object $\varprojlim_I \mathfrak{F}$ in degree -1. Let $\widetilde{C}(J, \mathfrak{F})$ be the co-augmented complex. In degree -1 it has the same term as the co-augmented complex $\overline{C}(I, \mathfrak{F})$ corresponding to the left side of (6.13.1). So we obtain an extension of (6.13.1),

$$\overline{C}(I, \mathfrak{F}) \to \widetilde{C}(J, \mathfrak{F}), \tag{6.14.1}$$

and (6.13.1) is a homotopy equivalence if and only if (6.14.1) is.

Let us evaluated the right side of (6.14.1) at a coinduced system of the form $\rho_z A$. In degree -1 we obtain $\varprojlim_I \rho_z A = A$. Clearly, for the restriction, we have

$$(\mathrm{Res}\,\rho_z A)_v = (\rho_z A)_{\varphi v} = \prod_{\varphi v \to z} A;$$

the product on the right side is exactly over all objects in the fiber J/z. More generally, it follows easily that $C^n(J, \mathrm{Res}\,\rho_z A) = C^n(J/z, A)$. In fact, it is easy to obtain the equality of complexes,

$$\widetilde{C}(J, \rho_z A) = \widetilde{C}(J/z, A), \tag{6.14.2}$$

where the right side is the reduced complex of the constant system A on the fiber J/z as in (6.11).

Now, for the equivalence of the two conditions, assume (ii). To prove that (6.14.1) is a homotopy equivalence, if suffices to prove that the right side is resolvent for I. Clearly, each functor $C^i(J, \mathrm{Res}(\ ))$ is relative exact and commutes with products. So it remains to verify that each complex $\widetilde{C}(J, \rho_z A)$ is contractible. The verification is immediate, given (6.14.2) and (ii).

Conversely, assume that (6.14.1) is a homotopy equivalence. Then, in particular, the left side of (6.14.2) is homotopy equivalent to $\overline{C}(I, \rho_z A)$. The latter is contractible, since $\rho_z A$ is a coinduced I-system. So the right side of (6.14.2) is contractible, that is, (ii) holds. ∎

(6.15) Example. A category I with an initial object b is acyclic. This is contained in Lemma (6.12)(2). It may also be obtained from (6.14). Indeed, the left fibers of the inclusion $\{b\} \to I$ are one-point categories, and hence trivially acyclic. So the restriction morphism (6.14.1), for a constant I-system A, is a homotopy equivalence from $\overline{C}(I, A)$ to the contractible complex $\widetilde{C}(\{b\}, A)$. It follows easily that $\overline{C}(I, A)$ is contractible and equal to $\widetilde{C}(I, A)$.

As a second application, consider the complex in (6.7.1). Assume first that $\mathfrak{s} = (\mathbf{ss})$. Consider each finite set $[p]$ as a partially ordered set, and hence as a category, denoted $[p]_{\mathbf{ss}}$. A morphism $f : n \to p$ in $(\mathbf{ss})$ may be identified with a weakly increasing sequence,

$$f_0 \leqslant f_1 \leqslant \cdots \leqslant f_n,$$

of elements of $[p]$, and hence as an n-string in the category $[p]_{\mathbf{ss}}$. It follows easily that the complex $\widetilde{C}(p, A)$ of (6.7.1), for $\mathfrak{s} = (\mathbf{ss})$, is equal to the complex $\widetilde{C}([p]_{\mathbf{ss}}, A)$, obtained by reaugmentating the standard complex. Hence it is contractible, because the category $[p]_{\mathbf{ss}}$ has an initial object.

For the simplicial category $(\mathbf{s})$, view the set $[p]$ is a groupoid, denoted $[p]_{\mathbf{s}}$, with one arrow $i \to j$ for any pair of elements i, j in $[p]$. Again, the complex $\widetilde{C}(p, A)$ of (6.7.1), for $\mathfrak{s} = (\mathbf{s})$, is equal to the complex $\widetilde{C}([p]_{\mathbf{s}}, A)$, and hence contractible, because $[p]_s$ has an initial object.

Finally, assume that $\mathfrak{s} = (\mathbf{sss})$. Up to homotopy equivalence, the complex $\widetilde{C}(I, A)$ may be obtained by reaugmentating the canonical complex instead. Clearly, with this definition, for $I = [p]_{\mathbf{ss}}$, the complex $\widetilde{C}([p]_{\mathbf{ss}}, A)$ is equal to the complex $\widetilde{C}(p, A)$ for $\mathfrak{s} = (\mathbf{sss})$. So $\widetilde{C}(p, A)$ is contractible, because the category $[p]_{\mathbf{ss}}$ is acyclic.

(6.16) Definition. Obviously, if the category $\mathfrak{A}^I$ has finite relative cohomological dimension d, that is, if any I-system $\mathfrak{F}$ has a relatively exact relatively injective resolution of length d, then the cohomology $H^p(I, \mathfrak{F})$ vanishes for $p > d$. This is in particular the case when I has finite *Krull dimension* d, that is, when d is the longest length of string of nontrivial arrows of I. Indeed, in this case, we have $C^p_{\mathrm{can}} = 0$ for $p > d$.

7. Cohomology of simplicial objects.

8. Cohomology of simplicial sets.

9. Cohomology of pos's.

10. Cohomology of combinatorial spaces.

Examples; illustrations

1. The Koszul komplex.

(1.1) Setup. Fix a commutative ring A and an r-tuple $\mathbf{f} = (f_1, \dots, f_r)$ of elements of A. Let I_p be the set of all subsets of cardinality p of the the set $\{1, \dots, r\}$. For every A-module M, consider the product,

$$K^p_{\mathbf{f}}(M) := M^{I_p}, \quad \text{the } A\text{-module of all maps } x\colon I_p \to M.$$

Identify the elements of I_p with p-tuples $(i_1, \dots, i_p)$ of integers $i_1 < \dots < i_p$ in the interval $[1, r]$, and hence the elements of M^{I_p} with functions $x(i_1, \dots, i_p)$. In this notation define $\partial\colon K^p \to K^{p+1}$ by

$$\partial x(i_0, \dots, i_p) := \sum_{\nu=0}^{p} (-1)^\nu f_{i_\nu} x(i_0, \dots, \widehat{i_\nu}, \dots, i_p), \tag{1.1.1}$$

where the "hat" indicates an omitted index. Then,

$$K_{\mathbf{f}}(M): \quad 0 \to K^0_{\mathbf{f}} \to K^1_{\mathbf{f}} \to \dots \to K^r_{\mathbf{f}} \to 0,$$

is a positive complex, the *Koszul* (cochain)*complex* of M. with *Koszul cohomology groups*,

$$\mathrm{H}^p_{\mathbf{f}}(M) = \mathrm{H}^p(K_{\mathbf{f}}(M)).$$

Note that I_0 and I_r are one-point-sets consisting, respectively, of the empty sequence $\emptyset$ and the full sequence $1, 2, \dots, r$. Hence we may identify $K^0 = M$ and $K^r = M$. Clearly, $H^0 \subseteq M$ is the submodule consisting of elements $x \in M$ with $f_\nu x = 0$ for all f_ν, and H^r is the quotient $H^r = M/\mathbf{f}M$.

The dual construction leads to the *Koszul chain complex*: Consider the direct sum $M^{\oplus I_p}$ with canonical embeddings $\iota_{i_1,\dots,i_p}\colon M \to M^{\oplus I_p}$. Then there is a chain complex,

$$K^{\mathbf{f}}_\bullet(M): \quad 0 \to K^{\mathbf{f}}_r \to \dots \to K^{\mathbf{f}}_1 \to K^{\mathbf{f}}_0 \to 0, \quad K^{\mathbf{f}}_p := M^{\oplus I_p}.$$

with differential $\partial\colon K_{p+1} \to K_p$ given by the formula,

$$\partial\, \iota_{i_0,\dots,i_p} := \sum_{\nu=0}^{p} (-1)^\nu f_{i_\nu}\, \iota_{i_0,\dots,\widehat{i_\nu},\dots,i_p}\,. \tag{1.1.2}$$

Let us make it a little more concrete: The module $K_{p+1} = M^{\oplus I_{p+1}}$ is the direct sum of identical copies of M, say $M_{i_0,\dots,i_p} = M$, indexed by sequences $(i_0,\dots,i_p) \in I_{p+1}$. So for any $x \in M$ there is an element $x_{i_0,\dots,i_p} \in M_{i_0,\dots,i_p} \in M$, and the elements in K_{p+1} are sums of elements of this form, for varying x and $i_0,\dots,i_p$. To define the differential $\partial K_{p+1} \to K_p$, it suffices to define it on an element of the form $x_{i_0,\dots,i_p}$, for $x \in M$. The formula (1.1.2) yields this:

$$\partial\, x_{i_0,\dots,i_p} := \sum_{\nu=0}^{p} (-1)^\nu f_{i_\nu}\, x_{i_0,\dots,\widehat{i_\nu},\dots,i_p}\,. \tag{1.1.3}$$

It should be emphasized that in spite of the striking similarity between (1.1.1) and (1.1.3), the objects that appear in the formulas are of a very different nature: In the first formula, x is a function $I_p \to M$, in the second, x is an element of M.

The case $M := A$ leads to the chain complex $K_\bullet^{\mathbf{f}}(A)$ and the augmented chain complex,

$$0 \to K_r(A) \to \cdots \to K_1(A) \to K_0(A) \to A/(\mathbf{f}) \to 0. \tag{1.1.4}$$

It is easy to obtain isomorphisms,

$$K_\bullet^{\mathbf{f}}(M) = K_\bullet^{\mathbf{f}}(A) \otimes M, \qquad K_{\mathbf{f}}(M) = \operatorname{Hom}_A(K_\bullet^{\mathbf{f}}(A)), M).$$

The modules $K_p(A) = A^{\oplus I_p}$ er free A-modules of rank $|I_p| = \binom{r}{p}$, and so the differentials are described by matrices of various sizes. For instance, if $r = 4$, the Koszul chain complex has the form

$$\cdots \to 0 \to A \xrightarrow{\partial_4} A^4 \xrightarrow{\partial_3} A^6 \xrightarrow{\partial_2} A^4 \xrightarrow{\partial_1} A \longrightarrow 0 \longrightarrow \cdots$$

The module $K_2 = A^{\binom{4}{2}}$ has basis e_{i_1,i_2} $(= 1_{i_1,i_2}$ in the notation of (1.1.3)), say in the order $e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}$, and K_1 has basis e_1, e_2, e_3. So $\partial e_{12} = f_1e_2 - f_2e_1$, etc. So it it immediate to write up the matrix for ∂_1:

$$\partial_2 = \begin{bmatrix} -f_2 & -f_3 & -f_4 & 0 & 0 & 0 \\ f_1 & 0 & 0 & -f_3 & -f_4 & 0 \\ 0 & f_1 & 0 & f_2 & 0 & -f_4 \\ 0 & 0 & f_1 & 0 & f_2 & f_3 \end{bmatrix}.$$

As we will see, the sequence (1.1.4) is exact, when $\mathbf{f}$ is a regular sequence in A. If (1.1.4) is exact, then the Koszul chain complex $K_\bullet^{\mathbf{f}}(A)$ is a resolution of $A/(\mathbf{f})$, and we obtain isomorphims,

$$\mathrm{H}^i_{\mathbf{f}}(M) = \operatorname{Ext}^i_A(A/\mathbf{f}, M), \quad \mathrm{H}_i^{\mathbf{f}}(M) = \operatorname{Tor}_i^A(A/\mathbf{f}, M).$$

(1.2) The Koszul complex of a complex. It is a terrific excercise to prove the following: *For any sequence $f_0, f_1, \dots, f_r$ of $r+1$ elements of A there is a canonical isomorphism of chain complexes,*

$$K^{f_0,f_1,\dots,f_r}(M) = \operatorname{Con}(f_0, K^{f_1,\dots,f_r}(M)), \tag{1.2.1}$$

where $\operatorname{Con}(f, X)$, *for a complex* X *and* $f \in A$, *denotes the mapping cone of multiplication by* f *on* X.

Hint. Let $\mathbf{f}$ be the sequence $(f_1, \dots, f_r)$ and indicate with a prime objects associated to the extended sequence $\mathbf{f}' = (f_0, \mathbf{f})$, with indices $0, \dots, r$. Hence I'_p consist of alle sequences $i_1, \dots, i_p$ with $0 \leqslant i_1, \dots \leqslant i_p \leqslant r$, etc. In particular, the I'_{p+1} is the disjoint union of two subsets I_p and I_{p+1} consisting, respectively, of sequences $0, i_1, \dots, i_p$ and of sequences $i_0, i_1, \dots, i_r$ with $i_0 > 0$. Accordingly, we may split: $K'_{p+1} = K_p \oplus K_{p+1}$. Now check that the differential ∂'_p under the splitting corresponds to the differential of the mapping cone of $f_0 \colon K \to K$. □

It is natural to take (1.2.1) as the definition of the Koszul complex of a complex X of A-modules. So we define:

$$K^f(X) := \operatorname{Con}(f, X), \qquad K^{f_1,\dots,f_r}(X) := K^{f_1}(K^{f_2,\dots,f_r}(X));$$

for the Koszul co-chain complex we use the cocone:

$$K_f(X) := \mathring{\mathrm{C}}\mathrm{on}(f, X), \qquad K_{f_1,\dots,f_r}(X) := K_{f_1}(K_{f_2,\dots,f_r}(X));$$

(1.3) Observations. (1) *The formation of the Koszul complex* $K^{\mathbf{f}}(X) = K^{f_1,\dots,f_r}(X)$ *is functorial with respect to* X, *and defines an additive functor* $K^{\mathbf{f}} \colon \mathfrak{Mod}_A^\bullet \to \mathfrak{Mod}_A^\bullet$.

(2) *The functor* $K^{\mathbf{f}}$ *is exact.*

(3) *The functor* $K^{\mathbf{f}}$ *commutes with formation of mapping cones.*

(4) *The functor* $K^{\mathbf{f}}$ *respects homotopy.*

(5) *The functor* $K^{\mathbf{f}}$ *respects quasi-isomorphisms. In particular, if* X *is acyclic, then* $K^{\mathbf{f}}(X)$ *is acyclic.*

Hints. For all five observations it suffices to treat the case $r = 1$, $f_1 = f$. (1) and (2) are obvious: A morphism $\varphi \colon X \to Y$ commutes with multiplication by f, because it is linear. Hence it induces the diagonale morphism on the cones: $K^f(\varphi) \colon K^f(X) \to K^f(Y)$.

Consider (3). Let Z be the mapping cone of φ. Then there is natural isomorphism of complexes from the mapping cone of $K^f(\varphi)$ to the Koszul complex $K^f(Z)$. Indeed, the two complexes, and their differentials, are the following:

$$\begin{matrix} X(2) \\ \oplus\, X(1) \\ Y(1) \\ Y \end{matrix}, \quad \begin{pmatrix} \partial & 0 & 0 & 0 \\ -f & -\partial & 0 & 0 \\ \varphi & 0 & -\partial & 0 \\ 0 & \varphi & f & \partial \end{pmatrix}, \qquad \begin{matrix} X(2) \\ \oplus\, Y(1) \\ X(1) \\ Y \end{matrix}, \quad \begin{pmatrix} \partial & 0 & 0 & 0 \\ -\varphi & -\partial & 0 & 0 \\ f & 0 & -\partial & 0 \\ 0 & f & \varphi & \partial \end{pmatrix},$$

and an isomorphism from the first to the second is given by the matrix,

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consider (4). Assume that $\varphi = \partial s + s\partial$, with a homotopy $s\colon X \to Y(1)$. Use that s is A-linear to prove that

$$K^{\mathbf{f}}(\varphi) = \begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix} = \begin{pmatrix} -s & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} -\partial_X & 0 \\ f & \partial_X \end{pmatrix} + \begin{pmatrix} -\partial_Y & 0 \\ f & \partial_Y \end{pmatrix}\begin{pmatrix} -s & 0 \\ 0 & s \end{pmatrix}.$$

Finally, consider (5). A quasi-isomorphism $\varphi\colon X \to Y$ is characterized by the condition that the mapping cone $Z = \operatorname{Con}\varphi$ is acyclic. Therefore, by (3), it suffices to prove the special case. Again, we may assume that $r = 1$. Then, since $H^n X = 0$, it follows from the long exact cohomology sequence associated to the cone of $f\colon X \to X$ that $H^n(K^f(X)) = 0$. Hence $K^f(X)$ is acyclic. □

Corollary 1. *The Koszul complex $K^{f_1,\dots,f_r}(X)$ is, up to canonical isomorphism, invariant under permutation of the f_i.*

Hint. We may assume that $r = 2$, and then the isomorphism $K^f(K^g(X)) = K^g(K^f(X))$ is a special case of (3). □

Corollary 2. *Multiplication by f_i in $K^{\mathbf{f}}(X)$ is homotopic to zero. In particular, the homology modules $H_p^{\mathbf{f}}(X)$ are annihilated by the f_i and hence by all elements in the ideal $(f_1, \dots, f_r)A$.*

Hint. By (4), we may assume that $r = 1$. Now check the equation,

$$\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -\partial & 0 \\ f & \partial \end{pmatrix} + \begin{pmatrix} -\partial & 0 \\ f & \partial \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

□

(1.4) Definition. The cokernel of multiplication by f on X is denoted X/f. Component for component it is the quotient $X^n/f := X^n/fX^n$. The complex $(X/f)/g$ is, component for component, equal to $X^n/(f,g)X^n$; we denote it $X/(f,g)$, and define $X/(f_1, \dots, f_r)$ inductively. There is a natural morphism,

$$K^{f_1,\dots,f_r}(X) \to X/(f_1, \dots, f_r)$$

defined inductively: $K^f(X)$ is the cone of $f\colon X \to X$, so there is an induced morphism $K^f(X) \to X/f$. If the morphism (1.4.1) is defined for r elements we define it for $r+1$ elements as a composition:

$$K^{f_0,f_1,\dots,f_r}(X) = K^{f_0}(K^{f_1,\dots,f_r}(X)) \to K^{f_0}(X/(f_1, \dots, f_r)) \to X/(f_0, f_1, \dots, f_r).$$

An element $f \in A$ is said to be *regular* on X, if multiplication by f on X is injective. The sequence $(f_1, \dots, f_r)$ is said to be an *X-regular sequence*, if f_1 is regular on X, and f_2 is regular on X/f_1, etc, that is, f_{i+1} is regular on $X/(f_1, \dots, f_i)$ for $0 \leqslant i < r$.

Proposition. *If $(f_1, \dots, f_r)$ is an X-regular sequence, then the canonical morphism is a quasi-isomorphism $K^{f_1,\dots,f_r}(X) \to X/(f_1, \dots, f_r)$. If $(f_1, \dots, f_r)$ is an M-regular sequence, then the Koszul complex is a left resolution of $M/(f_1, \dots, f_r)$,*

$$0 \to K_r(M) \to \cdots K_1(M) \to K_0(M) \to M/(f_1, \dots, f_r) \to 0.$$

(1.5) Excercises.

1. Describe the matrix ∂_3 in the Kozsul complex $K^{f_1,f_2,f_3,f_4}(A)$, cf. (1.1)

2. The de Rham complex.

(2.1) Setup. Let M be a smooth r-manifold ($\mathcal{C}_\infty$-manifold or, simply, an open subset $M \subseteq \mathbb{R}^r$). Denote by $\mathcal{C}^p(M)$ the vector space of p-forms on M. So a p-form ω has, locally, in local coordinates $x_1, \dots, x_r$, a unique expansion as a sum,

$$\omega = \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant r} f_{i_1,\dots,i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

where the coefficients $f_{i_1,\dots,i_p}$ are smooth functions. The differential $d \colon \mathcal{C}^p(M) \to \mathcal{C}^{p+1}(M)$ is a linear map, defined by additivity from the expansion as follows:

$$d\Big(f\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}\Big) = df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

where $df = \sum_{j=1}^r \frac{\partial f}{\partial x_j}\, dx_j$.

The *de Rham complex* of M is the complex,

$$\mathcal{C}_{\mathrm{dR}}(M): \qquad \cdots \to 0 \to \mathcal{C}^0(M) \to \mathcal{C}^1(M) \to \cdots \to \mathcal{C}^r(M) \to 0 \to \cdots .$$

Acyclicity of the complex is essentially commutation: $\partial^2 f/dx_i dx_j = \partial^2 f/dx_j dx_i$. The cohomology is the *de Rham cohomology* $H^p_{\mathrm{dR}}(M)$.

The degree-0 part, $\mathcal{C} := \mathcal{C}^0(M)$, is the algebra of smooth functions on M and each $\mathcal{C}^p(M)$ is a $\mathcal{C}$-module. Note that, locally, the top part $\mathcal{C}^r(M)$ is free of rank 1 as a $\mathcal{C}$-module.

The differential of a function is zero, if and only if the function is locally constant. So, the zeroth cohomology $H^0_{\mathrm{dR}}(M)$ is the vector space of locally the constant functions, of dimension equal to the number of connected components. In particular, the de Rham complex may be co-augmented with the vector space of constant functions (say $\mathbb{R}$ if we consider real valued functions on M). In other words there is a morphism of complexes $\mathbb{R}(0) \to \mathcal{C}_{\mathrm{dR}}$; its mapping cone is the *reduced de Rham complex*,

$$\tilde{\mathcal{C}}_{\mathrm{dR}}(M): \qquad \cdots \to 0 \to \mathbb{R} \to \mathcal{C}^0(M) \to \mathcal{C}^1(M) \to \cdots \to \mathcal{C}^r(M) \to 0 \to \cdots ,$$

with $\tilde{\mathcal{C}}^{-1}(M) = \mathbb{R}$. The reduced complex is exact if and only if the deRham complex $\mathcal{C}_{\mathrm{dR}}$ is a resolution of $\mathbb{R}$. According to the Poincaré Lemma, this is case when M is an open interval in $\mathbb{R}^r$ ($r \geqslant 0$); indeed, as is easily seen, then the reduced complex $\tilde{\mathcal{C}}_{\mathrm{dR}}(M)$ is contractible.

(2.2) Note. With some knowledge of sheaf theory you will realize that the de Rham complex is really a complex of sheaves on the manifold,

$$\tilde{\mathcal{C}}_{\mathrm{dR}}: \qquad \cdots \to 0 \to \mathbb{R}^{\#} \to \mathcal{C}^0 \to \mathcal{C}^1 \to \cdots \to \mathcal{C}^r \to 0 \to \cdots ,$$

where $\mathbb{R}^{\#}$ is the sheaf of locally constant functions. The reduced complex is an exact complex of sheaves on M.

3. The Euler characteristic.

(3.1) Setup. For simplicity, work with real vector spaces. A complex X is said to be *perfect*, or to *have an index*, if its cohomology is finite, that is, $H^p(X)$ is finite dimensional for all p and non-zero only for finitely many p. The *index* of a perfect complex X is the integer given as the alternating sum,

$$\chi(X) := \sum_i (-1)^i \dim H^i(X).$$

(3.2) The additivity properties of the index.

(1) *If X is perfect, then so are its shifts $X(n)$, and*

$$\chi(X(n)) = (-1)^n \chi(X).$$

(2) *Consider an exact triangle,*

$$\begin{tikzcd} & X'' \arrow[dl, squiggly] & \\ X' \arrow[rr] & & X \arrow[ul] \end{tikzcd}$$

If two of its vertices, X, X', and X'', are perfect, then so is the third, and

$$\chi(X) = \chi(X') + \chi(X'').$$

(3) *If X is a finite comples, that is, each X^p is finite dimensional and only finitely many X^p are nonzero, then X is perfect, and*

$$\chi(X) = \sum (-1)^i \dim X^i.$$

(3.3) The Euler characteristic. *If M is a compact manifold, then the de Rham complex $\mathcal{C}_{\mathrm{dR}}(M)$ is perfect, and*

$$\chi(\mathcal{C}_{\mathrm{dR}}(M)) = \chi_{\mathrm{E}}(M),$$

where $\chi_{\mathrm{E}}(M)$ is the Euler–Poincaré index, defined from a triangulation of M as the alternating sum,

$$\chi_{\mathrm{E}}(M) = \#(0\text{-simplices}) - \#(1\text{-simplices}) + \#(2\text{-simplices}) - \#(3\text{-simplices}) \pm \cdots .$$

(3.4) Example. The circle S^1 is triangulated as the boundary of the triangle: 3 vertices and 3 edges: $\chi_{\mathrm{E}}(S^1) = 3 - 3 = 0$.

The sphere S^2 is triangulated as the boundary of a tetrahedron: 4 vertices, 6 edges, and 4 faces: $\chi_{\mathrm{E}}(S^2) = 4 - 6 + 4 = 2$.

4. Oriented chains on a triangulated space.

(4.1) The affine simplices. For $p = 0, 1, 2, \dots$ consider the *affine p-simplex* Δ^p defined as a subset of $\mathbb{R}^{p+1}$:

$$\Delta^n = \{ (t_0, \dots, t_p) \mid t_i \geqslant 0,\ \sum t_i = 1\}.$$

In particular, Δ^p is a compact topological space (a metric space) with the structure induced from $\mathbb{R}^{p+1}$.

The vertices of Δ^p are the $p+1$ points denoted simply as follows:

$$0 = (1, 0, \dots, 0),\ 1 = (0, 1, \dots, 0),\ \dots,\ p = (0, 0, \dots, 1).$$

As the notation indicates, the vertices of Δ^p will often be identified with the finite set of $p+1$ integers $[p] := \{0, 1, \dots, p\}$.

Every map of sets $\varphi\colon [p] \to [q]$ induces an affine map φ_* or simply $\varphi\colon \Delta^p \to \Delta^q$; the image of a point $t \in \Delta^p$ is the point $s = \varphi(t) \in \Delta^q$ with j'th coordinate s_j defined as a sum,

$$s = \varphi(t), \qquad s_j = \sum_{i \mapsto j} t_i,$$

where the sum, as indicated, is over all $i = 0, \dots, p$ such that $\varphi(i) = j$.

In particular, for $p \geqslant 1$, let $\partial_k = \partial_k^{p-1}\colon [p-1] \to [p]$ be the strictly increasing injection avoiding k for $k = 0, \dots, p$. These $p+1$ injections induce $p+1$ affine embeddings $\partial_k\colon \Delta^{p-1} \to \Delta^p$ of Δ^{p-1} as a *face* of Δ^p.

(4.2) Regular simplices of a topological space. Let X be a topological space. A (*regular*) *p-simplex* s in X is an equivalence class of embeddings $\sigma\colon \Delta^p \hookrightarrow X$; two embeddings $\sigma, \sigma'\colon \Delta^p \hookrightarrow X$ are equivalent, if $\sigma' = \sigma\alpha$ with a permutation α of $[p]$. So s is represented by $(p+1)!$ embeddings $\sigma\colon \Delta^p \hookrightarrow X$; they have the same image in X, denoted $\bar{s}$, and they all map the set of vertices of Δ^p to the same set of $p+1$ different points of X, called the set of *vertices* of s; the set of vertices of s will be denoted $\{s\}$. Each vertex $x \in \{s\}$ has an *opposite face*, denoted s_x: if s is represented by $\sigma\colon \Delta^p \hookrightarrow X$ and $x = \sigma(k)$, then s_x is the $(p-1)$-simplex represented by $\sigma\partial_k$. The union of the faces s_x is the *boundary of* s, denote $\dot{s}$, and its complement in $\bar{s}$ is the *interior of* s, denoted $\mathring{s}$,

$$\dot{s} := \bigcup\nolimits_x s_x, \qquad \mathring{s} := \bar{s} - \dot{s}.$$

A 0-simplex in X is just a point x of X; it has one vertex, it has one (empty) face and hence empty boundary, and it is equal to its interior.

Note that the vertices of a p-simplex s is an unordered set; in fact, a choice of an order on the set $\{s\}$ of vertices, $\{s\} = \{x_0, \dots, x_p\}$, is the same as a choice of a representative $\sigma\colon \Delta^p \to X$ for s. Two representatives $\sigma, \sigma'\colon \Delta^p \to X$ of s differ by a unique permutation α of $[p]$, that is, $\sigma = \sigma'\alpha$. We will define an *orientation of* X as an equivalence class of pairs (σ, ε), where σ is a representative of s and ε is a sign, equal to ± 1. Two pairs (σ, ε)

and (σ', ε') represent the same orientation of s if the unique permutation α with $\sigma = \sigma'\alpha$ has sign equal to $\varepsilon\varepsilon'$. There are two possible orientations. If $p > 0$, an orientation may always be given by a pair with $\varepsilon = 1$, an so we may think of the two orientations as the division of the representatives of s into two classes, each of which consist of embeddings differing by an even permutation. If $p = 0$, then s is simply a point x of X, with the unique representative $x : 0 \mapsto x$, and it has two orientations: $(x, 1)$ and $(x, -1)$.

An orientation of a p-simplex s (for $p \geqslant 1$) induces an orientation of each of its faces as follows:

Consider the face s_x opposite a vertex x of s. Let the given orientation of s be determined by the pair (σ, ε), where $\sigma : \Delta^p \hookrightarrow X$ represents s. Say $x = \sigma(i)$. Then the face s_x is represented by the embedding $\sigma_i := \sigma\partial_i^p$. We orient s_x by the pair $(\sigma_i, (-1)^i\varepsilon)$. To prove that the orientation is independent of the choice, consider at second pair (σ', ε') representing the given orientation. Then $\sigma = \sigma'\alpha$ for a permutation α of $[p]$, and $\varepsilon = \varepsilon' \operatorname{sign}\alpha$. If $j = \alpha(i)$, then $\sigma'(j) = \sigma(i) = x$. Hence s_x is represented by $\sigma'_j = \sigma'\partial_j$. So we want to prove that the pairs $(\sigma_i, (-1)^i\varepsilon)$ and $(\sigma'_j, (-1)^j\varepsilon)$ determine the same orientation of s_x. Equivalently, if α_0 is the unique permutation of $[p-1]$ determined by the equation $\alpha\partial_i = \partial_j\alpha_0$, then we want to prove the equation $\operatorname{sign}\alpha_0 = \varepsilon\varepsilon'(-1)^{i+j} \operatorname{sign}\alpha$.

Check it!.

(4.3) Triangulations. Let Σ be a *triangulation* of X, that is, a collection Σ of regular simplices of X satisfying the following conditions:

(1) X is the disjoint union
$$X = \bigcup_{s \in \Sigma} \overset{\circ}{s} \ ,$$

(2) If s is in Σ then every face of s is in Σ.

(3) The map $s \mapsto \{s\}$ is injective on Σ, that is, if two simplices of σ has the same set of vertices, then they are equal.

Chose once and for all an orientation of every simplex s of Σ. There is no harm assuming that 0-simplices, the points x of X, are given their positive orientation, determined by $(x, 1)$.

Let Σ_p denote the set of p-simplices of Σ (so $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \cdots$). Define
$$C_p^{\text{orient}}(\Sigma, X, k) := k^{\oplus\Sigma_p};$$
in other words, $C_p = C_p^{\text{orient}}(\Sigma, X, k)$ is the free k-module generated by the p-simplices of Σ; its elements are formal k-linear combinations $\sum \lambda_s s$ of p-simplices s from Σ. The elements of C_p are called *oriented p-chains* on X (with respect to Σ).

Let $\partial : C_p \to C_{p-1}$ (for $p \geqslant 1$) be the k-linear map defined on the generators s of C_p as the sum,
$$\partial(s) = \sum_{x \in \{s\}} \pm s_x.$$
The sum is over the $p+1$ vertices x of s, and s_x is the face of s opposite of x. The sign in front of s_x is '+' or '−' according to whether the orientation on s_x induced from the orientation of s is equal to or opposite to the orientation of s_x as an element of Σ.

It is easy to see that the sequence,

$$\cdots \to C_p \xrightarrow{\partial} C_{p-1} \longrightarrow \cdots \longrightarrow C_0 \to 0 \to 0 \to \cdots,$$

is a complex, the *oriented chain complex* $C^{\text{orient}}(\Sigma, X)$ of the triangulation, with homology modules $H_p^{\text{orient}}(\Sigma, X)$.

The complex $C^{\text{orient}}(\Sigma, X)$ is essentially independent of the chosen orientations of Σ. If the ring k is a field, and the complex $C^{\text{orient}}(\Sigma, X)$ is perfect, then its Euler–Poincaré index is defined:

$$\chi(\Sigma, X) := \sum\nolimits_p (-1)^p \dim H_p(\Sigma, X).$$

If the triangulation Σ is finite, then each C_p is of finite dimension, equal to $|\Sigma_p|$, and so

$$\chi(\Sigma, X) = \sum\nolimits_p (-1)^p |\Sigma_p|,$$

independent of k.

(4.4) Stoke's Theorem. Assume that X in addition is a smooth manifold, and that the triangulation is smooth, that is, the simplices of Σ are represented by smooth embeddings $\sigma\colon \Delta^p \hookrightarrow X$. Note that Δ^p is a manifold with boundary: a map $\sigma\colon \Delta^p \to X$ is smooth if it is, locally on Δ^p, the restriction of a smooth map defined on an open subset of $\mathbb{R}^{p+1}$. Then a p-form $\omega \in C^p_{dR}(X)$ may be integrated over a regular p-simplex $s \in \Sigma_p$: The pull back $s^*\omega$ is a p-form on Δ^p. If we identify Δ^p with the subset of $\mathbb{R}^p$ obtained by discarding the 0'th coordinate:

$$\Delta^p = \{\, (t_1, \dots, t_p) \mid t_i \geqslant 0, \quad \sum t_i \leqslant 1\}$$

then $s^*\omega$ is a p-form in the variables $t_1, \dots, t_p$, and hence of the form $f\, dt_1 \wedge \cdots \wedge dt_p$. We set

$$\langle \omega, s\rangle := \int_s \omega := \int_{\Delta^p} f dt_1 \cdots dt_p.$$

Extending by linearity from p-simplices $s \in \Sigma_p$ to arbitrary chains in $C_p^{\text{orient}}(\Sigma, X)$, we obtain a pairing (a bilinear form),

$$\mathcal{C}^p_{\text{dR}}(X) \times C_p^{\text{orient}}(\Sigma, X) \to \mathbb{R},$$

or, equivalently, a linear map from the vector space of p-forms to the dual of the vector space of oriented p-chains,

$$\mathcal{C}^p_{\text{dR}}(X) \to \big(C_p^{\text{orient}}(\Sigma, X)\big)^*.$$

It is a consequence of Stoke's Theorem that the linear maps, for varying p, form a map of complexes,

$$\mathcal{C}_{\text{dR}}(X) \to \big(C^{\text{orient}}(\Sigma, X)\big)^*.$$

5. Combinatorial chains on a triangulated space.

(5.1) Combinatorial simplices. Let X be a topological space with a given triangulation Σ. Then a *combinatorial p-simplex* of X (or better, of (X, Σ)), is a $(p+1)$-tuple $(x_0, \dots, x_p)$ of points of X such that $\{x_0, \dots, x_p\}$ is the set of vertices of some (regular) simplex s of Σ, necessarily uniquely determined by the points x_i. It is not assumed that the x_i are different, and so s may be a q-simplex for some $q \leqslant p$. If $q < p$ then the combinatorial simplex is called *degenerate*. Note that there are degenerate combinatorial p-simplices of any dimension p. Let Σ_p^{comb} be the set of combinatorial p-simplices, and let,

$$C_p^{\text{comb}}(\Sigma, X, k) := k^{\oplus \Sigma_p^{\text{comb}}}.$$

Let $\partial\colon C_p \to C_{p-1}$ (for $p \geqslant 1$) be the k-linear map defined on the generators $(x_1, \dots, x_p)$ of C_p as the sum,

$$\partial(x_1, \dots, x_p) = \sum_{i=0}^{p} (-1)^i (x_1, \dots, \widehat{x_i}, \dots, x_p),$$

where the hat indicates an omitted coordinate.

It is easy to see that the sequence,

$$\cdots \to C_p \xrightarrow{\partial} C_{p-1} \longrightarrow \cdots \longrightarrow C_0 \to 0 \to 0 \to \cdots,$$

is a complex, the *combinatorial chain complex* $C^{\text{comb}}(\Sigma, X)$ of the triangulation, with homology modules $H_p^{\text{comb}}(\Sigma, X)$.

(5.2) Comparison. A linear map $h\colon C_p^{\text{comb}}(\Sigma, X) \to C^{\text{orient}}(\Sigma, X)$ is defined by the following values on the generators $(x_0, \dots, x_p) \in \Sigma_p^{\text{comb}}$. If $(x_0, \dots, x_p)$ is degenerate, $h(x_0, \dots, x_p) = 0$. Otherwise $\{x_0, \dots, x_p\}$ is the set of vertices of a regular p-simplex $s \in \Sigma$. There is a unique representative $\sigma\colon \Delta^p \hookrightarrow X$ of s such that $x_i = \sigma(i)$ for $i = 0, \dots, p$. Set

$$h(x_0, \dots, x_p) := \pm s$$

where the sign equals '$+$' if $(\sigma, 1)$ determines the given orientation of s, and equals '$-$' otherwise.

Note that the function $h(x_0, \dots, x_p)$ is alternating in its domain of definition: for $i \neq j$ the function vanishes if $x_i = x_j$ and it changes sign if x_i and x_j are interchanged.

It is easy to see that the linear maps, from combinatorial chains to oriented chains, form a map of complexes,

$$C^{\text{comb}}(\Sigma, X) \to C^{\text{orient}}(\Sigma, X)\big).$$

6. Singular chains on a topological space.

(6.1) Singular simplices. Let X be a topological space. A *singular p-simplex* on X is a continuous map $\sigma\colon \Delta^p \to X$. The set of singular p-simplices is denoted $\Delta_p(X)$, that is,

$$\Delta_p(X) := \mathrm{Hom}_{\mathrm{cont}}(\Delta^p, X).$$

For each map $\varphi\colon [q] \to [p]$ there is a continuous map $\varphi\colon \Delta^q \to \Delta^p$ and hence an induced map of sets $\Delta_p(X) \to \Delta_q(X)$ given by $\sigma \mapsto \sigma\varphi$. In particular, the maps $\partial_i^{p-1}\colon [p-1] \to [p]$ (for $i = 0, \dots, p$) induce maps $\sigma \mapsto \sigma\partial_i^p$ from $\Delta_{p-1}(X)$ to $\Delta_{p-1}(X)$. The $(p-1)$-simplex $\sigma_i := \sigma\partial_i$ is the i'th face of σ.

Let $C_p = C_p^{\mathrm{sing}}(X, k)$ be the free module generated by the singular p-simplices,

$$C_p^{\mathrm{sing}}(X, k) := k^{\oplus\Delta_p(X)},$$

and let $\partial\colon C_p \to C_{p-1}$ (for $p \geqslant 1$) be the k-linear map given on the generators σ of C_p as the sum,

$$\partial(\sigma) = \sum_{i=0}^{p}(-1)^i\sigma_i.$$

It is easily seen that the following sequence is a complex, the *singular chain complex* of X:

$$\cdots \to C_p \xrightarrow{\partial} C_{p-1} \longrightarrow \cdots \longrightarrow C_0 \to 0 \to 0 \to \cdots.$$

The homology of the complex is the *singular homology* of X with coefficients in k,

$$H_p^{\mathrm{sing}}(X, k) := H_p(C^{\mathrm{sing}}(X, k)).$$

(6.2). Assume that a triangulation Σ of X is given. Let $(x_0, \dots, x_p)$ be a combinatorial p-simplex. The the set $\{x_0, \dots, x_p\}$ is the set of vertices of a unique regular q-simplex s in Σ. Chose a representative $\sigma\colon \Delta^q \to X$ of s. Then there is a unique surjective map $\varphi\colon [p] \to [q]$ such that $x_i = \sigma(\varphi i)$ for $i = 0, \dots, p$. Moreover, the composition $\sigma\varphi\colon \Delta^p \to \Delta^q \to X$, which is a singular p-simplex in X, is independent of the choice of representative σ. So there is a well defined map $\Sigma_p^{\mathrm{comb}} \to \Delta^(X)$ from the set of combinatorial p-simplices to the set of singular p-simplices. It induces a linear map $C_p^{\mathrm{comb}}(\Sigma, X, k) \to C_p^{\mathrm{sing}}(X, k)$, and in fact a map of complexes,

$$C^{\mathrm{comb}}(\Sigma, X, k) \to C^{\mathrm{sing}}(X, k).$$

7. Fundamental theorems of singular homology.

(7.1) The reduced singular chain complex. Let X be a topological space. The definition of Δ^p and $\Delta_p(X)$ makes sense also when $p = -1$: The affine simplex Δ^{-1} is the empty set (defined as the empty subset of $\mathbb{R}^0$), and there is exactly one (-1)-simplex $\emptyset \to X$. Hence $C^{\mathrm{sing}}_{-1}(X, k) = k$, and we may consider the augmented complex $\widetilde{C} = \widetilde{C}^{\,\mathrm{sing}}(X, k)$,

$$\widetilde{C}: \quad \cdots \to C_p \xrightarrow{\partial} C_{p-1} \longrightarrow \cdots \longrightarrow C_0 \to C_{-1} \to 0 \to \cdots .$$

Its homology, denoted $\widetilde{H}^{\mathrm{sing}}_p(X, k)$, is the *reduced singular homology* of X.

Note that $C(X)$ and $\widetilde{C}(X)$ are covariant as functors of X: A continuous map $f\colon X \to Y$ induces maps of sets $\Delta_p(X) \to \Delta_p(Y)$, defined by $\sigma \mapsto f\sigma$, and hence maps of free modules $\widetilde{C}_p(X, k) \to \widetilde{C}_p(Y, k)$, forming a map of complexes $\widetilde{C}(X, k) \to \widetilde{C}(Y, k)$.

(7.2) Fundamental theorems.

Theorem I, the extremes. $\widetilde{C}(\emptyset) = k(-1)$, $\widetilde{C}(\mathrm{pt}) \backsimeq 0$.

Theorem II, the homotopy axiom. *A homotopy of maps $f_0 \backsimeq f_1\colon X \to Y$, induces a homotopy of chain maps $\widetilde{C}(f_0) \simeq \widetilde{C}(f_1)\colon \widetilde{C}(X) \to \widetilde{C}(Y)$.*

Theorem III, the axiom of small simplices. *Assume that $X = \bigcup_{U\in\mathcal{U}} U$ is an open covering of X. Let $\widetilde{C}(\mathcal{U}, X)$ be the subcomplex of $\widetilde{C}(X)$ generated by singular simplices $\sigma\colon \Delta^p \to X$ which are $\mathcal{U}$-small, that is, have their image contained in U for some $U \in \mathcal{U}$. Then the inclusion $\widetilde{C}(\mathcal{U}, X) \to \widetilde{C}(X)$ is a homotopy equivalence.*

Theorem IV, the Künneth formula. *For any two spaces X, Y there is a homotopy equivalence $C(X \times Y) \backsimeq C(X) \otimes C(Y)$.*

(7.3) And their consequences.

A. The equivalence theorem. *A homotopy equivalence $X \backsimeq Y$ induces a homotopy equivalence $\widetilde{C}(X) \backsimeq \widetilde{C}(Y)$.*

B. The Mayer–Vietoris theorem. *For open subsets $U_1, U_2 \subseteq X$ there is an exact triangle in the homotopy category,*

$$\begin{tikzcd}
 & \widetilde{C}(U_1 \cup U_2) \arrow[dl, squiggly] & \\
\widetilde{C}(U_1 \cap U_2) \arrow[rr] & & \widetilde{C}(U_1) \oplus \widetilde{C}(U_2). \arrow[ul]
\end{tikzcd}$$

C. The cone theorem. *Let $f : X \to Y$ be a continuous map, and let $Z_f = \mathrm{Con}(f)$ denote the mapping cone of f, see below. Then there is a homotopy equivalence,*

$$\widetilde{C}(\mathrm{Con}(f)) = \mathrm{Con}(\widetilde{C}(f)),$$

In particular, there is an exact triangle in the homotopy category,

$$\begin{tikzcd}
 & \widetilde{C}(Z_f) \arrow[dl, squiggly] & \\
\widetilde{C}(X) \arrow[rr, "C(f)"] & & \widetilde{C}(Y). \arrow[ul]
\end{tikzcd}$$

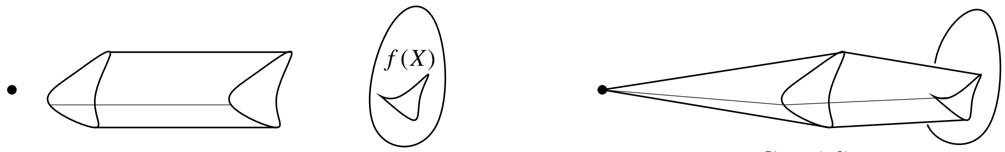
D. The suspension theorem. *Let $X(1) = SX$ denote the suspension of X, see below. Then there is a homotopy equivalence*

$$\widetilde{C}(X(1)) \backsimeq \widetilde{C}(X)(1).$$

(7.4) Cone and suspension. Let $f : X \to Y$ be a map. Then the *mapping cone* of f, denoted $Z = \mathrm{Con}(f)$ is the topological space,

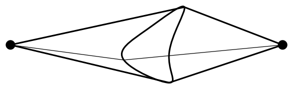
$$\mathrm{Con}(f) := \quad \frac{\{p_0\} \cup \; X \times I \; \cup Y}{p_0 = (x, 0), \;\; (x, 1) = fx},$$

obtained from the disjoint union of a point p_0, the product $X \times [0, 1]$, and the space Y, by identifying for all x the point $(x, 0)$ with p_0 and the point $(x, 1)$ with fx in Y.



p_0 $X \times I$ Y $\mathrm{Con}(f)$

The mapping cone of the constant map $X \to \mathrm{pt}$ is the (double) *suspension* of X, denoted SX or $X(1)$,



$SX = X(1)$

For pointed spaces the definitions are similar: The pointed cone, or simply the cone, of a pointed map,

$$\mathrm{Con}(f) := \quad \frac{X \times I \; \cup \; Y}{(x, 0) = (x_0, t), \, (x, 1) = fx},$$

and $SX :=$ (pointed) cone of constant map $X \to \mathrm{pt}$.

(7.5) Example. For $n \geqslant 0$ there is an explicit identification of the pointed suspension of the n-sphere,

$$S\,S^n = S^{n+1}. \tag{7.5.1}$$

Indeed, an isomorphism is given af follows: Let D^n be the equatorial disk of S^{n+1}, with boundary S^n, let $p_0 = (1, 0, \dots, 0)$ be the base point and $n = (0, \dots, 0, 1)$ the North pole. Let $\pi_+ \colon D^n \to S^n_+$ be the inverse stereographic projection centered at the South pole $-n$, from the disk onto the Northern hemisphere; it is given, for points x in the equatorial hyperplane orthogonal to n, by the formula $\pi_+(x) = (\lambda - 1)n + \lambda x$, where $\lambda = 2/(1 + |x|^2)$. Then the isomorphism (7.5.1) is given by the expression,

$$(z, s) \mapsto \begin{cases} \pi_+(\mu_t z), & t = 2s, \qquad\ \ \text{for } 0 \leqslant s \leqslant \frac{1}{2}, \\ \pi_-(\mu_t z), & t = 2 - 2s, \ \ \text{for } \frac{1}{2} \leqslant s \leqslant 1; \end{cases}$$

here μ_t is the multiplication $\mu_t z = tz + (1 - t)p_0$ around the center p_0, and π_- is the projection onto the Southern hemisphere.

Stereographic projection preserves spheres. In particular, under the isomorphism the sphere $S^n \times s \subset S\,S^n$ is mapped to a small sphere in S^{n+1}. In fact, the image $\pi_+\mu_t(S^n)$ is the intersection of S^{n+1} and the hyperplane through p_0 orthogonal to $tn + (1 - t)p_0$.

A direct proof of the latter fact is a simple computation: Let

$$x := \mu_t z = tz + (1 - t)p_0, \quad \text{and} \quad u = \pi_+ x = (\lambda - 1)n + \lambda x,$$

where $\lambda = 2/(1 + |x|^2)$. Then,

$$\begin{aligned} &(u - p_0) \cdot n = \lambda - 1, \\ &(u - p_0) \cdot p_0 = \lambda x \cdot p_0 - 1 = -\lambda t\big(1 - z \cdot p_0\big) + (\lambda - 1). \end{aligned}$$

Clearly,

$$|x|^2 = t^2 + (1 - t)^2 + 2t(1 - t)\, z \cdot p_0 = 1 - 2t(1 - t)\big(1 - z \cdot p_0\big).$$

Set $c := 1 + |x|^2$. Then $c\lambda = 2$, and $c(\lambda - 1) = 1 - |x|^2 = 2t(1 - t)\big(1 - z \cdot p_0\big)$. Hence,

$$\begin{aligned} &c\,(u - p_0) \cdot n = 2t(1 - t)\big(1 - z \cdot p_0\big), \\ &c\,(u - p_0) \cdot p_0 = -2t\big(1 - z \cdot p_0\big) + 2t(1 - t)\big(1 - z \cdot p_0\big) = -2t^2\big(1 - z \cdot p_0\big). \end{aligned}$$

As a consequence, $u - p_0$ is orthogonal to $tn + (1 - t)p_0$, as asserted. $\square$

8. Homology and cohomology of the spheres.

(8.1) Example. It may be proved for the *punctured plane* $\dot{\mathbb{R}}^2 := \mathbb{R}^2 \setminus \{0\}$ that de Rham cohomology $H_{\mathrm{dR}}(\dot{R}^2)$ is given by $H^p = \mathbb{R}$ for $p = 0, 1$ and $H^p_{\mathrm{dR}} = 0$ for all other p. Equivalently, the reduced de Rham cohomology is given by $\widetilde{H}^*_{\mathrm{dR}} = \mathbb{R}(-1)$. More generally, reduced de Rham cohomology of the punctured $(n+1)$-space is given by the equation,

$$\widetilde{H}^*_{\mathrm{dR}}(\dot{\mathbb{R}}^{n+1}) = \mathbb{R}(-n) \tag{8.1.1}$$

It follows that up to quasi-isomorphism, $\widetilde{\mathcal{C}}_{\mathrm{dR}}(\dot{\mathbb{R}}^{n+1}) = \mathbb{R}(-n)$.

(8.2) Example. The affine n-simplex Δ^n with its obvious triangulation has $H_0^{\mathrm{orient}} = H_0^{\mathrm{comb}} = k$ and $H_p = 0$ for all other p, that is,

$$H_*^{\mathrm{orient}}(\Delta^n, k) = H_*^{\mathrm{orient}}(\Delta^n, k) = k(0). \tag{8.2.1}$$

In fact, it is easy to see that the reduced complexes $\widetilde{C}^{\mathrm{orient}}(\Delta^n, k)$ and $\widetilde{C}^{\mathrm{orient}}(\Delta^n, k)$ are contractible.

Work with the oriented complexes $\widetilde{C} = \widetilde{C}^{\mathrm{orient}}$, and consider the boundary $\dot{\Delta}^{n+1}$ of the $(n+1)$-simplex. Its triangulation is obtained from the triangulation of Δ^{n+1} by omitting the top simplex, corresponding to the identity $\Delta^{n+1} \to \Delta^{n+1}$. So the two complexes $\widetilde{C}(\Delta^{n+1}, k)$ and $\widetilde{C}(\dot{\Delta}^{n+1}, k)$ agree in degrees $p \leqslant n$. Moreover, under the identifications $C_{n+1}(\Delta^{n+1}) = k$ and $C_n(\Delta^{n+1}) = C_n(\dot{\Delta}^{n+1})$, the $(n+1)$'st boundary map $\partial_{n+1}\colon$ may be viewed as a linear map $\partial : k \to C_n(\dot{\Delta}^{n+1}, k)$. Equivalently, ∂ may be viewed as a map of complexes $\partial\colon k(n) \to \widetilde{C}(\dot{\Delta}^{n+1}, k)$, and $\widetilde{C}(\Delta^{n+1}, k)$ is the mapping cone of ∂. As the cone is contractible, it follows that ∂ defines a homotopy equivalence,

$$k(n) \xrightarrow{\ \simeq\ } \widetilde{C}^{\mathrm{orient}}(\dot{\Delta}^{n+1}, k). \tag{8.2.2}$$

A similar result may be proved for the combinatorial simplex $\widetilde{C}^{\mathrm{orient}}(\dot{\Delta}^{n+1}, k)$.

(8.3) Example. The n-sphere $S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ may be defined inductively as a suspension: $S^{-1} := \emptyset$ and $S^n := S(S^{n-1})$ for $n \geqslant 0$. Hence, by EXPL(7.2)I and (7.3)D, we have

$$\widetilde{C}^{\mathrm{sing}}(S^n) \simeq k(n). \tag{8.3.1}$$

Note that, up to homotopy,

$$\dot{\mathbb{R}}^{n+1} \simeq \dot{\Delta}^{n+1} \simeq S^n.$$

(8.4) Example. Real projective n-space $I\!P^n = I\!P^n(\mathbb{R})$ may be defined as the quotient $I\!P^n := S^n/\pm 1$ of S^n modulo the cyclic group $\mathrm{C}_2 = \pm 1$ acting via the antipodal involution.

The quotient map $S^n \to \mathbb{P}^n$ maps the equatorial S^{n-1} onto a $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$, and, clearly, $\mathbb{P}^n$ is the mapping cone of the quotient map $S^{n-1} \to \mathbb{P}^{n-1}$. Consequently, there is an exact triangle of reduced chain complexes in the homotopy category,

$$\begin{tikzcd}
& \widetilde{C}^{\mathrm{sing}}(\mathbb{P}^{n+1}, k) \arrow[dl, squiggly] & \\
\widetilde{C}^{\mathrm{sing}}(S^n, k) \arrow[rr] & & \widetilde{C}^{\mathrm{sing}}(\mathbb{P}^n, k). \arrow[ul]
\end{tikzcd}$$

Now, $\widetilde{C}^{\mathrm{sing}}(S^n, k) = k(n)$. It follows by induction on n that there is a chain complex of the form

$$\widetilde{D}: \qquad \cdots \to k \xrightarrow{d_3} k \xrightarrow{d_2} k \longrightarrow 0 \to 0 \to \cdots,$$

with the rightmost k in homological degree 1, and a homotopy equivalence,

$$\widetilde{C}^{\mathrm{sing}}(\mathbb{P}^n, k) \simeq \widetilde{D}_{\leqslant n},$$

where $\widetilde{D}_{\leqslant n} = \widetilde{D}^{\geqslant -n}$ is the n'th chain truncation of $\widetilde{D}$. For the non reduced complex there is a similar homotopy equivalence,

$$C^{\mathrm{sing}}(\mathbb{P}^n, k) \simeq D_{\leqslant n},$$

where D is obtained from $\widetilde{D}$ by replacing 0 by k in degree 0. It may by proved that the maps d_i for odd i vanish and for even i are multiplication by 2. So D has the following form,

$$\widetilde{D}: \qquad \cdots \to k \xrightarrow{2} k \xrightarrow{0} k \xrightarrow{2} k \xrightarrow{0} k \to 0 \to \cdots,$$

with the rightmost k in degree 0. Note in particular that $H^n(\mathbb{P}^n, k) = k$ when n is odd and $H^n(\mathbb{P}^n, k) = {}_2k$ when n is even.

9. Some comparison theorems.

(9.1). Let Σ be a triangulation of a topological space X, with each $s \in \Sigma$ oriented. For a combinatorial simplex $(x_0, \dots, x_p)$ of (X, Σ), denote by $s = t(x_0, \dots, x_p)$ the regular simplex in Σ with vertices $\{x_0, \dots, x_p\}$; the combinatorial simplex is degenerated if $s \in \Sigma_n$ with $n < p$. Let $\tilde{t}\colon C_p^{\mathrm{comb}} \to C_p^{\mathrm{orient}}$ be the linear map given on generators by the formula,

$$\tilde{t}(x_0, \dots, x_p) = \varepsilon t(x_0, \dots, x_p),$$

where ε is 0 if $(x_0, \dots, x_p)$ is degenerated and, in the non degenerated case, ε is $+1$ or -1 according as the orientation of $t(x_0, \dots, x_p) \in \Sigma$ is equal to or opposite to the orientation given by the ordering $x_0, \dots, x_p$ of the vertices.

It is easily seen that $\tilde{t}$ is a map of complexes,

$$\tilde{t}\colon C^{\mathrm{comb}}(X, \Sigma, k) \to C^{\mathrm{orient}}(X, \Sigma, k); \tag{9.1.1}$$

it is in fact a homotopy equivalence (The Principle of alternating degeneration).

(9.2). On the other hand, associate with the combinatorial p-simplex $(x_0, \dots, x_p)$ in (X, Σ) the singular p-simplex

$$\tau(x_0, \dots, x_p)\colon \Delta^p \to X$$

defined as follows: Let $n + 1$ be the cardinality of the set $\{x_0, \dots, x_p\}$, and let $s = t(x_0, \dots, x_p) \in \Sigma_n$ be the regular n-simplex determined by the set of vertices x_i. Let $\sigma\colon \Delta^n \to X$ be a representative of s, determined by some ordering of the $n + 1$ vertices in s, say $\{y_0, \dots, y_n\} = \{x_1, \dots, x_p\}$. Then there is a corresponding surjection $\beta\colon [p] \to [n]$ such that $x_i = y_{\beta i}$, and the composition $\tau = \sigma\beta\colon \Delta^p \to \Delta^n \to X$ is independent of σ. It is easy to see that the map determined by $(x_0, \dots, x_p) \mapsto \tau(x_1, \dots, x_p)$ is a map of complexes,

$$\tau\colon C^{\mathrm{comb}}(X, \Sigma, k) \to C^{\mathrm{sing}}(X, k). \tag{9.2.1}$$

(9.3) Lemma. *If Σ is a finite triangulation of X, then the map τ of* (9.2.1) *is a homotopy equivalence.*

Proof. If $X = \Delta^n$, with the natural triangulation, then via the augmentations, both complexes are homotopy equivalent to $k(0)$. Indeed, the reduced singular complex is contractible, because Δ^n is contractible (isotopic to a point), and for the reduced combinatorial complex of Δ^n a contraction γ is defined as follows: The n-simplex Δ^n has $n + 1$ vertices (you may identify them with the numbers $0, 1, \dots, n$) and a combinatorial simplex is an arbitrary sequence $(x_0, \dots, x_p)$ (with $p \geqslant -1$) of these vertices. Fix a vertex y and define $\gamma\colon C_p^{\mathrm{comb}} \to C_{p+1}^{\mathrm{comb}}$ by

$$\gamma(x_0, \dots, x_p) = (y, x_0, \dots, x_p);$$

it is easy to check that $\partial\gamma + \gamma\partial = 1$. Hence the reduced combinatorial complex is $\tilde{C}^{\mathrm{comb}}$ is contractible.

So, the augmentations of the source and target of τ are homotopy equivalences. Since τ respects the augmentations, it follows that τ is a homotopy equivalence.

In the general case, proceed by induction on the cardinality of Σ. Chose $s \in \Sigma_n$ with maximal n. Then the subspaces $X_1 := \bar{s}$, $X_2 := X - \mathring{s}$, and $X_0 := X_1 \cap X_2$, are naturally triangulated, and $X = X_1 \cup X_2$. For the pair (X_1, X_2) there is an obvious Meyer–Vietoris triangle for the combinatorial complexes C^{comb}. Assume there is a similar sequence for the singular complexes C^{sing}. Then τ is a morphism of triangles. By induction, τ is a homotopy equivalence at two of the vertices,

$$C^{\text{comb}}(X_1 \cap X_2) \xrightarrow{\sim} C^{\text{sing}}(X_1 \cap X_2), \quad \text{and}$$
$$C^{\text{comb}}(X_1) \oplus C^{\text{comb}}(X_2) \xrightarrow{\sim} C^{\text{sing}}(X_1) \oplus C^{\text{sing}}(X_2).$$

Hence, at the third vertex τ is a homotopy equivalence,

$$C^{\text{comb}}(X) \xrightarrow{\sim} C^{\text{sing}}(X),$$

To obtain the Meyer–Vietoris triangle for the (closed) pair, we use a metric on X defined as follows: Clearly, for any simplex $s \in \Sigma$, there is a well-defined distance $\operatorname{dist}_s(x, y)$ defined for points $x, y \in \bar{s}$, independent of s. So define $\operatorname{dist}(x, y) := \operatorname{dist}_s(x, y)$ if x, y both belong to $\bar{s}$ for some $s \in \Sigma$, and $\operatorname{dist}(x, y) := \infty$ otherwise. Then the subset $U_0 := \{x \mid \operatorname{dist}(x, X_0) < \varepsilon\}$ is open, and so are the subsets $U_1 := X_1 \cup U_0$ and $U_2 := X_2 \cup U_0$. If ε is small (less than the distance from the center of Δ^n to the boundary), then the inclusions $X_0 \hookrightarrow U_0$, $X_1 \hookrightarrow U_1$, and $X_2 \hookrightarrow U_2$, are homotopy equivalences. Hence the required Meyer–Vietoris triangle for (X_1, X_2) is obtained from the triangle for the open pair (U_1, U_2), cf. EXPL(7.2)B. ▯

(9.4). Assume that Σ is a $\mathcal{C}_\infty$-triangulation of a $\mathcal{C}_\infty$-manifold (so for a simplex $s \in \Sigma$ the representatives $\sigma\colon \Delta^p \hookrightarrow X$ are $\mathcal{C}_\infty$-mappings). Then a p-form $\omega \in \mathcal{C}^p(X)$ can be integrated over a p-simplex s (oriented as usual); the result is the integral,

$$\langle \omega, s\rangle = \int_\sigma \omega.$$

Accordingly there is a pairing $\mathcal{C}^p(X) \otimes_{\mathbb{R}} C_p^{\text{orient}}(X, \Sigma, \mathbb{R}) \to \mathbb{R}$, or, equivalently, an $\mathbb{R}$-linear map,

$$\mathcal{C}^p(X) \to C^p_{\text{orient}}(X, S, \mathbb{R}), \tag{9.4.1}$$

where C^p_{orient} on the right side is the dual of the vector space C_p^{orient}. By Stoke's Theorem, the maps (9.4.1) define a map of complexes,

$$\mathcal{C}^\bullet_{\text{dR}}(X) \to C^\bullet_{\text{orient}}(X, S, \mathbb{R}). \tag{9.4.2}$$

It is the contents of de Rham's theorem, for a finite triangulation Σ, that the map (9.4.2) is a quasi-isomorphism. The dual of the homotopy equivalence (9.2.1) is a homotopy equivalence $C^\bullet_{\text{sing}}(X, \mathbb{R}) \xrightarrow{\sim} C^\bullet_{\text{orient}}(X, \Sigma, \mathbb{R})$; so there is an inverse homotopy equivalence $C^\bullet_{\text{orient}}(X, \Sigma, \mathbb{R}) \xrightarrow{\sim} C^\bullet_{\text{sing}}(X, \mathbb{R})$. Whence, by de Rham's Theorem, there is a quasi-isomorphism,

$$\mathcal{C}^\bullet_{\text{dR}}(X) \to C^\bullet_{\text{sing}}(X, \mathbb{R}). \tag{9.4.3}$$

10. Some limits.

(10.1). Let I be a partially ordered set. Let $i \mapsto \mathfrak{X}_i$ be an I-system in the category **Sets** such that the morphisms of the system are inclusions of subsets. To be precise, assume there is given a set E, and that $i \mapsto \mathcal{X}_i$ is an I-system in the category $\mathcal{P}(E)$ of subsets of E (with inclusions as morphisms). Clearly,

$$\varinjlim_i \mathcal{X}_i = \bigcup_i \mathcal{X}_i, \qquad \varprojlim_i \mathcal{X}_i = \bigcap_i \mathcal{X}_i, \quad \text{in the category } \mathcal{P}(E).$$

However, if the system is considered in the category **Sets**, then the equality $\varinjlim \mathcal{X}_i = \bigcup \mathcal{X}_i$ holds in general only if I is filtering, and the equality $\varprojlim \mathcal{X}_i = \bigcap \mathcal{X}_i$ holds in general only if I is connected.

Assume instead that $i \mapsto \mathcal{X}_i$ is a system of quotients with projections, that is, assume the following: Let E be a fixed set, and let $\mathfrak{Q}(E)$ be the category of quotients of E: An object of $\mathfrak{Q}(E)$ is a quotient E/R of E modulo an equivalence relation R, and there is a morphism $E/R' \to E/R''$ if and only if $R' \subseteq R''$. Let $i \mapsto \mathcal{X}_i = E/R_i$ be an I-system in $\mathfrak{Q}(E)$. Clearly,

$$\varinjlim_i E/R_i = E/\textstyle\bigcup_i R_i, \qquad \varprojlim_i E/R_i = E/\textstyle\bigcap_i R_i, \quad \text{in the category } \mathcal{P}(E).$$

If the system is considered in the category **Sets**, then the equality $\varinjlim E/R_i = \varinjlim E/\bigcup R_i$ holds in general. In contrast, the $\varprojlim E/R_i = \varprojlim E/\bigcap R_i$ is requires strong conditions on the system.

11. The derived functors of the limit functor.

(11.1) Setup. Let I be a (small) index category, and $\mathfrak{A}$ an abelian category with exact $\prod_I$'s. Then the limit is a functor,

$$\varprojlim_I \colon \mathfrak{A}^I \to \mathfrak{A};$$

it associates with an I-system $\mathcal{X}\colon I \to \mathfrak{A}$ the limit $\varprojlim_i \mathcal{X}_i$. We will describe in (11.5) below a resolvent complex for the limit functor. Its construction is a standard construction in simplicial cohomology. Parts of that theory is sketched in (11.2) and (11.4).

(11.2). Denote by $|I|$ the set of objects of I, viewed as a discrete category. Then, corresponding to the inclusion $|I| \hookrightarrow I$, there is a forgetful restriction functor $\lambda\colon \mathfrak{A}^I \to \mathfrak{A}^{|I|}$; it associates with an I-system $\mathcal{X}\colon I \to \mathfrak{A}$ the family $\{\mathcal{X}_i\}$ of objects (with no transition morphisms).

The forgetful functor λ has a right adjoint ρ:

$$\mathfrak{A}^I \underset{\rho}{\overset{\lambda}{\rightleftarrows}} \mathfrak{A}^{|I|}.$$

It is defined as follows: For $i \in I$ we let

$$(\rho\mathcal{Y})_i := \prod_{i\to j} \mathcal{Y}_j;$$

the index set for the product is the set of morphisms $\alpha\colon i \to j$ in I *from the given object* i; it is also denoted i/I. The morphisms in the I-system $\rho\mathcal{Y}$ are obtained by simple projections: A morphism $\gamma\colon i \to k$ in I induces a map sets $k/I \to i/I$ (defined by $\alpha \mapsto \alpha\gamma$), from the index set used for the product $(\rho\mathcal{Y})_k$ to the index set used for $(\rho\mathcal{Y})_i$. Accordingly, there is a morphism induced by γ:

$$\gamma\colon (\rho\mathcal{Y})_i = \prod_{i\to j} \mathcal{Y}_j \to (\rho\mathcal{Y})_k = \prod_{k\to j} \mathcal{Y}_j, \qquad \text{given by} \quad \mathrm{pr}_\alpha\, \gamma = \mathrm{pr}_{\alpha\gamma} \ \text{ for } \alpha\colon k \to j.$$

It is easy to see that $\rho\mathcal{Y}$ is an I-system, and that ρ is a functor. A direct system $\mathcal{X}$ in $\mathfrak{A}^I$ of the form $\mathcal{X} = \rho\mathcal{Y}$ with an I-family $\mathcal{Y}$ will be called a *trivial* or *co-induced* system. It is easy to determine the limit of a trivial system:

$$\varprojlim_{i\in I} (\rho\mathcal{Y})_i = \prod_{i\in I} \mathcal{Y}_i. \tag{11.2.1}$$

Do it!

(11.3). For the composition of functors $\rho\lambda\colon \mathfrak{A}^I \to \mathfrak{A}^{|I|} \to \mathfrak{A}^I$ there is a natural "adjunction" morphism,

$$\epsilon\colon \mathcal{X} \to \rho\lambda\mathcal{X}, \tag{11.3.1}$$

(the unit of the adjunction) from the system $\mathcal{X}$ to the trivial system $\rho\lambda\mathcal{X}$. The morphism ϵ is, at an index i, the morphism,

$$\epsilon_i\colon \mathcal{X}_i \to \prod_{i\to j} \mathcal{X}_j, \tag{11.3.1}$$

whose projection $\mathrm{pr}_\varphi\,\epsilon_i$ at an index $\varphi\colon i \to j$ of the product is equal to the transition morphism $\varphi_*\colon X_i \to X_j$. The identity $1 = 1_i$ of i is among the indices, and $\mathrm{pr}_1\,\epsilon_i$ is the identity of X_i. So pr_1 is a retraction for ϵ_i. In particular, ϵ_i is a monomorphism for every i. Therefore, the morphism (11.3.1) is a monomorphism of systems.

In particular, every I-system $\mathcal{X}$ in $\mathfrak{A}$ admits a monomorphism into a trivial I-system.

(11.4) The standard complex. Let $\pi := \rho\lambda$ be the composition $\mathfrak{A}^I \to \mathfrak{A}^{|I|} \to \mathfrak{A}^I$. Then, with the compositions of functors,

$$\pi^{[n]} := \overbrace{\pi\,\pi\,\cdots\pi}^{n+1},$$

there is an associated coaugmented "standard complex", with $\pi^{[p]}\mathcal{X}$ in degree p:

$$\tilde{\pi}\mathcal{X}\colon \qquad 0 \to \mathcal{X} \xrightarrow{\epsilon} \pi^{[0]}\mathcal{X} \to \pi^{[1]}\mathcal{X} \to \pi^{[2]}\mathcal{X} \to \cdots. \tag{11.4.1}$$

It may be described explicitely as follows: In degree p,

$$\big(\pi^{[p]}\mathcal{X}\big)_i = \prod_{i\to j_p\to\ \cdots\ \to j_0} \mathcal{X}_{j_0}.$$

The index set is the set of all composable p-strings $(\varphi_0, \dots, \varphi_p)$ from the given index i:

$$i \xrightarrow{\varphi_p} j_p \xrightarrow{\varphi_{p-1}} \cdots \xrightarrow{\varphi_0} j_0.$$

The differential $d^p\colon \pi^{[p]}\mathcal{X} \to \pi^{[p+1]}\mathcal{X}$ in the complex is determined by its projections onto the factors of the target. Corresponding to the index $i \xrightarrow{\varphi_{p+1}} j_{p+1} \xrightarrow{\varphi_p} \cdots \xrightarrow{\varphi_0} j_0$ the projection is given by the expression,

$$\mathrm{pr}_{\varphi_0,\dots,\varphi_{p+1}}\, d = \varphi_0\ \mathrm{pr}_{\varphi_1,\dots,\varphi_{p+1}} + \sum_{\nu=1}^{p+1}(-1)^\nu\,\mathrm{pr}_{\varphi_0,\dots,\varphi_{\nu-1}\varphi_\nu,\dots,\varphi_{p+1}}.$$

The indices on the projections in the sum are the strings obtained from the given string by replacing to consecutive morphisms by their composition.

It is a general fact that *the standard complex* (11.4.1) *is contractible when $\mathcal{X}$ is a trivial system*, that is, if $\mathcal{X} = \rho\mathcal{Y}$ for some family $\mathcal{Y}$.

As every I-system admits a monomorphism into a trivial I-system, and the functor π, and hence all the functors $\pi^{[n]}$ are exact, we conclude that the standard complex (11.4.1) defines a resolution of the identity of $\mathfrak{A}^I$. In fact, for any additive functor $T\colon \mathfrak{A}^I \to \mathfrak{B}$ from $\mathfrak{A}^I$ to an abelian category $\mathfrak{B}$ such that the functors $T\pi$ is exact, it follows that the coaugmented complex $T\tilde{\pi}\chi$ defines a resolvent complex for the functor $\chi \to T\chi$. In particular, then T is (uniformly) derivable and the trivial I-systems are T-acyclic.

(11.5) Proposition. *A resolvent complex for the limit functor* $\varprojlim_I \colon \mathfrak{A}^I \to \mathfrak{A}$ *is given by the following coaugmented complex of functors* Π^i *defined on systems* $\mathcal{X} \colon I \to \mathfrak{A}$,

$$\widetilde{\Pi}\mathcal{X}: \qquad 0 \to \varprojlim_I \mathcal{X} \xrightarrow{\epsilon} \Pi^0\mathcal{X} \xrightarrow{d^0} \Pi^1\mathcal{X} \xrightarrow{d^1} \Pi^2\mathcal{X} \xrightarrow{d^2} \cdots, \tag{11.5.1}$$

where

$$\Pi^p\mathcal{X} = \varprojlim \pi^{[p]}\mathcal{X} = \prod_{j_p \to j_{p-1} \to \cdots \to j_0} \mathcal{X}_{j_0},$$

and the differential $d \colon \Pi^p \to \Pi^{p+1}$ *is given by the projections,*

$$\mathrm{pr}_{\varphi_0,\dots,\varphi_p}\, d = \varphi_0\, \mathrm{pr}_{\varphi_1,\dots,\varphi_p} + \sum_{\nu=1}^{p}(-1)^\nu\, \mathrm{pr}_{\varphi_0,\dots,\varphi_{\nu-1}\varphi_\nu,\dots,\varphi_p} + (-1)^{p+1}\, \mathrm{pr}_{\varphi_0,\dots,\varphi_{p-1}}.$$

Proof. The complex is obtained by applying the limit functor $\varprojlim_I$ to the complex (11.4.1). Each functor Π^p is exact, since $\mathfrak{A}$ has exact $\prod_I$'s. It was noted in (11.3) that every I-system admits a monomorphism into a trivial I-system. Moreover, if $\mathcal{X}$ is a trivial system, then the complex (11.5.1) is exact, since it is obtained from the contractible complex (11.4.1). Thus the conditions for a resolvent complex of functors have been verified. □

(11.6). It follows from this result that the functor $\varprojlim \colon \mathfrak{A}^I \to \mathfrak{A}$ is uniformly derivable. The p'the derived functor is denoted $\varprojlim_I^{(p)}$ or $H^p(I;\;)$; its value $\varprojlim_I^{(p)} \mathcal{X} = H^p(I, \mathcal{X})$ is the p'th *cohomology of the category* I with coefficients in the system $\mathcal{X}$. It is, by (11.5), equal to the cohomology of the complex in (11.5.1) without coaugmentation,

$$\varprojlim_I^{(p)} \mathcal{X} = H^p(I, \mathcal{X}) := \mathrm{H}^p(\Pi X). \tag{11.6.1}$$

It is a consequence of the result that trivial I-systems are acyclic for the limit functor $\varprojlim_I$.

Note that any I-system $\mathcal{X}$ has a *canonical embedding* into a trivial system, namely the canonical embedding $\mathcal{X} \to \mathcal{Y} := \rho\lambda\mathcal{X}$ described in (11.5.2).

(11.7) Example. For special index categories there may be other exact resolving complexes.

Consider for instance the category $I = (0 \rightrightarrows 1)$, with two morphisms in addition to the two identities. An I-system $\mathcal{X}$ is a pair of morphisms $f', f'' \colon \mathcal{X}_0 \rightrightarrows \mathcal{X}_1$, and its limit is the coequalizer,

$$\varprojlim\,(\mathcal{X}_0 \overset{f'}{\underset{f"}{\rightrightarrows}} \mathcal{X}_1) = \mathrm{Ker}(f', f").$$

The following coaugmented complex,

$$0 \to \mathrm{Ker}(f', f") \to \mathcal{X}_0 \xrightarrow{f'-f"} \mathcal{X}_1 \to 0 \to \cdots,$$

is a an exact resolving complex for $\varprojlim_I$. Indeed, the functors $\mathcal{X} \mapsto \mathcal{X}_0$ and $\mathcal{X} \mapsto \mathcal{X}_1$ are exact functors $\mathfrak{A}^I \to \mathfrak{A}$ and evaluation at a trivial system $\rho\mathcal{Y}$ is the exact complex,

$$0 \to \mathcal{Y}_0 \times \mathcal{Y}_1 \to \mathcal{Y}_0 \times \mathcal{Y}_1 \times \mathcal{Y}_1 \xrightarrow{\mathrm{pr}' - \mathrm{pr}''} \mathcal{Y}_1 \to 0 \to \cdots,$$

where pr' and pr'' are the projections on the second and third factor. Hence, for the derived functors,

$$R^p \operatorname{Ker}(\mathcal{X}_0 \underset{f"}{\overset{f'}{\rightrightarrows}} \mathcal{X}_1) = \begin{cases} \operatorname{Ker}(f', f") & \text{if } p = 0, \\ \operatorname{Cok}(f', f") & \text{if } p = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(11.8) Example. As a second example, consider the category $I := 1' \to 0 \leftarrow 1''$. Then an I-system $\mathcal{X}$ is a diagram,

$$\begin{array}{ccc} & & \mathcal{X}'' \\ & & \downarrow{\scriptstyle f''} \\ \mathcal{X}' & \xrightarrow{f'} & \mathcal{X}_0; \end{array}$$

the limit is the fibered product $\mathcal{X}' \times_{\mathcal{X}_0} \mathcal{X}''$. It is easy to see that the following coaugmented complex,

$$0 \to \mathcal{X}' \times_{\mathcal{X}_0} \mathcal{X}'' \to \mathcal{X}' \times \mathcal{X}'' \xrightarrow{f'\,\mathrm{pr}' - f''\,\mathrm{pr}''} \mathcal{X}_0 \to 0 \to \cdots,$$

defines a resolvent complex for the limit. So, for this index category I,

$$H^0(I, \mathcal{X}) = \mathcal{X}' \times_{\mathcal{X}_0} \mathcal{X}'', \quad H^1(I, \mathcal{X}) = \mathcal{X}_0/(\operatorname{Im} f' + \operatorname{Im} f''),$$

and $H^p = 0$ for $p > 1$.

(11.9) Remark. The cohomology to the category I sketched here, with coefficients in an I-system $\mathcal{X}$, is one out of four parallel theories: For an *inverse* I-system $\mathcal{Z}\colon I \to \mathfrak{A}$, the cohomology is defined by replacing $\mathcal{Z}$ by the direct system ${}^{\mathrm{op}}\mathcal{Z}\colon I^{\mathrm{op}} \to \mathfrak{A}$, that is, $H^p(I, \mathcal{Z}) := H^p(I^{\mathrm{op}}, {}^{\mathrm{op}}\mathcal{Z})$.

In addition, a direct system $\mathcal{X}\colon I \to \mathfrak{A}$ may be viewed as a direct system ${}^{\mathrm{op}}\mathcal{X}^{\mathrm{op}}\colon I^{\mathrm{op}} \to \mathfrak{A}^{\mathrm{op}}$, with values in the dual category $\mathfrak{A}^{\mathrm{op}}$; the p'th cohomology of the latter system, as an object in $\mathfrak{A}$, is called the p'th *homology of the category I* with coefficients in the system $\mathfrak{X}$, and denoted $H_p(I, \mathcal{X})$. There is a similar definition of homology with coefficients in an inverse system.

(11.10) Exercises.

1. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of I-systems in $\mathfrak{C}$. Assume for every i that $f_i\colon \mathcal{X}_i \to \mathcal{Y}_i$ is a monomorphism. Prove that f is a monomorphism in the category $\mathfrak{C}^I$. What about the converse?

2. Let $\mathcal{Q} = \{\mathcal{Q}_i\}$ be a family (indexed by the objects of I) of injective objects of $\mathfrak{A}$. Prove that the family is an injective object in the abelian category $\mathfrak{A}^{|I|}$ of families. Prove that the trivial I-system $\rho\mathcal{Q}$ is an injective object in the category $\mathfrak{A}^I$ of I-systems.

12. The spectral sequences of Hom(B,lim).

13. The derived functors of the limit functor over N.

(13.1) Setup. Let $\mathfrak{A}$ be an abelian category with exact $\prod_{\mathbb{N}}$'s. View the set $\mathbb{N}$ with its usual order as a category. Then an inverse $\mathbb{N}$-system $\mathcal{X}$ in $\mathfrak{A}$ is a system of morphisms,

$$\cdots \xrightarrow{\varphi_3} \mathcal{X}_3 \xrightarrow{\varphi_2} \mathcal{X}_2 \xrightarrow{\varphi_1} \mathcal{X}_1 \,. \tag{13.1.1}$$

Equivalently, we will in this section view $\mathbb{N}$ with the order $\cdots \geqslant 3 \geqslant 2 \geqslant 1$, and consider (13.1.1) as an $\mathbb{N}$-system in $\mathfrak{A}$.

A trivial $\mathbb{N}$-system, defined from a family $\mathcal{Y}_n$ for $n \in \mathbb{N}$, has the form $\mathcal{X} = \rho\mathcal{Y}$ with

$$(\rho\mathcal{Y})_n = \prod_{n \to j} \mathcal{Y}_j = \prod_{j \leqslant n} \mathcal{Y}_j = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n.$$

So, inductively, $\mathcal{X}_1 = \mathcal{Y}_1$ and $\mathcal{X}_n = \mathcal{X}_{n-1} \times \mathcal{Y}_n$. Equivalently, at system $\mathcal{X}$ is trivial if every transition morphism $\mathcal{X}_n \to \mathcal{X}_{n-1}$ is a split epimorphism.

(13.2) Observation. *The following augmented complex of functors* $\Pi^0, \Pi^1 \colon \mathfrak{A}^{\mathbb{N}} \to \mathfrak{A}$ *defines an exact resolvent complex for* $\varprojlim_{\mathbb{N}}$*:*

$$\overline{\Pi}(\mathcal{X}) : \cdots \to 0 \to \varprojlim_{\mathbb{N}} \mathcal{X} \to \Pi^0\mathcal{X} \xrightarrow{d} \Pi^1\mathcal{X} \to 0 \to \cdots , \tag{13.2.1}$$

where $\Pi^0\mathcal{X} = \Pi^1\mathcal{X} := \prod \mathcal{X}_n$ *and* $d \colon \Pi^0\mathcal{X} \to \Pi^1\mathcal{X}$ *is given by its n'th projection,*

$$\mathrm{pr}_n\, d = \varphi_n\, \mathrm{pr}_{n+1} - \mathrm{pr}_n, \qquad \text{for } n \geqslant j.$$

Proof. Indeed, Π^0 and Π^1 are clearly exact functors, and $\varprojlim_{\mathbb{N}} \mathcal{X}$ is the kernel of the differential d. Assume that $\mathcal{X} = \rho\mathcal{Y}$ is trivial. Then

$$\varprojlim_{\mathbb{N}} \rho\mathcal{Y} = \prod_n \mathcal{Y}_n, \qquad \text{and} \qquad \Pi^0\mathcal{Y} = \Pi^1\mathcal{Y} = \prod\nolimits_{n \geqslant j} \mathcal{Y}_j,$$

where the product is over all pairs (n, j) with $n \geqslant j$. The differential d is given by its projections:

$$\mathrm{pr}_{n,j}\, d = \mathrm{pr}_{n+1,j} - \mathrm{pr}_{n,j},$$

corresponding to the index (n, j). So it remains to prove that d is an epimorphism. In fact, it is easy to prove that d is a split epic with a section $s \colon \Pi^1 \to \Pi^0$ defined by its projections,

$$\mathrm{pr}_{n,j}\, s = \sum\nolimits_{n > k \geqslant j} \mathrm{pr}_{k,j}\,.$$

□

It is a consequence of the result that the derived functors of $\varprojlim_{n\in\mathbb{N}}$ are determined as follows: $\varprojlim_n^{(0)} \chi_n = \varprojlim_n \chi_n = \operatorname{Ker} d$, $\varprojlim_n^{(1)} \chi_n = \operatorname{Cok} d$, and $\varprojlim_n^{(p)} \chi_n = 0$ for $p > 1$. In particular, an an $\mathbb{N}$-system $\mathcal{Q}$ is $\varprojlim_{\mathbb{N}}$-acyclic if and only if the morphism $d = d(\mathcal{Q})$ in (13.2.1) is an epimorphism.

(13.3) Application. *Assume the conditions of* (13.1). *Consider an* $\mathbb{N}$*-system* $\mathcal{X}$ *in* $\mathfrak{A}^\bullet$*, that is, an* $\mathbb{N}$*-system of complexes,*

$$\cdots \xrightarrow{\varphi_3} \mathcal{X}_3 \xrightarrow{\varphi_2} \mathcal{X}_2 \xrightarrow{\varphi_1} \mathcal{X}_1 . \tag{13.3.1}$$

Assume that in each degree p *the* $\mathbb{N}$*-system* $\cdots \to \mathcal{X}_3^p \to \mathcal{X}_2^p \to \mathcal{X}_1^p$ *is* $\varprojlim_{\mathbb{N}}$*-acyclic. Then there is for every* p *an exact sequence in* $\mathfrak{A}$*:*

$$0 \to \varprojlim_n{}^{(1)} H^{p-1}(\mathcal{X}_n) \to H^p(\varprojlim_n \mathcal{X}_n) \to \varprojlim_n H^p(\mathcal{X}_n) \to 0. \tag{13.3.2}$$

Proof. Consider the sequence of complexes,

$$0 \to \varprojlim_n \mathcal{X}_n \to \prod_n \mathcal{X}_n \xrightarrow{d} \prod_n \mathcal{X}_n \to 0 \to \cdots . \tag{13.3.3}$$

Limits of complexes are obtained degree by degree. Hence the sequence (13.3.3) is in degree p the sequence corresponding to the $\mathbb{N}$-system $n \mapsto \mathcal{X}_n^p$. By assumption, the sequence (13.3.3) is exact in each degree. Therefore it is an exact sequence of complexes. Consider the corresponding long exact cohomology sequence. The cohomology of a product is the product of cohomology since $\mathfrak{A}$ has exact $\prod_{\mathbb{N}}$'s. So the long exact sequence has this form

$$\prod_n H^{p-1}(\mathcal{X}_n) \xrightarrow{d} \prod H^{p-1}(\mathcal{X}_n) \to H^p(\varprojlim_n \mathcal{X}_n) \to \prod_n H^p(\mathcal{X}_n) \xrightarrow{d} \prod_n H^p(\mathcal{X}_n).$$

The short exact sequence (13.3.2) is a consequence. ∎

Corollary. *If the system* (13.3.1) *is* $\varprojlim_{\mathbb{N}}$*-acyclic in each degree and if each complex* $\mathcal{X}_n$ *is exact, then the limit* $\varprojlim_n \mathcal{X}_n$ *is an exact complex.*

(13.4). Under the hypothesis in (13.1), we may define the functor Tot from bifamilies $X = \{X^{pq}\}$ to families,

$$(\operatorname{Tot} X)^p = \prod_{j \in \mathbb{Z}} X^{p-j,j}.$$

With the usual definition of the differential we may view Tot as a functor $\mathfrak{A}^{\bullet,\bullet} \to \mathfrak{A}^\bullet$, from bicomplexes to complexes.

Corollary. *If a bicomplex* $X \in \mathfrak{A}^{\bullet,+}$ *has exact rows, then the total complex* $\operatorname{Tot} X$ *is exact.*

Proof. We may assume that $X \in \mathfrak{A}^{\bullet,\geqslant 1}$. Consider for each $n \geqslant 1$ the truncated bicomplex $X^{\bullet,\leqslant n}$. Then there is an $\mathbb{N}$-system of bicomplexes,

$$\cdots \to X^{\bullet,\leqslant n+1} \to X^{\bullet,\leqslant n} \to \cdots \to X^{\bullet,\leqslant 1},$$

and an $\mathbb{N}$-system of complexes,

$$\cdots \to \operatorname{Tot} X^{\bullet,\leqslant n+1} \to \operatorname{Tot} X^{\bullet,\leqslant n} \to \cdots \to \operatorname{Tot} X^{\bullet,\leqslant 1}.$$

The limit is easily computed: In degree degree p, the $\mathbb{N}$-system is the following:

$$n \mapsto (\operatorname{Tot} X^{\bullet,\leqslant n})^p = \prod_{n \geqslant j} X^{p-j,j},$$

and hence equal to the trivial $\mathbb{N}$-system determined by the family $X^{p-j,j}$ for $j \in \mathbb{N}$. So the limit is the product $\prod_j X^{p-j,j}$; in fact, it is easy to see that

$$\varprojlim_n \operatorname{Tot} X^{\bullet,\leqslant n} = \operatorname{Tot} X.$$

Now, the bicomplex $X^{\bullet,\leqslant n}$ has at most n non-vanishing rows, and they are exact by hypothesis. Hence $X^{\bullet,\leqslant n} \in \mathfrak{A}^{\bullet;\bullet}_{\divideontimes}$, and, by The Row Theorem, the total complex $\operatorname{Tot} X^{\bullet,\leqslant n}$ is exact. Moreover, we noticed above that the $\mathbb{N}$-system of complexes in each degree is a trivial $\mathbb{N}$-system, and hence $\varprojlim_{\mathbb{N}}$-acyclic. Therefore, the assertion follows from the Corollary in (13.3). $\square$

14. Mittag-Leffler systems.

(14.1) Setup. Let ${}_k\mathbf{Mod}$ be the category of modules over the commutative ring k, for instance (with $k = \mathbb{Z}$) the category **Ab** of abelian groups. Consider an $\mathbb{N}$-system in ${}_k\mathbf{Mod}$,

$$\cdots \longrightarrow \chi_3 \xrightarrow{\varphi_3} \chi_2 \xrightarrow{\varphi_2} \chi_1.$$

For simplicity we will write φ for the transition map $\chi_{n+1} \to \chi_n$ whenever the source is obvious from the context.

Recall that the system χ is $\varprojlim_{\mathbb{N}}$-acyclic if and only if the following map is surjective:

$$d\colon \prod_n \chi_n \to \prod_n \chi_n, \quad \text{given by } \mathrm{pr}_n\, d = \varphi\, \mathrm{pr}_{n+1} - \mathrm{pr}_n\,.$$

In coordinates the map d is described as follows:

$$(x_1, x_2, x_3, \dots) \mapsto (\varphi x_2 - x_1, \varphi x_3 - x_2, \varphi x_4 - x_3, \dots).$$

So d is surjective if and only if for any sequence $(a_1, a_2, a_3, \dots)$ with $a_n \in \chi_n$, the following system of equations has a solution $(x_1, x_2, x_3, \dots)$:

$$\varphi x_2 - x_1 = a_1, \quad \varphi x_3 - x_2 = a_2, \quad \varphi x_4 - x_3 = a_3, \quad \dots .$$

(14.2). The solvability is obvious if all the transition maps $\varphi\colon \chi_{n+1} \to \chi_n$ are surjective. A more general condition is contained in the following definition.

Definition. The system $n \mapsto \chi_n$ is called a *Mittag-Leffler system* if for every n the following descending sequence is stationary:

$$\chi_n \supseteq \varphi(\chi_{n+1}) \supseteq \varphi^2(\chi_{n+2}) \supseteq \cdots .$$

Lemma. *A Mittag-Leffler system $\cdots \to \chi_3 \to \chi_2 \to \chi_1$ is $\varprojlim_{\mathbb{N}}$-acyclic.*

Proof. Case 1: Assume that every transition morphism is surjective. As noted above, the solvability is trivial in this case.

Case 2: Assume for every n that the composition $\varphi^p\colon \mathcal{X}_{n+p} \to \mathcal{X}_n$ is zero when $p \gg 0$. Then a solution to the equations are given by the finite sums,

$$x_n = a_n + \varphi a_{n+1} + \varphi^2 a_{n+2} + \cdots .$$

The general case: Form, for every n, the intersection,

$$\chi_n' := \bigcap_p \varphi^p(\chi_{n+p}) \subseteq \chi_n.$$

Clearly, the χ_n' form a subsystem of the system χ_n. So there is an exact sequence of $\mathbb{N}$-systems,

$$0 \to \chi' \to \chi \to \chi/\chi' \to 0.$$

Use the Mittag-Leffler conditions on $\mathcal{X}$ to see that χ' falls under case 1, and, again by the Mittag-Leffler conditions, that χ/χ' falls under case 2. Therefore, from the exact sequence

$$\varprojlim^{(1)}_{\mathbb{N}} \chi' \to \varprojlim^{(1)}_{\mathbb{N}} \chi \to \varprojlim^{(1)}_{\mathbb{N}} \chi/\chi',$$

it follows that $\lim^{(1)}_{\mathbb{N}} \chi = 0$. ∎

(14.3) Corollary. *If, in a short exact sequence of* $\mathbb{N}$*-systems in* ${}_k\mathbf{Mod}$,

$$0 \to \mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \to 0,$$

the system $\mathcal{X}$ *is a Mittag-Leffler system, then the following sequence is exact:*

$$0 \to \varprojlim_n \mathcal{X}_n \to \varprojlim_n \mathcal{Y}_n \to \varprojlim_n \mathcal{Z}_n \to 0.$$

15. Cohomology of groups.

(15.1) Setup. Let G be a group, and view G as a category (also denoted G) with one object, and with the elements of G as the endomorphisms; composition of endomorphisms is multiplication in the group. In particular, in the category G every morphism s is invertible, and the assignment $s \mapsto s^{-1}$ identifies G and the dual category G^{op}.

Let $\mathfrak{C}$ be an arbitrary category. Then a G-system X in $\mathfrak{C}$ is a functor $X\colon G \to \mathfrak{C}$; it is is also called a G-object in $\mathfrak{C}$ (strictly speaking, a G-object ought to be given as a contravariant functor, but the two notions are equivalent via the identification of G and G^{op}). A G-object in $\mathfrak{C}$ is specified by an object X of $\mathfrak{C}$ (the image of the single object in the category G) and an additional monoid map $G \to \mathrm{End}_{\mathfrak{C}}(X)$. As all morphisms in G are invertible, the monoid map is a group homomorphism; it is called the associated *representation* of G,

$$G \to \mathrm{Aut}_{\mathfrak{C}}(X) \quad \text{usually denoted } s \mapsto s_X.$$

Let X be a G-object of $\mathfrak{C}$. We shall use the following notations for the limit and the colimit,

$$\varprojlim_G X = X^G = \Gamma^G X, \quad \text{and} \qquad \varinjlim_G X = X/G = \Gamma_G X.$$

(Don't confuse the notation X^G with the product of identical copies of X indexed by the elements of G; in connection with G-objects, the product will play a role, and it will be denoted $\prod_{s\in G} X$.)

A common source for a given G-object X is given by an object A of $\mathfrak{C}$, and a morphism $a\colon A \to X$ satisfying the compatibility conditions: $s_X a = a$ for all $s \in G$. If the limit X^G exists, then the canonical projection (there is only one) $\epsilon\colon X^G \to X$ is in fact a monomorphism with $s_X\epsilon = \epsilon$ for all $s \in G$. If $\mathfrak{C}$ has equalizers and intersections of subobjects, then X^G is the intersection of the equalizers of all pairs $(1_X, s_X)$ for $s \in G$. If $\mathfrak{C}$ has $\prod_G$'s and equalizers, then X^G is the equalizer of the pair of morphisms,

$$X \underset{\partial^1}{\overset{\partial^0}{\rightrightarrows}} \prod_{s\in G} X,$$

given by $\mathrm{pr}_s\,\partial^0 = 1_X$, and $\mathrm{pr}_s\,\partial^1 = s_X$.

(15.2). Fix an abelian category $\mathfrak{A}$ with exact $\prod_G$'s. Let A be a G-object of $\mathfrak{A}$. The p'th *cohomology* of G with coefficients in A, denoted $H^p(G, A)$ is the the value at A of the p'th derived functor $\varprojlim_G$:

$$H^p(G, A) := \varprojlim_G{}^{(p)} A.$$

It may be defined as the cohomology of the standard complex associated to the limit functor over an arbitrary index category. For the category G, the coaugmented standard complex has the following form:

$$\widetilde{\Pi}\colon \quad 0 \to \Gamma^G A \xrightarrow{\epsilon} \Pi^0 A \xrightarrow{d} \Pi^1 A \xrightarrow{d} \Pi^2 A \longrightarrow \cdots,$$

where $\Pi^p A = \prod_{(s_0,\dots,s_p)} A$ (the product is over all $(p+1)$-sets of elements of G) and $d\colon \Pi^p \to P^{p+1}$ is given by the projections,

$$\mathrm{pr}_{s_1,\dots,s_{p+1}} d = s_1 \,\mathrm{pr}_{s_2,\dots,s_{p+1}} + \sum_{\nu=1}^{p} (-1)^\nu \,\mathrm{pr}_{s_1,\dots,s_\nu,s_{nu+1},s_{p+1}} + (-1)^{p+1} \,\mathrm{pr}_{s_1,\dots,s_p} .$$

Note that a trivial G-object is determined from an object Y of $\mathfrak{A}$: It is the G-object

$$\rho Y = \prod_{s\in G} Y,$$

where the transition morphism $t_*\colon \rho Y \to \rho Y$ corresponding to $t \in G$ is obtained by permutation of the coordinates of the product: $\mathrm{pr}_s\ t_* = \mathrm{pr}_{st}$. The trivial G-objects are Γ^G-acyclic.

(15.3) Example. Assume that $\mathfrak{A} = {}_k\mathbf{Mod}$ is the category of modules over the commutative ring. Let kG be the group algebra of G over k. Then, clearly, the category of ${}_k\mathbf{Mod}^G$ of G-objects may be identified with the category ${}_{kG}\mathbf{Mod}$ of left kG-modules. View k as a constant G-object. Then it is easy to identify,

$$A^G = \mathrm{Hom}_{kG}(k, A),$$

and consequently we may identify the derived functors,

$$H^p(G, A) = \mathrm{Ext}^p_{kG}(k, A).$$

Note that $\Pi^p(A)$ is the product of identical copies of A over the index set of all $(s_1,\dots,s_p)$; as such it may be identified with the k-module of all functions $(s_1,\dots,s_p) \mapsto f(s_1,\dots,s_p)$ with values in A. In low degrees, the differential is given as follows:

$$\begin{aligned}
(d^0 a)(s) &= sa - a, \quad s \in \Pi^0 = A,\\
d^1 f(s,t) &= sf(t) - f(st) + f(s), \quad f \in \Pi^1,\\
d^2 f(s,t,u) &= sf(t,u) - f(st,u) + f(s,tu) - f(s,t), \quad f \in \Pi^2.
\end{aligned}$$

In particular, the degree-1 cycles are the maps $f(s)$ for which $d^1 f = 0$; they are called *crossed homomorphisms* $G \to A$, and the degree-1 boundaries are the *principal crossed homomorpisms*, of the form $s \mapsto sa - a$ for $a \in A$.

16. Cohomology of some special groups and monoids.

(16.0) Exercise. Read the examples on the cohomology of a group G via standard resolution assuming only that G is a monoid. Did you find any reservations?

(16.1) Setup. Let G be a monoid and let $\mathfrak{A}$ be an abelian category with exact $\prod_G$'s. A standard complex defines a resolvent complex for the limit functor $\Gamma^G = \varprojlim_G \colon \mathfrak{A}^G \to \mathfrak{A}$, defined on the category $\mathfrak{A}^G$ of co-G-objects.

For special groups or monoids there may be special constructions of a coaugmented resolvent complex for Γ^G:

$$\widetilde{\Pi}A: \qquad 0 \to \Gamma^G A \to \Pi^0 A \to \Pi^1 A \to \cdots,$$

where each Π^i is a functor $A \mapsto \Pi^i A$, defined on co-G-objects of $\mathfrak{A}$. In each of the examples below we construct such a complex where each Π^i is exact and such that the coaugmented complex is contractible when evaluated on a trivial G-object. Then it results from the general theory that the cohomology $H^p(\Pi A)$ of the complex ΠA is the cohomology $\varprojlim_G^{(p)} A = H^p(G, A)$.

(16.2) Example. Let G be the free (multiplicative) monoid with a single generator f (in additive notation, G is the monoid $\mathbb{N}_0$ of nonnegative integers). Then a G-object A is an object A of $\mathfrak{A}$ with a given endomorphism $f = f_A \colon A \to A$. The following coaugmented complex:

$$\widetilde{\Pi}A: \qquad 0 \to \Gamma^G A \to A \xrightarrow{f-1} A \to 0 \to \cdots,$$

is a resolvent complex. Indeed, the functors $\Pi^0 = \Pi^1$ are exact, given by $A \mapsto A$ (and forget the endomorphism f). A coinduced object has the form $\rho B = \prod_{n\geqslant 0} B$, where the endomorphism $f = f_{\rho B}$ is determined by $\mathrm{pr}_n f = \mathrm{pr}_{n+1}$. The complex $\widetilde{\Pi}(\rho B)$ is the following:

$$\widetilde{\Pi}(\rho B): \qquad 0 \to B \underset{\tau}{\overset{\epsilon}{\rightleftarrows}} \prod_{n\geqslant 0} B \underset{\sigma}{\overset{f-1}{\rightleftarrows}} \prod_{n\geqslant 0} B \to 0 \to \cdots,$$

split by the indicated morphisms defined by $\tau = \mathrm{pr}_0$ and $\mathrm{pr}_n \sigma = \sum_{j<n} \mathrm{pr}_n$. Do check it! As a consequence, for this monoid G,

$$H^0(G, A) = \Gamma^G A = \mathrm{Ker}(A \xrightarrow{f-1} A), \qquad H^1(G, A) = \mathrm{Cok}(A \xrightarrow{f-1} A) = A/G,$$

and $H^p = 0$ for $p > 1$.

(16.3) Example. Let G be the free group with a single generator e (in additive notation, G is the group $\mathbb{Z}$ of integers). Then a G-object A is an object A of $\mathfrak{A}$ with a given automorphism $e = e_A \colon A \to A$. The the following coaugmented complex,

$$\Pi A: 0 \to \Gamma^G A \to A \xrightarrow{e-1} A \to 0 \to \cdots,$$

is a resolvent complex. Indeed, the functors $\Pi^0 = \Pi^1$ are exact, given by $A \mapsto A$ (and forget the automorphism e). A coinduced object has the form $\rho B = \prod_n B$ (where the product is over $n \in \mathbb{Z}$), where the endomorphism $e = e_{\rho B}$ is determined by $\mathrm{pr}_n e = \mathrm{pr}_{n+1}$. The complex $\widetilde{\Pi}(\rho B)$ is the following:

$$\widetilde{\Pi}(\rho B): \qquad 0 \to B \underset{\tau}{\overset{\epsilon}{\rightleftarrows}} \prod_n B \underset{\sigma}{\overset{e-1}{\rightleftarrows}} \prod_n B \to 0 \to \cdots,$$

split by the indicated morphisms defined by $\tau = \mathrm{pr}_0$ and $\mathrm{pr}_n \sigma = \sum_{0 \leqslant j < n} \mathrm{pr}_n$ for $n \geqslant 0$ and $\mathrm{pr}_n \sigma = -\sum_{n \leqslant j < 0}$ for $n \leqslant 0$. Do check it! As a consequence, for this group G,

$$H^0(G, A) = \Gamma^G A = \mathrm{Ker}(A \xrightarrow{e-1} A), \qquad H^1(G, A) = \mathrm{Cok}(A \xrightarrow{e-1} A) = A/G,$$

and $H^p = 0$ for $p > 1$.

(16.4) Example. Let G be a free (noncommutative) group with generators e_i for $i \in I$; denote by $\emptyset$ the neutral element of G (the empty word). Then a G-object A is an object $A \in \mathfrak{A}$ with a given family of automorphisms $e_i = e_{i,A}$ for $i \in I$. Consider the complex,

$$0 \to \Gamma^G A \xrightarrow{\epsilon} A \xrightarrow{d} \prod_{i \in I} A \to 0 \to \cdots,$$

where d is determined by its projections: $\mathrm{pr}_i d = e_{i,A} - 1_A$ for $i \in I$. Assume that $A = \rho B$ is the trivial G-object determined by an object B of $\mathfrak{A}$. Then $A = \prod_{w \in G} B$, and G acts by permutation of the coordinates. We want to prove that the complex is contractible when evaluated on ρB, so we want to define homotopies σ, τ:

$$0 \to B \underset{\sigma}{\overset{\epsilon}{\rightleftarrows}} \prod_{w \in G} B \underset{\tau}{\overset{d}{\rightleftarrows}} \prod_{i \in I} \prod_{w \in G} B \to 0,$$

The morphism σ is the projection on the index $\emptyset$ (the unit of the group G), that is, $\sigma = \mathrm{pr}_\emptyset$. The morphism τ is determined by its projections $\mathrm{pr}_w \tau$, and they are defined inductively on the length of the word w. For the empty word $\mathrm{pr}_\emptyset \tau = 0$, and

$$\mathrm{pr}_{e_i w} \tau = \mathrm{pr}_w \tau + \mathrm{pr}_{i,w}, \qquad \mathrm{pr}_{e_i^{-1} w} \tau = \mathrm{pr}_w \tau - \mathrm{pr}_{i, e_i^{-1} w}.$$

(16.5) Example. Let G be the free abelian (multiplicative) monoid with basis $f_1, \dots, f_r$ (the additive version of G is the monoid $\mathbb{N}_0^r$ of r-sets $(n_1, \dots, n_r)$ of nonnegative integers). Then a G-object is an object A of $\mathfrak{A}$ with a given family of commuting endomorphisms f_i. The trivial G-object ρB determined by an object B is the product $\rho B = \prod_{n_1, \dots, n_r} B$, over $\mathbb{N}_0^r$; the endomorphism $f_i : \rho B \to \rho B$ is determined by its projections,

$$\mathrm{pr}_{n_1, \dots, n_r} f_i = \mathrm{pr}_{n_1, \dots, n_i + 1, \dots, n_r}. \tag{16.5.1}$$

Consider the functor $\Gamma_{\mathbf{f}} = \Gamma_{f_1,\dots,f_r} : \mathfrak{A}^G \to \mathfrak{A}$ defined by

$$\Gamma_{f_1,\dots,f_r} A = \bigcap \operatorname{Ker}(f_{i,A}) = \operatorname{Ker}(A \xrightarrow{\mathbf{f}} A^{\oplus r}),$$

where $\mathbf{f} = \mathbf{f}_A : A \to A^{\oplus r}$ is the morphism with coordinates $f_{i,A}$. Clearly, in this notation,

$$\Gamma^G A = \Gamma_{f_1-1,\dots,f_r-1}(A).$$

Note that the Koszul cochain complex $K^\bullet(X) = K^\bullet(f_1, \dots, f_r; X)$ is defined in this general setup for any complex X of G-objects. For the G-object A, viewed as a complex in degree 0, the morphism $K^0(A) \to K^1(A)$ is the morphism $\mathbf{f} : A \to A^{\oplus r}$; hence $\Gamma_{\mathbf{f}}(A) = H^0(\mathbf{f}, A)$.

Lemma. *The following coaugmented complex* the reduced Koszul complex),

$$\widetilde{K}_{\mathbf{f}}(A): \qquad 0 \to \Gamma_f A \to K^0(A) \to \cdots \to K^r(A) \to 0 \to \cdots, \tag{16.5.2}$$

defines a resolvent complex for the functor $\Gamma_{\mathbf{f}}$.

Hint. In degree i the functor K^i is given by $K^i(A) = A^{\oplus\binom{r}{i}}$; it is clearly exact. So it remains to prove that the *reduced Koszul complex* (16.5.1) is exact when evaluated at a coinduced object ρB. Note that $\Gamma_{\mathbf{f}}(\rho B) = B$, as it follows from the description (16.5.1).

Exactness is proved by induction on r. It $r = 1$, the coinduced object has the form $\rho_1 B = \prod_{n\geqslant 0} B$, and the reduced Koszul complex has the following form:

$$\widetilde{K}(\rho_1 B): \qquad 0 \to B \underset{\tau}{\overset{\iota}{\rightleftarrows}} \prod_{n\geqslant 0} B \underset{\sigma}{\overset{f}{\rightleftarrows}} \prod_{n\geqslant 0} B \to 0 \to \cdots,$$

where $\mathrm{pr}_0\,\iota = 1$ and $\mathrm{pr}_n\,\iota = 0$ for $n > 0$. It is split by the indicated morphisms, defined by $\tau = \mathrm{pr}_0$ and $\mathrm{pr}_0\,\sigma = 0$ and $\mathrm{pr}_n\,\sigma = \mathrm{pr}_{n-1}$ for $n > 0$. Do check it!

Now, for $r > 1$, the monoid G is the product $G = G' \times G_1$ where G' is the submonoid generated by $f_1, \dots, f_{r-1}$ and G_1 is generated by f_r; both submonoids are free. Accordingly, a co-G-object may be viewed as a co-G_1-object in the category of co-G'-objects, $\mathfrak{A}^G = (\mathfrak{A}^{G'})^{G_1}$, and the functor ρ is a composition $\rho = \rho_1\rho'$:

$$\rho : \mathfrak{A} \xrightarrow{\rho'} \mathfrak{A}^{G'} \xrightarrow{\rho_1} (\mathfrak{A}^{G'})^{G_1} = \mathfrak{A}^G.$$

The Koszul cochain complex may be defined by a similar recursion: $K_{\mathbf{f}} X = K_{\mathbf{f}',f_r} X = K_{\mathbf{f}'} K_{f_r} X$. Now, let B be an object of $\mathfrak{A}$, and set $B_1 := \rho' B$; then $\rho B = \rho_1 B_1$. The following two morphisms of complexes are homotopy equivalences,

$$B \to K_{\mathbf{f}'}(\rho' B) = K_{\mathbf{f}'}(B_1) \to K_{\mathbf{f}'} K_{f_r}(\rho_1 B_1) = K_{\mathbf{f}}(\rho B),$$

the first by the induction hypothesis, the second because it is obtained by applying $K_{\mathbf{f}'}$ to the morphism $B_1 \to K_{f_r}(\rho_1 B_1)$ which is a homotopy equivalence by the case $r = 1$. ∎

Corollary. *For the free abelian monoid G with generators $f_1, \dots, f_r$, the cohomology with coefficients in a co-G-object A is equal to the Koszul cohomology with respect to the sequence $f_1 - 1, \dots, f_r - 1$,*

$$H^p(G, A) = H^p_{f_1-1,\dots,f_r-1}(A).$$

(16.6) Example. Let $G = \mathbb{Z}^r$ be the free rank-r abelian group, with multiplicative generators $e_1, \dots, e_r$. As in (16.5), the group cohomology is given by the Koszul cohomology,

$$H^p(G, A) = H^p_{e_1-1,\dots,e_r-1}(A).$$

(16.7) Example. If A is a co-G-object in $\mathfrak{A}$, then the group homomorphism $G \to \mathrm{Aut}(A)$ extends to a ring homomorphism $\mathbb{Z}G \to \mathrm{End}(A)$, denoted $\lambda \mapsto \lambda_A$, from the group ring of G to the endomorphism ring of A.

Assume that G is a finite group. Then the *norm* N is the element $N = \sum_{u \in G} u \in \mathbb{Z}G$. So the norm defines an endomorphism $N = N_A \colon A \to A$. Again, since G is finite, there is a morphism $D = D_A \colon \prod_{s \in G} A \to A$ defined by $D = \sum_{s \in G}(s_A - 1_A)\,\mathrm{pr}_s$.

Clearly, since $N(s-1) = (s-1)N$ in the group algebra, it follows that $ND \colon \prod_s A \to A$ and $\varepsilon N \colon A \to \prod_s A$ are zero. So the image $NA := \mathrm{Im}\, N_A$ of the norm is a subobject of $\Gamma^G A$, and the image $DA := \mathrm{Im}\, D_A$ is contained in the kernel ${}_N A := \mathrm{Ker}\, N_A$ of the norm.

Lemma. *If A is co-G-induced, $A = \rho B$, then the following two zero sequences are split by the morphisms defined in the proof:*

$$A \xrightarrow{N} A \xrightarrow{\varepsilon} \prod_{s \in G} A, \qquad \prod_{s \in G} \xrightarrow{D} A \xrightarrow{N} A,$$

Proof. With $A = \rho B$ the first sequence is the following, split by the indicated morphisms:

$$\prod_t B \underset{\tau}{\overset{N}{\rightleftarrows}} \prod_t B \underset{\sigma}{\overset{\epsilon}{\rightleftarrows}} \prod_s \prod_t B;$$

here N and ε satisfy the equations $\mathrm{pr}_t\, N = \sum_u \mathrm{pr}_{tu}$ and $\mathrm{pr}_t\, \mathrm{pr}_s\, \varepsilon = \mathrm{pr}_{ts} - \mathrm{pr}_t$. The morphisms τ and σ are determined by the projections, $\mathrm{pr}_1\, \tau = \mathrm{pr}_1$ and $\mathrm{pr}_t\, \tau = 0$ when $t \neq 1$ and $\mathrm{pr}_t\, \sigma = \mathrm{pr}_1\, \mathrm{pr}_t$. It is easy to verify the equation $N\tau + \sigma\varepsilon = 1$.

The second sequence is the dual of the first; so the result for the second second is a consequence of the first. ∎

(16.8) Example. Let $G = \mathrm{C}_d$ be the finite cyclic group of order d with a generator e. (So the additive version of G is the group $\mathbb{Z}/d\mathbb{Z}$.) Then the following coaugmented complex defines a resolvent complex for Γ^G:

$$0 \to \Gamma^G A \to A \xrightarrow{e-1} A \xrightarrow{N} A \xrightarrow{e-1} A \xrightarrow{N} \cdots.$$

As a consequence, for $p > 0$,

$$H^{2p-1}(G, A) = {}_N A/DA, \qquad H^{2p}(G, A) = \Gamma^G A/NA.$$

(16.9) Exercises.

1. Prove for a finite group G and a G-object A that the p'th cohomology $H^p(G, A)$ for $p > 0$ is killed by the order of G. [Hint: Consider for the standard complex ΠA the morphisms $\sigma^{p+1}\colon \Pi^{p+1} \to \Pi^p$ determined by the projections,

$$\mathrm{pr}_{s_1,\dots,s_p}\, \sigma^{p+1} = (-1)^{p+1} \sum_{s \in G} \mathrm{pr}_{s_1,\dots,s_p,s}\,.$$

Prove that $\sigma\partial + \partial\sigma = |G|$. Is it unfair not to specify the range of p?]

2. (1) Let G be the group of order 2. Give an example of a commutative group with a G-action such that $H^p(G, A) \neq 0$ for every $p \geqslant 0$.

Let G be the monoid of order 2 generated by an f with $f^2 = f$. Prove for any co-G-object A that $H^p(G, A) = 0$ for all p. [Hint: prove that Γ^G is exact.]

17. Thge Lyndon spectral sequence.

18. The spectral sequence of a Galois covering.

(18.1) Setup. Consider for a topological space X the set of singular p-simplices in X,

$$\Delta_p(X) = \mathrm{Hom}_{\mathbf{Top}}(\Delta^p, X) \qquad p \geqslant -1.$$

Let $\mathfrak{A}$ be an abelian category with $\prod$'s, and A an object in $\mathfrak{A}$. We write $C_{\mathrm{sing}}(X, A)$ for the product $A^{\Delta_p(X)}$. Then there is a positive complex $C = C_{\mathrm{sing}}(X, A)$ with differentials defined by formulas analogous to those defining the chain complex $C^{\mathrm{sing}}(X, \mathbb{Z})$. With an obvious coaugmentation from $C^{-1} = A$ there is a similar *reduced singular cochain complex* $\widetilde{C}_{\mathrm{sing}}(X, A)$.

Alternatively the differentials between the objects of $C_{\mathrm{sing}}(X, A)$, and other related morphisms, may be defined by the following process of of transposing linear maps between the modules in the chain complex $C^{\mathrm{sing}}(X, \mathbb{Z})$: For any set I the projections $\mathrm{pr}_i\colon A^I \to A$ for $i \in I$ form a family of morphisms in the set $\mathrm{Hom}_{\mathfrak{A}}(A^I, A)$, that is, $i \mapsto \mathrm{pr}_i$ is a map of sets from I to $\mathrm{Hom}_{\mathfrak{A}}(A^I, A)$. So it extends to a homomorphism of abelian groups,

$$\mathbb{Z}^{\oplus I} \to \mathrm{Hom}_{\mathfrak{A}}(A^I, A);$$

naturally, the image of en element $c \in \mathbb{Z}^{\oplus I}$ will be denoted $\mathrm{pr}_c\colon A^I \to I$. If c is the finite linear combination $c = \sum_i c_i i$, then pr_c is the sum morphism $\mathrm{pr}_c = \sum_{i \in I} c_i\, \mathrm{pr}_i$ in the group $\mathrm{Hom}_{\mathfrak{A}}(A^I, A)$. With this notation there is for every linear map $\varphi\colon \mathbb{Z}^{\oplus I} \to \mathbb{Z}^{\oplus J}$ an associated *transposed morphism*,

$$\varphi^{\mathrm{tr}}\colon A^J \to A^I, \qquad \text{defined by } \mathrm{pr}_i\, \varphi^{\mathrm{tr}} = \mathrm{pr}_{\varphi i}\,.$$

It is easy to see that transposing is functorial:

$$(\varphi\psi)^{\mathrm{tr}} = \psi^{\mathrm{tr}}\varphi^{\mathrm{tr}}.$$

The differentials in the cochain complex $C_{\mathrm{sing}}(X, A)$ may by obtained by transposing the differentials of $C^{\mathrm{sing}}(X, \mathbb{Z})$.

(18.2) Example. Other linear maps may be transposed. For instance: For the n-sphere ($n \geqslant 0$) there is a homotopy equivalence $C^{\text{sing}}(X, \mathbb{Z}) = \mathbb{Z}(0) \oplus \mathbb{Z}(n)$. Consequently, for the general cochain complex there is a homotopy equivalence

$$C_{\text{sing}}(X, A) = A(0) \oplus A(-n).$$

(18.3) Setup. A map $f: X \to Y$ (of topological spaces) is a *covering projection* if Y may be covered by open subsets U such that the restricted map $f_U: f^{-1}U \to U$ is isomorphic to a projection $U \times J \to U$ with a discrete set J (equivalently, if $f^{-1}U$ is a disjoint union of open sets U_α each being mapped homeomorphically onto U. The covering is *trivial*, if f is isomorphic to a projection $Y \times J \to Y$.

It is a standard fact that every covering of the unit square $[0.1] \times [0, 1]$ is trivial. It is a consequence that every covering of a 1-connected (i.e., path connected and simply connected) space Y_0 trivial. It follows that a covering has the following *lifting property*: for every pair of based maps p, φ:

$$\begin{tikzcd} & (X, x) \arrow[d, "p"] \\ (Y_0, y_0) \arrow[ur, dashed] \arrow[r, "\varphi"'] & (Y, y), \end{tikzcd}$$

where p is a covering and Y_0 is 1-connected there is a unique map $(Y_0, y_0) \to (X, x)$ making the diagram commutative.

Consider a topological space X with a *properly discontinuous* action of a group G, in other words, every point $x \in X$ has on open neighborhood U such that $U \cap sU = \emptyset$ for all elements $s \neq 1$ in G. It follows easily that the quotient map,

$$X \to X/G,$$

is a covering projection.

Clearly, G acts on each set $\Delta_p(X)$ of singular p-simplices. By the lifting property, the map induced by $X \to X/G$ is surjective:

$$\Delta_p(X) \to \Delta_p(X/G),$$

Let $T_p \subseteq \Delta_p(X)$ be a subset mapped bijectively onto $\Delta_p(X/G)$. Assume that $p \geqslant 0$ so that $\Delta^p \neq \emptyset$. Then, by uniqueness of the lifting, $\Delta_p(X)$ is the disjoint union of 'translates' of T_p,

$$\Delta_p(X) = \bigvee_{t \in G} t(T_p).$$

The action of G on the set $\Delta_p(X)$ induces an action of G on the product $A^{\Delta_p(X)}$. Moreover, it is easy to see that the differentials in the complex commute with the action of G; hence the cochain complex C_{sing} may be viewed as a complex of objects from $\mathfrak{A}^G$. Moreover, by the description above,

$$A^{\Delta_p(X)} = \prod_{t \in G} A^{T_p};$$

hence each $A^{\Delta_p(X)}$ is a trivial G-object, induced by the object $A^{T_p} = A^{\Delta_p(X/G)}$. In particular, there is an isomorphism,

$$\Gamma^G C_{\text{sing}}(X, A) = C_{\text{sing}}(X/G, A). \tag{18.3.1}$$

As the objects of $C_{\text{sing}}(X, A)$ are co-induced G-objects, and hence acyclic for Γ^G, the left side of (18.3.1) is the hyper derived of Γ^G evaluated at the complex $C_{\text{sing}}(X, A)$:

$$R\Gamma^G C_{\text{sing}}(X, A) = C_{\text{sing}}(X/G, A). \tag{18.3.2}$$

A 2-spectral sequence falls out:

$$H^p(G, H^q(X, A) \Rightarrow H^n(X/G, A). \tag{18.3.3}$$

19. Some special Galois coverings.

(19.1) Setup. Let A be an object in an abelian category $\mathfrak{A}$. Assume that the group G acts properly discontinuously on a topological space X. Then there is an isomorphism of complexes,

$$R\Gamma^G C_{\text{sing}}(X, A) = \Gamma^G C_{\text{sing}}(X, A) \simeq C_{\text{sing}}(X/G, A); \tag{19.1.1}$$

the first equality because the complex $C_{\text{sing}}(X, A)$ on the left side is consists of co-induced G-objects which are acyclic for Γ^G. The functor $R\Gamma^G$ respects quasi-isomorphisms and exact triangles. Hence, from the mapping cone,

$$\begin{tikzcd} & \widetilde{C}_{\text{sing}}(X, A) \arrow[dl, squiggly] & \\ A(0) \arrow[rr] & & C_{\text{sing}}(X, A), \arrow[ul] \end{tikzcd}$$

and the isomorphism (19.1.1), there is an induced exact triangle,

$$\begin{tikzcd} & R\Gamma^G \widetilde{C}_{\text{sing}}(X, A) \arrow[dl, squiggly] & \\ R\Gamma^G A(0) \arrow[rr] & & C_{\text{sing}}(X/G, A). \arrow[ul] \end{tikzcd} \tag{19.1.2}$$

(19.2) Example. The group $G = \mathbb{Z}$ acts as translations on the space $\mathbb{R}$ of reals. The quotient is the 1-sphere: $\mathbb{R}/\mathbb{Z} = S^1$. The space $\mathbb{R}$ is contractible, and so there is a homotopy equivalence of chain complexes $C^{\text{sing}}(\mathbb{R}, \mathbb{Z}) \simeq \mathbb{Z}(0)$. Hence there is an induce equivalence of cochain complexes $C_{\mathrm{sing}}(\mathbb{R}, A) \simeq A(0)$. It is easily seen to be G-invariant, when A is viewed as a constant G-object. Consequently, by (19.1.1),

$$C_{\mathrm{sing}}(\mathbb{R}/\mathbb{Z}, A) = R\Gamma^G A(0).$$

So the cohomology of S^1 is the cohomology $H^*(G, A)$ which, with the constant action of $G = \mathbb{Z}$ on A, is the following:

$$H^0_{\mathrm{sing}}(S^1, A) = H^1_{\mathrm{sing}}(S^1, A) = A, \quad H^p_{\mathrm{sing}}(S^1, A) = 0 \text{ for } p > 1.$$

(19.3) Example. The group $G = \mathbb{Z}^r$ acts on r-space $\mathbb{R}^r$. The quotient is a product of 1-spheres: $\mathbb{R}^r/\mathbb{Z}^r = (S^1)^r$. Hence $H^p_{\mathrm{sing}}((S^1)^r, A) = H^p(G, A)$ (where A is the constant G-object); the latter cohomology is the p'th Koszul cohomology of A corresponding to the sequence $\mathbf{f} = \mathbf{0}$. Hence,

$$H^p_{\mathrm{sing}}((S^1)^r) = A^{\binom{r}{p}}.$$

(19.4) Example. The cyclic group $G = \pm 1$ operates on S^r via the antipodal map $x \mapsto -x$; the quotient $S^r/\pm$ is the real projective r-space $\mathbb{P}^r = \mathbb{P}^r(\mathbb{R})$. There is a natural homotopy equivalence $\mathbb{Z}(r) \xrightarrow{\sim} \widetilde{C}^{\mathrm{sing}}(S^r, \mathbb{Z})$ and hence a homotopy equivalence $A(-r) \xrightarrow{\sim} \widetilde{C}_{\mathrm{sing}}(X, A)$. The equivalence is *not* G-invariant. In fact, it is easy to see that the induced action of the element $-1 \in G$ on $H^r_{\mathrm{sing}}(S^r, A) = A$ is multiplication by $(-1)^{r+1}$.

Let us write $A^{\pm}$ for A with this G-action (if r is odd, it is the constant action of G on A, and when r is even, the element $-1 \in G$ acts as multiplication by -1 on A). Then there is a quasi-isomorphism of complexes of G-objects $\widetilde{C}_{\mathrm{sing}}(X, A) \xrightarrow{\sim} A^{\pm}(-r)$. So the exact triangle (19.1.2) takes the following form,

$$\begin{tikzcd} & R\Gamma^G A^{\pm}(-r) \arrow[dl, squiggly] & \\ R\Gamma^G A(0) \arrow[rr] & & C_{\mathrm{sing}}(\mathbb{P}^r, A). \arrow[ul] \end{tikzcd} \tag{19.4.1}$$

The pth cohomology of the top vertex is $H^{p-r}(G, A^{\pm})$, and it vanishes when $p < r$. So the long exact cohomology sequence of the triangle yields isomorphisms,

$$H^p_{\mathrm{sing}}(\mathbb{P}^r, A) = H^p(G, A) = \begin{cases} A & \text{when } p = 0; \\ {}_2A & \text{when } 0 < p < r,\ p \text{ odd}; \\ A/2A & \text{when } 0 < p < r,\ p \text{ even}. \end{cases}$$

Without knowledge of the morphisms in the triangle, the exact sequence does not determine the cohomology $H^p_{\mathrm{sing}}(\mathbb{P}^r, A)$ for $p \geqslant r$. A triangulation of $\mathbb{P}^r$ may be obtained from a G-invariant triangulation of S^r; it is a consequence that $H^p_{\mathrm{sing}}(\mathbb{P}^r, A) = 0$ for $p > r$. Given this fact, the long exact sequence reduces to isomorphism $H^{p-r}(G, A^{\pm}) \xrightarrow{\sim} H^{p+1}(G, A)$ for $p > r$ and an exact sequence:

$$0 \to H^r(G, A) \to H^r_{\mathrm{sing}}(\mathbb{P}^r, A) \to H^0(G, A^{\pm}) \to H^{r+1}(G, A) \to 0.$$

In turn, depending on the parity of r, the exact sequence is the following:

$$\begin{aligned} 0 \to {}_2A \to H^r_{\text{sing}}(I\!P^r, A) \to A \to A/2A \to 0 \quad (r \text{ odd}),\\ 0 \to A/2A \to H^r_{\text{sing}}(I\!P^r, A) \to {}_2A \to {}_2A \to 0. \quad (r \text{ even}). \end{aligned} \tag{19.4.2}$$

The exact sequence determines the cohomology in important cases, like $A = \mathbb{Z}$, $A = \mathbb{R}$, or $A = \mathbb{F}_2$. It is natural to expect from (19.4.2) in general that $H^r_{\text{sing}}(I\!P^r, A) = A$ when r is odd, and $H^r_{\text{sing}}(I\!P^r, A) = A/2A$ when r is even. In fact, there is an isomorphism $H^p_{\text{sing}}(I\!P^r, A) = H^p(C^{\leqslant r})$ for all p, where $C^{\leqslant r}$ is the r'th cochain truncation of the following positive complex (with the first A in degree 0):

$$C: \quad 0 \to A \xrightarrow{0} A \xrightarrow{2} A \xrightarrow{0} A \xrightarrow{2} A \xrightarrow{0} A \longrightarrow \cdots .$$

20. Local systems; homotopy groups.

(20.1) Setup. Fix a topological space X and a decent category $\mathfrak{C}$. Assume in particular the $\mathfrak{C}$ has small limits, and denote by 0 the initial object of $\mathfrak{C}$. A $\mathfrak{C}$-valued *local system* on X is a functor,

$$\mathcal{G}: \mathcal{P}(X) \to \mathfrak{C},$$

where $\mathcal{P}(X)$, the *fundamental groupoid* of X, is the following category: The objects of $\mathcal{P} = \mathcal{P}(X)$ are the points of X, the morphisms in $\mathcal{P}$ from $a \in X$ to $b \in X$ are homotopy classes of paths from a to b, and composition in $\mathcal{P}$ is concatenation of paths. The category $\mathcal{P} = \mathcal{P}(X)$ is indeed a groupoid: every morphism is an isomorphism.

Fix a point b in X. The group $\mathrm{Aut}_{\mathcal{P}}(b)$ (equal to $\mathrm{End}_{\mathcal{P}}(b)$) is the *fundamental group* $\pi = \pi_1(X, b)$ of X at b. View the group π as a category with one object. Then the inclusion is a functor,

$$b: \pi \hookrightarrow \mathcal{P}, \tag{20.1.1}$$

from the fundamental group to the fundamental groupoid $\mathcal{P} = \mathcal{P}(X)$. The corresponding restriction functor,

$$b^*: \mathfrak{C}^{\mathcal{P}} \to \mathfrak{C}^{\pi}, \tag{20.1.2}$$

associates to a local system $\mathcal{G}$ the co-π-object $\mathcal{G}(b)$. By the Kan-construction, the restriction functor has a right adjoint functor $\rho_b: \mathfrak{C}^{\pi} \to \mathfrak{C}^{\mathcal{P}}$. It associates with a co-π-object A the local system given as a limit,

$$(\rho_b A)(a) = \varprojlim_{a/\pi} A, \tag{20.1.3}$$

where the index category a/π is the right fiber at a of the inclusion $\pi \to \mathcal{P}$: Its objects are the paths $\xi: a \to b$, and there is only one morphism from $\xi: a \to b$ to $\eta: a \to b$, which is the loop $\eta\xi^{-1}$. Consequently,

$$(\rho_b A)(a) = \begin{cases} A & \text{if } a, b \text{ belong to the same path component,} \\ 0 & \text{otherwise;} \end{cases}$$

an explicit isomorphism in the first case being given by a choice of a path class $a \to b$.

Hence. , *when X is path connected: For any $b \in X$ the functor $\mathfrak{F} \mapsto \mathfrak{F}(b)$ is an equivalence between local systems on X and co-$\pi(X, b)$-objects.*

If a local system $\mathfrak{G}$ has values in an abelian category with $\prod$'s we may form the complex $C(\mathfrak{P}, \mathfrak{G})$ with cohomology $H^p(\mathfrak{P}, \mathfrak{G})$. If $b \in X$ (and $\pi := \pi(X, b)$), we have the restriction,

$$C(\mathfrak{P}, \mathfrak{G}) \to C(\pi, \mathfrak{G}(b)),$$

and, for a co-π-object A the right adjunction,

$$C(\mathfrak{P}, \rho_b A) \to C(\pi, A).$$

As noted above these two maps of complexes are homotopy equivalences when X is path connected. In particular, in the path connected case, the cohomology $H^p(\mathfrak{P}, \mathfrak{G})$ is isomorphic to the group cohomology $H^p(\pi(X, b), \mathfrak{G}(b))$.

(20.2). Special local systems are the homotopy groups: Let S^n be the *n-sphere*,

$$S^n = \{ x \in \mathbb{R}^{n+1} \mid \sum_i x_i^2 = 1 \},$$

as a pointed topological space (pointed by the north pole $p = (0, \dots, 0, 1)$. Let $\pi_n(X, b)$ be the set of homotopy classes of maps (of pointed topological spaces) $\varphi\colon (S^n, p) \to (X, b)$. The class in $\pi_n(X, b)$ represented by φ is denoted $[\varphi]$. Clearly, $\pi_0(X, b)$ is the set of path components of X.

Assume that $n \geqslant 1$. Then there is a well defined composition in $\pi_n(X, b)$ determined as follows: Denote by S^n_-, S^n_0 and S^n_+ the subsets of S^n determined, respectively, by the relations $x_1 \leqslant 0$, $x_1 = 0$, and $x_1 \geqslant 0$. By squeezing the equator S^n_0 to a north pole we get a map,

$$*\colon S^n \to S^n \vee S^n,$$

4trucm

For maps $\varphi, \psi\colon (S^n, p) \to (X, b)$ we obtain a map $(\varphi, \psi)\colon S^n \vee S^n \to X$ and the composition in $\pi_n(X, b)$ is determined by the formula,

$$[\varphi] * [\psi] := [\varphi * \psi].$$

It is a standard fact that the composition is well defined, and is a group law on $\pi_n(X, b)$, abelian if $n \geqslant 2$. [Note that the obvious identification $\pi_1(X, b) = \pi(X, b)$ is an anti-isomorphism with respect to the group structures as defined here.]

For a morphism $\xi\colon a \to b$ in the path category $\mathfrak{P}$ and an element $z \in \pi_n(X, b)$ there is an element $\xi_* z \in \pi_n(X, a)$ determined similarly the the obvious map $S^n \to S^n \vee I$ squeezing the upper hemisphere to I. The map ξ_* is a group ismomorhism,

$$\xi_*\colon \pi_n(X, b) \to \pi_n(X, a),$$

and the formation of $\pi_n(X, b)$ is an inverse local system on X, denoted $\pi_n(X)$, with values in **Sets** when $n = 0$, in **Gr** when $n = 1$, and in **Ab** when $n \geqslant 2$.

Note that the isomorphism $\pi_1(X, a) \to \pi_1(X, b)$ corresponding to a morhism $\xi\colon a \to b$ is given by the formula,

$$\omega \mapsto \xi_*(\omega) = \xi^{-1} \circ \omega \circ \xi.$$

We will need a few properties of the π_n.

Fact 1. The Homotopy addition Lemma. *20.3*

21. Singular cohomology with coefficients.

22. The Hurewicz spectral sequence.

23. Homotopy axioms for singular cohomology.

24. Barycentric subdivision.

25. Cubical cohomology.

26. The Leray–Serre spectral sequence, version A.

27. The Leray–Serre spectral sequence, version B.

28. Geometric realization.

29. Base change, bundles, etc. for singular cohomology.

30. Cohomology of presheaves.

31. Cech cohomology of presheaves.

32. Homotopy axioms for Cech cohomology.

33. Alexander–Spanier cohomology.

34. Paracompact spaces and fine presheaves.

35. Duality on locally compact spaces.

35a. Verdier duality.

35b. The dualizing complex.

35c. Bivariant cohomology.

35d. Computations.

35e. Manifolds.

35f. Vector bundles.

35g. Algebraic cycles.

35h. Algebraic cocycles.

35i. Complete intersections.

Acyclic complexes

1. Constructions.

(1.1) Definition. Recall that if $f: X \to Y$ is a chain map, then the *cone* of f is the complex $\mathrm{Con}\, f$ with objects $X(1) \oplus Y$ and differential $\left(\begin{smallmatrix} -\partial & 0 \\ f & \partial \end{smallmatrix}\right)$. The cone of f fits into a triangle,

$$\begin{tikzcd} & \mathrm{Con}\, f \arrow[dl, squiggly, "{(1,0)}"'] & \\ X \arrow[rr, "f"] & & Y, \arrow[ul, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"'] \end{tikzcd}$$

where the notation $Z \rightsquigarrow X$ indicates indicates a chain map $Z \longrightarrow X(1)$.

Dually, define the *co-cone* of f as the complex $\mathring{\mathrm{C}}\mathrm{on}\, f$ with objects $X \oplus Y(-1)$ and differential $\left(\begin{smallmatrix} \partial & 0 \\ f & -\partial \end{smallmatrix}\right)$. The co-cone fits into a similar diagram. Note the simple connection between the cone and the co-cone:

$$\mathring{\mathrm{C}}\mathrm{on}\, -f = \mathrm{Con}\, f(-1). \tag{1.1.1}$$

(1.2) Lemma. *Consider chain maps $f: X \to Y$ and $g: Y \to Z$. Form the cone $\mathrm{Con}\, f$ of f and the co-cone $\mathring{\mathrm{C}}\mathrm{on}\, g$ of g. Consider the resulting diagram,*

$$\begin{tikzcd} X \arrow[dr, "f"] \arrow[dd, dashed, "\tilde{f}"'] & & \mathrm{Con}\, f \arrow[ll, squiggly] \arrow[dd, dashed, "\tilde{g}"] \\ & Y \arrow[ur] \arrow[dr, "g"] & \\ \mathring{\mathrm{C}}\mathrm{on}\, g \arrow[ur] & & Z \arrow[ll, squiggly] \end{tikzcd}$$

Then there is a unique correspondence between the following three sets: The liftings of f to the co-cone $\mathring{\mathrm{C}}\mathrm{on}\, g$, the extensions of g to the cone of f, and the homotopies from gf to 0. More precisely, if $h: X(1) \to Z$ is a family of maps, then the following conditions on h are equivalent:

(i) *The map $\left(\begin{smallmatrix} f \\ h \end{smallmatrix}\right)$ is a chain map $X \to \mathring{\mathrm{C}}\mathrm{on}\, g$.*
(ii) *The map (h, g) is a chain map $\mathrm{Con}\, f \to Z$.*
(iii) *The map h is a homotopy, $gf = h\partial + \partial h$.*

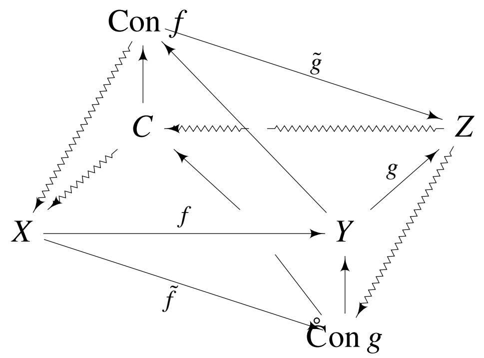
Moreover, if a lifting $\tilde{f}$ of f corresponds to an extension $\tilde{g}$ of g under the correspondance above (assuming necessarily that gf is homotopic to 0), then the cone C of $\tilde{f}$ is equal to the co-cone of $\tilde{g}$.

Proof. The proof is a simple computation. Note that C has $X(1) \oplus Y \oplus Z(-1)$ as objects, and the following differential

$$\begin{pmatrix} -\partial & 0 & 0 \\ f & \partial & 0 \\ h & g & -\partial \end{pmatrix}.$$

∎

(1.3) Remark. The maps of Lemma (1.2) appear as edges in the following octahedron



The octahedron has eight faces. Four of the faces are "triangular"; the other faces are commutative triangels. The square involving $\mathring{\mathrm{C}}\mathrm{on}\, g$, Y, C, and $\mathrm{Con}\, f$ is commutative. The square involving $\mathrm{Con}\, f$, X, Z, and $\mathring{\mathrm{C}}\mathrm{on}\, g$ is commutative up to the homotopy defined by $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

(1.4) Lemma. *Let $\mathfrak{K}$ be a full triangulated subcategory of complexes and $\mathfrak{I}$ a class of complexes such that every complex X in $\mathfrak{K}$ has a right $\mathfrak{I}$-resolution. Then, for every chain map $f\colon X \to Y$ in $\mathfrak{K}$ there is a factorization,*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, dashed] & & Y \\ & Z \arrow[ur, dashed] & \end{tikzcd} \qquad (1.4.1)$$

such that $X \to Z$ is a quasi-isomorphism and $Z \to Y$ is a semi split epic with kernel in $\mathfrak{I}$.

Proof. Form the cone $\mathrm{Con}\, f$ of f. Let $\tilde{g}\colon \mathrm{Con}\, f \to I$ be an arbitrary quasi-isomorphism. Consider the composition $g\colon Y \to \mathrm{Con}\, f \to I$. By construction, $\tilde{g}$ is an extension of g. Denote by $Z = \mathring{\mathrm{C}}\mathrm{on}\, g$ the co-cone of g. Then, by Lemma (1.2), there is a lifting $\tilde{f}$ of f corresponding to $\tilde{g}$. Hence we have obtained a diagram (1.4.1). The map $\tilde{f}$ is a quasi-isomorphism because its cone is equal to the co-cone of $\tilde{g}$. The map $Z \to Y$ is the projection from the co-cone Z of $g\colon Y \to I$. Hence the map $Z \to Y$ is a semi split epic with kernel equal to $I(-1)$. Thus the assertion of the Lemma follows, because by assumption I can be chosen such that $I(-1)$ belongs to $\mathfrak{I}$. ∎

(1.5) Construction. Let X be a complex. Recall that the truncated quotient complex $\tau_{\leqslant n}X$ (with a lower index) is the complex

$$\tau_{\leqslant n}X: \quad \cdots \longrightarrow 0 \longrightarrow Z_n \longrightarrow X_{n-1} \longrightarrow X_{n-2} \longrightarrow \cdots$$

where Z_n is the nth cocycle object, that is, the cokernel of the differential $X_{n+1} \to X_n$ (as allways, $X_n = X^{-n}$). The induced map $H_pX \to H_p(\tau_{\leqslant n}X)$ is an isomorphism when $p \leqslant n$; the homology $H_p(\tau_{\leqslant n}X)$ equals 0 for $p > n$. The truncated quotients fit into an inverse system

$$\cdots \longleftarrow \tau_{\leqslant n}X \longleftarrow \tau_{\leqslant n+1}X \longleftarrow \cdots \longleftarrow X$$

having X as inverse limit.

Let $\mathfrak{I}$ be a triangular class of complexes bounded below such that every bounded below complex X has a right $\mathfrak{I}$-resolution. Then it follows from Lemma (1.4) that there is a commutative diagram of inverse systems

$$\begin{tikzcd}
0 & \tau_{\leqslant 0}X \arrow[l] \arrow[d] & \cdots \arrow[l] & \tau_{\leqslant n}X \arrow[l] \arrow[d] & \tau_{\leqslant n+1}X \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 \arrow[u, leftarrow] & I_0 \arrow[l] & \cdots \arrow[l] & I_n \arrow[l] & I_{n+1} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

where the vertical maps are quasi-isomorphisms and the maps of bottom row are semi split surjections of objects of $\mathfrak{I}$. Indeed, assume that the I_p's are found for $p \leqslant n$. To define the I_{n+1} at the $n+1$'th level, apply Lemma (1.3) to the composition $\tau_{\leqslant n+1}X \to \tau_n X \to I_n$.

Consider the inverse limit $\varprojlim I_n$ and the corresponding map of complexes $X \to \varprojlim I_n$. *Question.* Under what conditions is the map $X \to \varprojlim I_n$ a quasi-isomorhism? Clearly, the homology of X is equal to the inverse limit of the homology of the I_n's, $H^pX = \varprojlim H^pI_n$. Fix a degree p. Note that we have a commutative diagram

$$\begin{tikzcd}
H^pX \arrow[r, "\sim"] \arrow[d] & \varprojlim H^p(\tau_n) \arrow[d, "\wr"] \\
H^pI \arrow[r] & \varprojlim H^p(I_n)
\end{tikzcd}$$

It follows that the map $H^pX \to H^pI$ is injective and that $H^pI \to \varprojlim H^p(I_n)$ is surjective. In particular, the map $X \to I$ is a quasi-isomorphism if and only iff for all p the canonical map

$$H^p(\varprojlim I_n) \to \varprojlim H^p(I_n)$$

is an isomorphim.

It follows that the question has an affirmative answer if the inverse limit $\varprojlim$ is exact. Assume more generally that the direct product $\prod_{\mathbf{N}}$ is exact. Then it is well known that for any inverse system over $\mathbf{N}$ of complexes and semi split epics there is an exact sequence

$$0 \longrightarrow \varprojlim^{(1)} H^{p-1}(I_n) \longrightarrow H^p(\varprojlim I_n) \longrightarrow \varprojlim H^p(I_n) \longrightarrow 0. \tag{1.5.1}$$

In the case at hand, the $H^{p-1}(I_n)$'s are essentially constant, and hence the $\varprojlim^{(1)}$ vanishes. Hence the answer is affirmative when the category has exact $\prod_{\mathbf{N}}$'s.

(1.6) Lemma. *Consider over the category (Ab) of $\mathbf{Z}$-modules an inverse system of complexes,*

$$0 \longleftarrow I_0 \longleftarrow \cdots \longleftarrow I_n \longleftarrow I_{n+1} \longleftarrow \cdots,$$

where the maps are semi split epics. Then the canonical map is surjective:

$$H^0(\varprojlim I_n) \to \varprojlim H^0(I_n). \tag{1.6.1}$$

Denote by K_n the kernel of the surjection $I_n \to I_{n-1}$. Assume that $H^0(K_n) = 0$ for $n \gg 0$. Then the canonical map is an isomorphism. If, in addition, $H^1(K_n) = 0$ for $n \gg 0$, then

$$H^0(\varprojlim I_n) \xrightarrow{\sim} H^0(I_n) \quad \text{for } n \gg 0.$$

Proof. The last assertion is a consequence of the preceding assertions since, under the additional assumptions, the inverse system $H^0(I_n)$ is essentially constant.

The two first assertions of the Lemma is contained in the statement of (1.5). In fact, the asserted surjectivity follows from the exact sequence (1.5.1). Morover, it follows from the vanishing of the $H^0(K_n)$'s that the maps $H^{-1}(I_n) \to H^{-1}(I_{n-1})$ are surjective when $n \gg 0$. Therefore, as is well known, the $\varprojlim^{(1)}$ in the exact sequence (1.5.1) vanishes, and hence the asserted bijectivity is a consequence.

Here is a direct proof of the first two assertions: By hypothesis, there is a chain map $f_n \colon I_n(-1) \to K_{n+1}$ and an identification of I_{n+1} with the cone of f_{n+1}. Hence $I_{n+1} = I_n \oplus K_{n+1}$ with the differential as in (1.1).

To prove the asserted surjectivity, represent an element on the right hand side of (1.6.1) as a sequence of cycles (of degree 0), $(c_0, c_1, \dots)$, such that the image of c_{n+1} in I_n is homologous to c_n. It is asserted that there is a sequence $(d_0, d_1, \dots)$ such that d_n is homologous to c_n and the image of d_{n+1} in I_n is equal to d_n. The elements d_n are constructed inductively. Assume that $d_0, \dots, d_n = d$ have been found. The element c_{n+1} has the form $\binom{c}{k}$, and the elements homologous to c_{n+1} have the form

$$\begin{pmatrix} c \\ k \end{pmatrix} + \begin{pmatrix} -\partial & 0 \\ f & \partial \end{pmatrix} \begin{pmatrix} c' \\ k' \end{pmatrix} = \begin{pmatrix} c - \partial c' \\ k + fc' + \partial k' \end{pmatrix}. \tag{1.6.2}$$

By assumption, the image of c_{n+1}, that is c, is homologous to d. Hence the exists an element c' in I_n of degree -1 such that $c - d = \partial c'$. It follows that an element on the right hand side of (1.6.2) has the form $\binom{d}{k''}$, as asserted.

To prove the asserted injectivity, consider an element in the kernel of (1.6.1). Represent the element as a cycle c (of degree 0), $(k_0, k_1, \dots)$. Then the finite column c_n with entries $k_0, \dots, k_n$ represents the image in $H^0(I_n)$. By assumption, the cycle c_n is a boundary, that is, it has the form $c_n = \partial d_n$, where d_n is an element of degree -1 in I_n, given by a sequence $l_0, \dots, l_n$. It is asserted that an infinite sequence $l_0, l_1, \dots$ can be found. The sequence is defined inductively. The element c_{n+1} is equal to $\binom{c_n}{k}$. Let $c_n = \partial d_n$. By assumption, the element c_{n+1} is a boundary, $c_{n+1} = \partial d_{n+1}$, where d_{n+1} has the form $\binom{x}{l}$. It is asserted that d_{n+1} dan be chosen of the form $\binom{d_n}{l}$. The assertion is easily verified. ∎

(1.7) Corollary. *Given an inververse system of semi split epics:*

$$0 \leftarrow I_0 \leftarrow \cdots \leftarrow I_n \leftarrow \cdots .$$

Assume that every I_n is homotopy injective. Then the inverse limit $I := \lim I_n$ is a homotopy injective complex.

Proof. Recall that a complex I is homotopy injective if, for every acyclic complex X we have that $\mathrm{Hot}(X, I) = 0$, or equivalently, if the functor $Hom^{\cdot}(-, I)$ takes acyclic complexes to acyclic complexes.

To prove the Lemma, let K_n be the kernel of the split epic $I_n \to I_{n-1}$. Then, each K_n is homotopy injective. Let X be any complex. Then the $Hom^{\cdot}(X, I_n)$ form an inverse system of split epics, and $Hom^{\cdot}(X, K_n)$ is the kernel of the N'th map of the system. The p'th cohomology of the complex $Hom^{\cdot}(X, Y)$ is $\mathrm{Hot}(X[-p], Y)$. Assume that X is acyclic. Then the cohomology of $Hom^{\cdot}(X, K_n)$ vanishes in all degrees and for all n. Therefore, by the Lemma, $\mathrm{Hot}(X, I)$ is equal to $\mathrm{Hot}(X, I_n)$ for $n \gg 0$, and hence $\mathrm{Hot}(X, I)$ is equal to 0. ∎

(1.8) Examples. (i) Every right complex of injectives is homotopy injective. The proof is easy and well known.

(ii) Assume that the underlying abelean category has finite injective dimension, that is, assume for some d that every object has an injective resolution of length at most d. Then every complex of injective objects is homotopy injective. In fact, if Q is any complex of injectives and X is an arbitrary complex, then

$$\mathrm{Hot}(X, Q) = \mathrm{Hom}_D(X, Q). \tag{1.8.1}$$

To prove the assertion, we need that any complex X admits a quasi isomorphism into a comples of injectives. The latter result follows from the lemma below.

Clearly, it suffices to establish the isomorphism (1.8.1). Let us first note that any acyclic complex Q of injectives is homotopy trivial. Indeed, for the n cycle object Z^n of Q we have an infinite left resolution,

$$\cdots \to Q^{n-2} \to Q^{n-1} \to Z_n \to 0.$$

Since the resolution has length at least equal to d, it follows that Z_n is injective. Therefore, the inclusion of Z_n in Q_n is split. Hence the acyclic complex Q is homotopy trivial.

Now, the arrows in the group on the right side of (1.8.1) are represented by diagrams,

$$\begin{tikzcd} & Q' & \\ X \arrow[ur] & & Q, \arrow[ul, "s"'] \end{tikzcd}$$

where s is a quasi-isomorphism. The morphism s may be followed by a quasi-isomorphism into a complex of injectives. Hence we may assume that Q' is a complex of injectives. It suffices to prove that the induced map, $\mathrm{Hot}(X, Q) \to \mathrm{Hot}(X, Q')$ is an isomorphism. Equivalently, if Q'' is the cone of s, it suffices to prove that $\mathrm{Hot}(X, Q'') = 0$. However, Q'' is a complex of injectives, and acyclic since s is a quasi-isomorphism. Hence Q'' is homotopy trivial as was observed above. Therefore, $\mathrm{Hot}(X, Q'') = 0$.

(1.9) Note. The result used in (1.8)(ii) that an acyclic complex of injectives is homotopy trivial does require some additional hypothesis. Consider for instance the ring $R = \mathbf{Z}/4$ and the infinite complex P with $P_n = R$ and multiplication by 2 as differentials. Then P is an acyclic comples of projectives. Obviously, P is not homptopy trivial.

(1.10) Lemma. *Let $\mathfrak{Q}$ be an additive class of objects such that every object has an embedding into an object of $\mathfrak{Q}$. Then every complex X has an embedding into a complex of objects from $\mathfrak{Q}$. Moreover, every positive complex has a quasi-isomorphism into a positive complex of objects from $\mathfrak{Q}$. Finally, if $\mathfrak{Q}$ has finite right dimension, then every complex has a quasi-isomorphism into af complex of objects in $\mathfrak{Q}$.*

Proof. Chose a family of embeddings, $f\colon X \hookrightarrow P$ into a family P of objects of $\mathfrak{Q}$. Consider P as a complex with zero differential, and form the cone Q of the identity $1\colon P \to P$. Thus Q has objects $P(1) \oplus P$, and differential $\binom{0\ 0}{1\ 0}$. Clearly, $\binom{f\partial}{f}$ is a chain map $X \to Q$. It is an embedding, because its second projection is the chosen family of embeddings $f\colon X \to P$.

To prove the second statement, let X be a positive complex. Chose as embedding $X \hookrightarrow Q$ into a complex Q of objects from $\mathfrak{Q}$, and truncate Q by defining $Q^p := 0$ for $p < 0$. Hence an embedding $f\colon X \to Q$ into a positive complex is obtained. Embedd similarly the cokernel of f, and continue to obtain an exact sequence of positive complexes,

$$0 \longrightarrow X \longrightarrow Q^0 \longrightarrow Q^1 \longrightarrow \cdots . \tag{1.10.1}$$

View the complex of Q^p's as a first quadrant bicomplex. Then the exact sequence defines a quasi-isomorphim into the total complex of this bicomplex.

To prove the last statement, note that a resolution similar to (1.10.1) can be formed for an arbitrary complex X. If $\mathfrak{Q}$ has finite dimension, the resolution may be chosen finite. Hence the same argument as above applies. ▯

(1.11) Lemma. *Let $\mathfrak{M}$ be thick sub-category, and $\mathfrak{I} \subseteq \mathfrak{M}$ a subclass of injective objects such that every object A in $\mathfrak{M}$ embeds into an object of $\mathfrak{I}$. Then every positive complex X with cohomology in $\mathfrak{M}$ admits a quasi-isomorphism $X \xrightarrow{\sim} I$ into a positive complex of objects from $\mathfrak{I}$.*

Proof. Let X be a positive complex with cohomology in $\mathfrak{M}$. The positive complex I and the quasi-isomorphism $X \to I$ will be constructed inductively. It suffices to prove the following: Given a complex of length n of objects of $\mathfrak{I}$,

$$I:\quad \cdots \longrightarrow 0 \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^n \longrightarrow 0 \longrightarrow \cdots$$

and a chain map $f\colon X \to I$ whose cone $\operatorname{Con} f$ has cohomology $H^p(\mathring{\operatorname{C}}\!\operatorname{on} f) = 0$ for $p < n$, then I can be extended with one extra object I^{n+1} in degree $n+1$ into a complex $\tilde{I}$ and f can be extended to a chain map $\tilde{f}\colon X \to \tilde{I}$ whose cone has cohomology $H^p(\operatorname{Con} \tilde{f}) = 0$ for $p < n + 1$.

To prove the assertion, embedd the n'th cohomology $H^n(\operatorname{Con} f)$ into an object J of $\mathfrak{I}$. If Z^n is the n'th cycle object of $\operatorname{Con} f$, there is an exact sequence

$$\operatorname{Con} f^{n-1} \longrightarrow Z^n \longrightarrow J.$$

Since J is injective, the map $Z^n \to J$ can be extended to a map $\operatorname{Con} f^n \to J$. By construction, the latter map defines defines a chain map $\tilde{g}\colon \operatorname{Con} f \to J(-n)$, such that $\tilde{g}$ induces an injection of cohomology in degree n. Since $\operatorname{Con} f$ has zero cohomology in degree $p < n$, it follows that the co-cone of $\tilde{g}$ has zero cohomology in degree $p \leqslant n$. Let g be the restriction of $\tilde{g}$ to I, and denote by $\tilde{I}$ the co-cone of g. By Lemma (1.1), the extension $\tilde{g}$ of g corresponds to a lifting $\tilde{f}\colon X \to \tilde{I}$ of f, and the cohomology of the cone of $\tilde{f}$ equals the cohomology of the co-cone of $\tilde{g}$; hence the cone of $\tilde{f}$ has cohomology equal to 0 in degree $p \leqslant n$. Thus $\tilde{f}$ has the required properties. ∎

(1.12) Lemma. *Consider a complex X of $\mathfrak{O}$-modules on a scheme. Assume that X has quasi-coherent cohomology. Chose as in* (1.5) *a quasi-isomorphism from the truncated quotients into bounded below complexes of injectives. Then the construction $\varprojlim I_n$ of* (1.5) *yields a quasi-isomorphim $X \to \varprojlim I_n$.*

Proof. It has to be shown that the canonical map (in degree 0 say), $H^0(\varprojlim I_n) \to \varprojlim H^0(I_n)$ is an isomorphism. Note that the inverse system on the right hand side is essentially constant: When $n \gg 0$, $H^0(I_n) = H^0(X)$. Hence it has to be proved that the canonical map

$$H^0(\varprojlim I_n) \to H^0(I_n) \tag{1.12.1}$$

is an isomorphism when $n \gg 0$. The two sides of (1.12.1) are cohomology groups of complexes of sheaves. Hence they are equal to the sheaffifications of the cohomology of the underlying complexes of presheaves. The map of presheaf cohomology is the map

$$H^0(\varprojlim I_n(U)) \to H^0(I_n(U)) \tag{1.12.2}$$

defined for all open subsets U of X. Hence it suffices to prove that the latter map is an isomorphism over every affine open U (when $n \geqslant n_0$, independent of U). The latter assertion follows from Lemma (1.6). Indeed, assume that U is affine. The complex of modules $K_n(U)$ is the complex of sections over U in an injective resolution K_n of $H^n(X)(-n)$. Therefore, since $H_n(X)$ is quasi-coherent and U is affine, the only cohomology of $K_n(U)$ is in degree $-n$. In particular, $H^0(K_n(U)) = 0$. Thus the conditions for applying Lemma (1.6) are satisfied. It follows from Lemma (1.6) that (1.12.2) is an isomorphism. ∎

2. Injective complexes.

(2.1) Setup. Given a triangulated category K and a multiplicative system S in K. Recall that an object q in K is called S-*injective* if $Hom(x, q) = 0$ for any object x which is S-equivalent to 0. It follows from the long exact sequence of Hom that if q is S-injective then, for any map $s : x \to y$ of S, the induced map $Hom(y, q) \to Hom(x, q)$ is an isomorphism. Similarly, if two vertices of an exact triangle are S-injective, then so is the third.

In addition, recall that

$$Ext^n_S(x, y) := Hom_D(x, y[n]) = Hom_D(x[-n], y),$$

where $D := KS^{-1}$ is the localized category.

(2.2) Definition. A complex q which as an object in the homotopy category $\mathrm{Hot}(\mathfrak{A})$ is injective with respect to the multiplicative system of quasi-isomorphisms is called *homotopy injective*.

(2.3) Lemma. *Any right complex q of injectives is homotopy injective.*

Proof. Let x be an acyclic complex and let $f : x \to q$ be a chain map. We have to prove that f is homotopy trivial, that is, we have to show that there exists a family of maps $s : x[1] \to q$ such that, for all n,

$$f_n = \partial_{n-1}s_{n-1} + s_n\partial_n. \tag{2.3.1}$$

The maps s_n are constructed inductively. Since q is a right complex, the equations (2.3.1) hold with $s_n := 0$ for $n \ll 0$. Assume that morphisms s_n are found for $n \leqslant p$ such that (2.3.1) holds. To construct s_{p+1}, consider the morphism $f' := f_{p+1} - \partial_p s_p$. As (2.3.1) holds for $n = p$, we have that

$$f'\partial_p = f_{p+1}\partial_p - \partial_p s_p \partial_p = f_{p+1}\partial_p - \partial_p(f_p - \partial_{p-1}s_{p-1}).$$

The right side vanishes because $\partial\partial = 0$ and $f\partial = \partial f$. Hence $f'\partial_p = 0$. It follows, since x is acyclic, that $f' : x^{p+1} \to q^{p+1}$ extends to a morphism defined on the $(p+2)$'th boundary object of x. The latter morphism extends, since q^{p+1} is injective, to a morphism defined on x^{p+2}. Thus f' extends to a map $s_{p+1} : x^{p+2} \to q_{p+1}$. From $f' = s_{p+1}\partial_{p+1}$ and the definition of f', it follows that (2.3.1) holds for $n = p + 1$. ∎

(2.4) Lemma. *Consider an inverse system of split surjective epimorphisms of complexes,*

$$0 \leftarrow q_0 \leftarrow g_1 \leftarrow \cdots .$$

Assume that $\mathfrak{A}$ has infinite products. If each q_n is homotopy injective, then the inverse limit $q = \varprojlim q_n$ is homotopy injective.

Proof. See [Der1]. ∎

(2.5) Lemma. *Fix a complex a. Let K be a triangular subcategory of* $\mathrm{Hot}(\mathfrak{A})$ *such if $x \to z$ is a quasi-isomorphim and x belongs to K, then there exists a quasi-isomorphism $z \to x'$ with x' in K. Assume that the functor $hom(a, -)$ is derivable on K. Then,*

$$R^n hom(a, x) = Ext^n(a, x). \tag{2.4.1}$$

In particular,

$$R^0 hom(a, x) = Hom_D(a, x), \tag{2.4.2}$$

where $D = D(\mathfrak{A})$.

Proof. Since $hom(a, -)$ is derivable on K, there is a triangular subclass Q of K such that (1) any complex x of K admits a quasi-isomorphism into a complex q of Q and (2) if q is an acyclic complex in Q, then $hom(a, q)$ is acyclic.

Clearly, it suffices to prove the special case (2.4.2). Moreover, by (1), it suffices to verify (2.4.2) when $x = q$ belongs to Q. As $Rhom(x, q) = hom(x, q)$, the left hand side of (2.4.2) is equal to $H^0 hom(a, q) = \mathrm{Hot}(a, q)$. Thus it suffices to prove, for q in Q, that the canonical map is an isomorphism,

$$\mathrm{Hot}(a, q) \to Hom_D(a, q).$$

By definition, $Hom_D(a, q)$ is the direct limit of $\mathrm{Hot}(a, q')$ over the index category whose objects are pairs $a \to q' \leftarrow q$ where $q \to q'$ is a quasi-isomorphism. By the hypothesis on K, we may restrict to pairs with q' in K, and by (1) we may even restrict to pairs with q' in Q. However, by (2), the direct system is constant on the restricted category; hence the direct limit is equalt to $\mathrm{Hot}(a, q)$, as asserted. ∎

(2.6) Example. Let T be a left exact functor defined on $\mathfrak{A}$. Assume that T is derivable, that is, assume that there is an additive subclass $\mathfrak{Q}$ of $\mathfrak{A}$ such that (1) any object of $\mathfrak{A}$ admits an embedding into an object of $\mathfrak{Q}$ and (2) if Q' is a subobject of Q and Q' and Q belongs to $\mathfrak{Q}$, then $Q'' := Q'/Q$ belongs to $\mathfrak{Q}$ and $TQ \to TQ''$ is an epimorphism. Then, as is well known, the functor T is derivable on $Hot^+(\mathfrak{A})$. Assume that the derived functor RT is of finite dimension, that is, assume that there is an integer d such that for any object A of $\mathfrak{A}$, we have that $R^n T(A) = 0$ for $n > d$. Then the functor T is derivable on all of $\mathrm{Hot}(\mathfrak{A})$. In fact, consider the class of objects Q such that $R^n T(Q) = 0$ for all $n > 0$. It contains the class $\mathfrak{Q}$ and, clearly, it has the properties (1) and (2). So we may replace $\mathfrak{Q}$ with the class of objects such that $R^n T(Q) = 0$ for $n > 0$. Then $\mathfrak{Q}$ has the following addtional property: (3) given a exact sequence,

$$0 \to A \to Q^0 \to Q^1 \to \cdots \to Q^d \to B \to 0.$$

If all the Q^n belongs to $\mathfrak{Q}$, then B belongs to $\mathfrak{Q}$.

Now it follows easily from (1) and (3) that any complex has a finite right resolution of length at most d with complexes of objects of $\mathfrak{Q}$. It follows that any complex admits a quasi-isomorphism into a complex of objects from $\mathfrak{Q}$. Moreover, if q is an acyclic complex of objects from $\mathfrak{Q}$, then the complex Tq is acyclic. Indeed, from (3) it follows that the cocycle objects of q are in $\mathfrak{Q}$. As T is short exact on $\mathfrak{Q}$ by (2), it follows that Tq is acyclic.