Schubert calculus on Grassmannians and exterior powers

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Abstract. We present a general formalism for homology and cohomology theories for Grassmannians, and show how the bivariant intersection theory of W. Fulton and R. MacPherson fits into this formalism

Introduction

Grassmannians have always provided a testing ground for cohomology theories. Under the name of *Schubert Calculus* there is a variety of descriptions of the classical cohomology, the Chow ring, bivariant intersection theories, quantum cohomology, equivariant cohomology, and equivariant quantum cohomology. The central parts of all these descriptions are *Pieri type formulas*, and variations on the *Giambelli formula*.

In [LT] we proposed a general algebraic model for Schubert Calculus in terms of the ring of symmetric functions in d variables over a ring A acting on the d'th exterior power of the polynomial ring $A[T]$ in one variable over A. This action defines a module structure on the exterior power such that the Pieri formula is part of the action, and the structure of the module is described by the Giambelli formula, or rather a more general *determinantal formula*.

The model of [LT] does not however, represent any of the cohomology theories mentioned above since it is given by a module of infinite rank. To obtain a generalization of these theories we must modify the general model by replacing the polynomial ring $A[T]$ by the residue ring $A[T]/(p)$ of a monic polynomial $p(T)$ of degree $n \geq d$, and the symmetric algebra by a *degree-d factorization algebra* of $p(T)$.

In this article we perform this modification. The main tools are splitting and factorization algebras of polynomials, and a considerable portion of the first part of this work consists in defining and giving the main properties of such algebras.

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In the second part we show how our model gives the bivariant intersection theory of W. Fulton and R. MacPherson for Grassmannians when the base scheme has a bivariant theory. In this representation the polynomial $p(T)$ mentioned above corresponds to the Chern polynomial of the locally free sheaf on the base defining the Grassmannian. In order to fully clarify the relation with our model we redo the theory of Chern classes in the language of bivariant theory.

It is worth mentioning that in order to show that our model gives the bivariant theory for Grassmannians we use the determinantal formula of Schubert calculus in bivariant intersection theory. This can however easily be avoided by a slight development of our model to include *Gysin maps*. Resolving the singularities of a Schubert scheme in the usual way one then proves that the class of the Schubert scheme in bivariant intersection theory corresponds to a determinantal expression in our model. In this way our formalism gives the full Schubert calculus of bivariant intersection theory, using only the main results for *projective bundles* used to define *Chern classes* (see Theorem 4.2, Corollary 4.3, and Section 4.4). To save space we have not included this material.

Part of our work is based upon an idea of L. Gatto [G]. It represents a crucial generalization and change of language of the material in [G]. After our text was made available a similar generalization was performed by Gatto and T. Santiago [GS] in the language of [G].

We now give a more detailed description of the approach sketched above, and we exhibit the main results of [LT] and of this work:

0.0 Setup. Let A be a commutative ring with 1. We fix the following notation: d is a positive integer and $A[X_1, X_2, \dots, X_d]$ is the polynomial algebra in d variables over A. Let $S := A[X_1, X_2, \dots, X_d]^{\mathrm{sym}}$ be the subalgebra of symmetric polynomials. Write $Q(T)$ for the following polynomial in one variable T,

$$Q(T) = (T - X_1) \cdots (T - X_d) = T^d - c_1 T^{d-1} + \cdots + (-1)^d c_d. \tag{0.0.1}$$

The coefficients $c_i = c_i(X_1, \dots, X_d)$ are the elementary symmetric functions, generating S as an A-algebra. In particular, $Q(T)$ is a monic polynomial with coefficients in S.

0.1 Laurent series. Denote by $A[T][[T^{-1}]]$ the algebra of *Laurent series* of finite degree. In other words, an element of the algebra $A[T][[T^{-1}]]$ is an infinite series of the form,

$$g = b_N T^N + b_{N-1} T^{N-1} + b_{N-2} T^{N-2} + \cdots ,$$

with each b_i in A. Note that the algebra $A[T]$ of polynomials is a subalgebra of $A[T][[T^{-1}]]$. When $b_N = 1$ the Laurent series is invertible in $A[T][[T^{-1}]]$. In particular, every monic polynomial $p(T)$ has an inverse in $A[T][[T^{-1}]]$. For instance, the inverse of the polynomial $Q(T)$ in (0.0.1) is the series in $S[T][[T^{-1}]]$,

$$\frac{1}{Q(T)} = T^{-d} \sum_{i=0}^{\infty} s_i T^{-i}, \tag{0.1.1}$$

where $s_j = s_j(X_1, \dots, X_d)$ is the j'th *complete symmetric function.*

0.2 The residue. For every sequence $g_1, \dots, g_d$ of d Laurent series in $A[T][[T^{-1}]]$,

$$g_i = b_{i,N_i}T^{N_i} + b_{i,N_i-1}T^{N_i-1} + b_{i,N_i-2}T^{N_i-2} + \cdots ,$$

we define the *residue* as the determinant in A,

$$\mathrm{Res}(g_1, \dots, g_d) := \begin{vmatrix} b_{1,-1} & b_{1,-2} & \dots & b_{1,-d} \\ b_{2,-1} & b_{2,-2} & \dots & b_{2,-d} \\ \vdots & \vdots & \ddots & \vdots \\ b_{d,-1} & b_{d,-2} & \dots & b_{d,-d} \end{vmatrix}.$$

Clearly, the residue is an A-multi-linear alternating function of $g_1, \dots, g_d$, and it vanishes if one of $g_1, \dots, g_d$ is a polynomial.

0.3 The symmetric structure. Denote by $A[X]$ the polynomial ring in the variable X over A. There is a canonical identification of $\bigotimes_A^d A[X]$ with $A[X_1, X_2, \dots, X_d]$. In particular, if M is any $A[X]$-module then $\bigotimes_A^d M$ has a natural structure as a module over $A[X_1, X_2, \dots, X_d]$, and hence as a module over the symmetric algebra $S = A[X_1, X_2, \dots, X_d]^{\mathrm{sym}}$. We recall from [LT, Proposition 1.6] that $\bigwedge_A^d M$ has a unique *symmetric structure* as an S-module such that the canonical surjection $\bigotimes_A^d M \to \bigwedge_A^d M$ is S-linear.

0.4 The Main Theorem of [LT]**.**

(1) *The exterior power $\bigwedge_A^d A[X]$ with its symmetric structure as a module over the algebra $S := A[X_1, \dots, X_d]^{\mathrm{sym}}$ is free of rank* 1*, generated by the d-vector $X^{d-1} \wedge \cdots \wedge X^0$.*

(2) (The determinantal formula) *For any sequence of d polynomials $f_1, \dots, f_d$ in $A[T]$ we have the equation in $\bigwedge_A^d A[X]$,*

$$f_1(X) \wedge \cdots \wedge f_d(X) = \mathrm{Res}\left(\frac{f_1}{Q}, \dots, \frac{f_d}{Q}\right) X^{d-1} \wedge \cdots \wedge X^0.$$

(3) (Gatto's formula) *For any weakly decreasing set of exponents $h_1 \geq \cdots \geq h_d$ we have the equation,*

$$X^{h_1+d-1} \wedge X^{h_2+d-2} \wedge \cdots \wedge X^{h_d} = s_{h_1,h_2,\dots,h_d} X^{d-1} \wedge \cdots \wedge X^0,$$

where $s_{h_1,h_2\dots,h_d} = s_{h_1,h_2\dots,h_d}(X_1, \dots, X_d) \in S$ is the Schur polynomial, defined as the following determinant in the complete symmetric functions $s_j = s_j(X_1, \dots, X_d)$:

$$s_{h_1,h_2\dots,h_d} = \begin{vmatrix} s_{h_1} & s_{h_1+1} & \dots & s_{h_1+d-1} \\ s_{h_2+1} & s_{h_2} & \dots & s_{h_2-d-2} \\ \vdots & \vdots & \ddots & \vdots \\ s_{h_d-d-1} & s_{h_d-d-2} & \dots & s_{h_d} \end{vmatrix}. \tag{0.4.1}$$

0.5 Splitting algebras and factorization algebras. Let p be a monic polynomial of degree $n \geq d$ in $A[T]$. We denote by $\mathrm{Split}^d(p)$ the *d'th partial splitting algebra* of p over A. If $\mathrm{Split}^n(p) = A[\xi_1, \dots, \xi_n]$ is the *full universal splitting algebra* of p over A, generated by the universal roots $\xi_1, \dots, \xi_n$ (see [B, IV 6.5], [PZ, 2.1], [EL], or [T2]) then $\mathrm{Split}^d(p)$ is the subalgebra,

$$\mathrm{Split}^d(p) = A[\xi_1, \dots, \xi_d],$$

generated by the first d roots. Alternatively, the splitting algebras may be constructed inductively: The first splitting algebra $A_1 = \mathrm{Split}^1(p)$ is the quotient $A_1 := A[X]/(p(X))$ and ξ_1 is the class of X modulo $p(X)$. Then p has the root ξ_1 in A_1, and consequently p has a splitting, $p = (T - \xi_1)p_1$, with a polynomial $p_1 \in A_1[T]$. Inductively, the d'th splitting algebra of p may be obtained as the $(d-1)$'th splitting algebra of p_1. The first splitting algebra A_1 is free of rank n as an A-module, with basis $1, \xi_1, \dots, \xi_1^{n-1}$; in particular, it contains A. By induction, the splitting algebra $\mathrm{Split}^d(p)$ is free of rank $n(n-1)\cdots(n-d+1)$ as an A-module, and it contains A.

Over $\mathrm{Split}^d(p)$ there is a universal splitting,

$$p = q\tilde{q}, \quad \text{where } q = (T - \xi_1)\cdots(T - \xi_d), \tag{0.5.1}$$

of p into a product of a set of d monic factors of degree 1 and one factor of degree $n - d$. Let $R \subset \mathrm{Split}^d(p)$ be the A-subalgebra generated by the coefficients of q; it follows from the equation $p = q\tilde{q}$ that R is also generated by the coefficients of $\tilde{q}$. We prove in Section 1 that the factorization $p = q\tilde{q}$ in $R[T]$ is the universal factorization of p into a product of two monic factors of degrees d and $n - d$. The subalgebra R is called the *degree-d factorization algebra* of p over A, and it is denoted $R = \mathrm{Fact}^d(p)$.

Note that under the surjection of algebras $A[X_1, X_2, \dots, X_d] \to \mathrm{Split}^d(p)$, mapping X_i to ξ_i, the symmetric algebra S, generated by the elementary symmetric functions, is mapped onto the factorization algebra R, that is, we have a canonical surjective algebra homomorphism,

$$S \to R.$$

As noted in the inductive construction, the first partial splitting algebra $\mathrm{Split}^1(p)$, equal to the degree-1 factorization algebra $\mathrm{Fact}^1(p)$, may be defined as the quotient $A[X]/(p(X))$. We write $\xi = \xi_1$ for the first universal root, equal to the class of X modulo $p(X)$. As observed in 0.3 above, the exterior power $\bigwedge_A^d A[\xi]$ has a natural symmetric structure of an S-module.

0.6 The Main Theorem. *Let $p \in A[T]$ be a monic polynomial of degree $n \geq d$, let $A[\xi] = A[T]/(p)$ be the first splitting algebra, and let $R = \mathrm{Fact}^d(p)$ be the degree-d factorization algebra, with the universal factorization $p = q\tilde{q}$ in $R[T]$. Then the symmetric action of S on $\bigwedge^d A[\xi]$ factors through the surjection $S \to R$ described in* 0.5*. Moreover, the following assertions hold:*

(1) *The exterior power $\bigwedge_A^d A[\xi]$ with its structure as an R-module is free of rank 1 generated by the d-vector $\xi^{d-1} \wedge \cdots \wedge \xi^0$.*

(2) (The determinantal formula) *For any sequence of polynomials* $f_1, \dots, f_d$ *in* $A[T]$ *we have the equation in* $\bigwedge_A^d A[\xi]$,

$$f_1(\xi) \wedge \cdots \wedge f_d(\xi) = \operatorname{Res}\left(\frac{f_1}{q}, \dots, \frac{f_d}{q}\right) \xi^{d-1} \wedge \cdots \wedge \xi^0. \tag{0.6.1}$$

(3) (Gatto's formula) *For any weakly decreasing set of exponents* $h_1 \geq \cdots \geq h_d$, *we have the equation in* $\bigwedge_A^d A[\xi]$,

$$\xi^{h_1+d-1} \wedge \xi^{h_2+d-2} \wedge \cdots \wedge \xi^{h_d} = s_{h_1,h_2,\dots,h_d}(\xi_1, \dots, \xi_d)\, \xi^{d-1} \wedge \cdots \wedge \xi^0, \tag{0.6.2}$$

where $s_{h_1,h_2\dots,h_d}(\xi_1, \dots, \xi_d) \in R$ *is the value of the Schur polynomial* 0.4.1 *at* $\xi_1, \dots, \xi_d$.

(4) *The* A*-linear map induced by the residue in* (0.6.1),

$$\bigwedge\nolimits_A^d A[\xi] \to R, \tag{0.6.3}$$

is an R*-linear isomorphism.*

0.7 Corollary. *The* $\binom{n}{d}$ *elements* $s_{h_1,\dots,h_d}(\xi_1, \dots, \xi_d) \in R$ *for* $n - d \geq h_1 \geq \cdots \geq h_d \geq 0$ *form a free* A*-basis for the factorization algebra* $R = \operatorname{Fact}^d(p)$.

The Corollary follows immediately from assertion (4) of the Theorem: the d-vectors $\xi^{h_1+d-1} \wedge \cdots \wedge \xi^{h_{d-1}+1} \wedge \xi^{h_d}$ in the range $n - d \geq h_1 \geq \cdots \geq h_d \geq 0$ form an A-basis for $\bigwedge_A^d A[\xi]$, and the elements $s_{h_1,\dots,h_d}(\xi_1, \dots, \xi_d)$ are their images in R.

0.8 Remark. The four assertions in 0.6 are not independent:

First, (2) and (3) are equivalent. Indeed, under the surjection $A[X_1, X_2, \dots, X_d] \to A[\xi_1, \dots, \xi_d]$, specializing $X_1, \dots, X_d$ to $\xi_1, \dots, \xi_d$, the polynomial $Q(T)$ specializes to $q(T)$. Hence we obtain from (0.1.1) that $T^h/q = \sum_{j=d-h}^{\infty} s_{h+j-d}(\xi_1, \dots, \xi_d)T^{-j}$. Hence Equation (0.6.2) is the special case of Equation (0.6.1) obtained with $f_i := T^{h_i+d-i}$. Thus (3) follows from (2). Conversely, (2) is a consequence of (3) since the two sides of (2) are A-multi-linear and alternating in the polynomials $f_1, \dots, f_d$.

Secondly (1) and (2) imply (4). In fact, consider (4). The residue considered in (2) is A-multi-linear and alternating in the polynomials $f_1, \dots, f_d$. Moreover, it vanishes if one of the f_i in the ring $R[T]$ is divisible by q; in particular, the residue vanishes if one of the f_i in the ring $A[T]$ belongs to the principal ideal (p). Therefore, the residue induces the A-linear map (0.6.3). Clearly, it maps $\xi^{d-1} \wedge \cdots \wedge \xi \wedge 1$ to $1 \in R$. If (1) and (2) holds, then the residue in (2) is the coefficient with respect to the basis element given in (1). Therefore, the map (0.6.3) is R-linear and bijective.

Conversely, if (4) holds, then obviously (1) and (2) hold. In fact, appealing only to the symmetric structure on $\bigwedge_A^d A[\xi]$, if the map (0.6.3) is an S-linear isomorphism, then all the assertions of the Theorem hold.

As we just saw, specializing of $X_1, \dots, X_d$ to $\xi_1, \dots, \xi_d$ we obtain, via the canonical surjection $\bigwedge_A^d A[X] \to \bigwedge_A^d A[\xi]$, that (2) and (3) of The Main Theorem follows from (2) and (3) of 0.4. So, to prove The Main Theorem by appealing to 0.4, it remains to verify that the symmetric action of S factors through R and that (1) holds.

1. The universal factorization algebra

The main purpose of this section is to give further descriptions and properties of the factorization algebra R introduced in 0.5. These properties are crucial in the proofs in Sections 2 and 3.

1.1 The universal factorization algebra. Let $p \in A[T]$ be a monic polynomial of degree n. The degree-d factorization algebra $R = \mathrm{Fact}^d(p)$ with the factorization $p = q\tilde{q}$, defined in 0.5 as a subalgebra $R \subset A[\xi_1, \dots, \xi_d]$ of the splitting algebra, is universal in the following sense:

Given any A-algebra B and a factorization in $B[T]$ of p into a product $p = \varphi\tilde{\varphi}$ of factors of degrees d and $n - d$, there exists a unique homomorphism of A-algebras $R \to B$ under which $q \in R[T]$ is mapped to $\varphi \in B[T]$.

Indeed, uniqueness is obvious, since R, by construction, is generated as an A-algebra by the coefficients of q. To prove existence of the map, consider the total splitting algebra of φ over B, generated by universal roots $\eta_1, \dots, \eta_d$. It contains B, and p splits over $B[\eta_1, \dots, \eta_d]$:

$$p = \varphi\tilde{\varphi} = (T - \eta_1) \cdots (T - \eta_d)\, \tilde{\varphi}.$$

By the universal property of splitting algebras, there is an A-algebra homomorphism $A[\xi_1, \dots, \xi_d] \to B[\eta_1, \dots, \eta_d]$ mapping ξ_i to η_i. It maps symmetric polynomials in the ξ_i to corresponding symmetric polynomials in the η_i. In particular, it maps the coefficients of q to the corresponding coefficients of φ; hence it defines the required A-algebra map $R \to B$.

It follows easily from the universal properties of splitting algebras that $A[\xi_1, \dots, \xi_d]$ is equal to $R[\xi_1, \dots, \xi_d]$, and, as an R-algebra is the total splitting algebra of the polynomial q.

1.2. There is a second construction of the factorization algebra R: Consider the generic monic degree-d polynomial over A, say $u = T^d + u_1 T^{d-1} + \cdots + u_d$, where the $u_1, \dots, u_d$ are algebraically independent variables over A. By the Division algorithm in the polynomial ring $A[u_1, \dots, u_d][T]$ there is an equation,

$$p = uv + r, \quad \text{where } r = r_0 + r_1 T + \cdots r_{d-1} T^{d-1}.$$

Then, clearly, R may be defined as the quotient algebra of $A[u_1, \dots, u_d]$ modulo the ideal generated by the d coefficients of $r_0, \dots, r_{d-1}$ of the remainder r.

The two constructions are connected via the algebra map $S \to R$ considered in 0.5. In fact, the elementary symmetric polynomials $c_1, \dots, c_d$ are algebraically independent generators of the A-algebra $S = A[X_1, X_2, \dots, X_d]^{\mathrm{sym}}$ of symmetric polynomials. So we may identify the polynomial algebra $A[u_1, \dots, u_d]$ with the algebra S so that $u_i = (-1)^i c_i$. Under this identification, the universal polynomial $u(T)$ corresponds to the polynomial $Q(T)$ of 0.1, and the quotient map $A[u_1, \dots, u_d] \to R$ corresponds to the surjective algebra map $S \to R$ described in 0.5. Viewing $r(T)$ as a polynomial in $S[T]$, it follows that the kernel of $S \to R$ is the ideal of S generated by the coefficients of $r(T)$.

2. First proof of the Main Theorem

Here we give a proof of the Main Theorem of 0.6 based upon the theory in [LT].

2.1 Notation. Keep the notation of Section 0.

2.2 Lemma. *Let $F(T) = \sum_j v_j T^j$ and $G(T) = \sum_j w_j T^j$ be polynomials in $S[T]$ such that $F(T) \equiv G(T) \pmod{Q(T)}$. Then, for all sequences of d non-negative integers $i_1, \dots, i_d$, the following equation holds in the S-module $\bigwedge_A^d A[X]$:*

$$\sum_j v_j X^{j+i_1} \wedge X^{i_2} \wedge \cdots \wedge X^{i_d} = \sum_j w_j X^{j+i_1} \wedge X^{i_2} \wedge \cdots \wedge X^{i_d}. \tag{2.2.1}$$

In particular, if $r(T) = \sum_{j=0}^{d-1} r_j T^j$ is the remainder of $p(T)$ after division by $Q(T)$, then

$$\big(p(X)X^{i_1}\big) \wedge X^{i_2} \wedge \cdots \wedge X^{i_d} = \sum_{j=0}^{d-1} r_j X^{j+i_1} \wedge X^{i_2} \wedge \cdots \wedge X^{i_d}. \tag{2.2.2}$$

Moreover, for $j = 0, \dots, d-1$,

$$r_j X^{d-1} \wedge \cdots \wedge X \wedge 1 = X^{d-1} \wedge \cdots \wedge p(X) \wedge \cdots \wedge 1, \tag{2.2.3}$$

where the wedge product on the right side is obtained from $X^{d-1} \wedge \cdots \wedge 1$ by replacing the factor X^j by $p(X)$.

Proof. Consider the surjection $A[X_1, X_2, \dots, X_d] \to \bigwedge_A^d A[X]$. By definition of the symmetric action, if $u \in S$ is a symmetric polynomial, then $uX^{i_1} \wedge \cdots \wedge X^{i_d}$ is the image of the product $uX_1^{i_1} \cdots X_d^{i_d}$. So the left side of (2.2.1) is the image of the polynomial $F(X_1)X_1^{i_1} \dots X_d^{i_d}$. The right hand side may be obtained similarly. Since $Q(X_1) = 0$, it follows that if $F(T) \equiv G(T)$ then $F(X_1) = G(X_1)$. Therefore (2.2.1) holds.

Clearly Equation (2.2.2) is a special case of (2.2.1).

Finally, Equation (2.2.3) is obtained from (2.2.2) by taking as exponents the sequence $(i_1, \dots, i_d) = (0, d-1, \dots, j+1, j-1, \dots, 0)$.

2.3 Proof of the Main Theorem. Let $r(T) = \sum_{i=0}^{d-1} r_i T^i$ be the remainder of $p(T)$ after division by $Q(T)$. Denote by $\mathfrak{J}$ the kernel of the surjection $S \to R$ described in 0.5. It follows from the construction of R in 2.1 that $\mathfrak{J}$ is the ideal generated by $r_0, \dots, r_{d-1}$.

As noted in Remark 0.8, it suffices to prove that the symmetric action of S on $\bigwedge_A^d A[\xi]$ factors through the surjection $S \to R$ and that $\bigwedge_A^d A[\xi]$ as an R-module is free with basis $\xi^{d-1} \wedge \cdots \xi \wedge 1$.

Consider the natural homomorphism,

$$\bigwedge_A^d A[X] \to \bigwedge_A^d A[\xi]. \tag{2.3.1}$$

It is surjective and S-linear. Moreover, by 0.4(1) its source $\bigwedge_A^d A[X]$ is a free S-module generated by $X^{d-1} \wedge \cdots \wedge X^0$. Consequently it suffices to show that the kernel of (2.3.1) is the S-module $\mathfrak{J}X^{d-1} \wedge \cdots \wedge X^0$.

Since $p(\xi) = 0$ it follows from Equation (2.2.3) that every element in $\mathfrak{J}X^{d-1} \wedge \cdots \wedge X^0$ is in the kernel of (2.3.1).

Conversely, we have that the kernel of (2.3.1) obviously is generated, as an A-module, by the elements on the left side of Equation (2.2.2), and each term on the right side is contained in $\mathfrak{J}X^{d-1} \wedge \cdots \wedge X^0$. Consequently the kernel of (2.3.1) is contained in $\mathfrak{J}X^{d-1} \wedge \cdots \wedge 1$, and we have proved the theorem.

3. A "self contained" proof of the Main Theorem

We give another proof of the Main Theorem of 0.6. It builds upon the divided differences introduced by M. Demazure, C. Chevalley, and I.N. Bernšteĭn, I.M. Gelfand and S.I. Gelfand (see [M, Chapter 2.3], [F2, §10.3] or [La, Chapter 7]). The proof of the basic Lemma 3.2 refers to [LT], but is an easy consequence of the properties of splitting algebras.

3.1 Notation and roots in splitting algebras. As noted in Section 1.1, the d'th partial splitting algebra $A[\xi_1, \dots, \xi_d] = R[\xi_1, \dots, \xi_d]$ of p over A may be identified with the total splitting algebra of q over R. In particular, we may identify the first splitting algebra $R[T]/(q) = R[\xi]$ with any of the subalgebras $R[\xi_i]$ of $A[\xi_1, \dots, \xi_d]$.

3.2 Lemma. *There is a well defined R-linear map,*

$$\partial \colon A[\xi_1, \dots, \xi_d] \to R,$$

such that, for any sequence of d polynomials $f_1, \dots, f_d \in R[T]$ we have the following equation in R:

$$\partial\big(f_1(\xi_1) \cdots f_r(\xi_r)\big) = \operatorname{Res}\big(\frac{f_1}{q}, \dots, \frac{f_d}{p}\big). \tag{3.2.1}$$

Proof. Note that $A[\xi_1, \dots, \xi_d] = R[\xi_1, \dots, \xi_d]$ is the total splitting algebra of the polynomial $q \in R[T]$. So the assertion is the contents of [LT, Proposition 6.4], applied to the polynomial q in $R[T]$ (and $r := d$) and polynomials $f_i \in A[T] \subseteq R[T]$.

3.3 Proof of the Main Theorem. Consider the following diagram,

$$\begin{array}{ccc} A[X_1, \dots, X_n] & \longrightarrow & A[\xi_1, \dots, \xi_d] \\ \big\downarrow & & \big\downarrow \partial \\ \bigwedge_A^d A[\xi] & \longrightarrow & R. \end{array}$$

The lower horizontal map is the A-linear map of (0.6.3) induced by the residue. As noted in 0.8, to prove the Theorem it suffices to prove that this map is an S-linear isomorphism.

First, we prove that the map is S-linear. To this end we note that it follows from the expression (3.2.1) that the diagram is commutative. The top horizontal map is the map of algebras mapping S into R, as described in 0.5; in particular, it is S-linear. The map ∂ is the R-linear map of Lemma 3.2; in particular, it is S-linear. By construction of the symmetric structure on $\bigwedge_A^d A[\xi]$, the left vertical map is S-linear and surjective, Therefore, the bottom horizontal map is S-linear.

Next, the map $\bigwedge_A^d A[\xi] \to R$ is surjective. Indeed, since the map is S-linear, it suffices to note that $1 \in R$ is in the image, namely equal to the image of $\xi^{d-1} \wedge \cdots \wedge \xi \wedge 1$.

Finally, we prove that the map is an isomorphism. The A-module $A[\xi]$ is free of rank n. Hence, the source $\bigwedge_A^d A[\xi]$ is a free A-module of rank $\binom{n}{d}$. So, with $h := \binom{n}{d}$ the map may be viewed as an A-linear surjection $A^h \to R$.

Now, the d'th partial splitting algebra $A[\xi_1, \dots, \xi_d] = R[\xi_1, \dots, \xi_d]$ is also the total splitting algebra of q; in particular, the algebra is free of rank $k := d!$ as an R-module, and free of rank $n(n-1)\cdots(n-d+1) = hk$ as an A-module. Hence there is an isomorphism of A-modules $R^k = A^{hk}$. Therefore, taking direct sums of the A-linear surjection $A^h \to R$ we obtain an A-linear surjection $(A^h)^k \to R^k = A^{hk}$. The latter surjection is necessarily an isomorphism; therefore, the surjection $A^h \to R$ is an isomorphism.

Thus the Theorem has been proved.

3.4 Note. The residue,

$$\operatorname{Res}(\varphi_1/q, \dots, \varphi_d/q),$$

is R-multi-linear and alternating in polynomials $\varphi_i \in R[T]$, and it vanishes if one of the φ_i is divisible by q. Hence the residue induces an R-linear map $\bigwedge_R^d R[\xi] \to R$. The R-module $R[\xi]$ is free of rank d, so $\bigwedge_R^d R[\xi]$ is free of rank 1. Moreover, with $\varphi_i := T^{d-i}$ the residue has the value 1. Hence, the map induced by the residue is an R-linear isomorphism $\bigwedge_R^d R[\xi] \xrightarrow{\sim} R$. Clearly, the map $\bigwedge_A^d A[\xi] \to R$ factors,

$$\bigwedge\nolimits_A^d A[\xi] \to \bigwedge\nolimits_R^d R[\xi] \xrightarrow{\sim} R, \tag{3.4.1}$$

where the first map is the A-linear map induced by the inclusion $A[\xi] \to R[\xi]$. By the Theorem, the composition is an R-linear isomorphism. Consequently, the first map is an R-linear isomorphism.

It should be noted that a priori it is not even obvious that the first map in (3.4.1) is S-linear. Indeed, the source is an S-module via the symmetric structure on $\bigwedge_A^d A[\xi]$. Similarly, the target $\bigwedge_R^d R[\xi]$ has a symmetric structure as a module over the algebra of symmetric polynomials over R. The latter algebra contains the algebra S. So both exterior powers are S-modules via their symmetric structure, and, clearly, the first map is S-linear with respect to these structures.

So the exterior power $\bigwedge_R^d R[\xi]$ has two actions of S, the symmetric action and the natural action via R on the exterior power of an R-module. The first map is S-linear with respect to the symmetric action on $\bigwedge_R^d R[\xi]$. By the Theorem, the first map is S-linear isomorphism with respect to the natural action. So the two actions are the same. Conversely, from the equality of the two actions it follows that the morphism $\bigwedge_A^d A[\xi] \to R$ is S-linear. From this S-linearity it is easy to obtain the assertions of the Main Theorem, as in the last parts of the proof 3.3.

4. Intersection Theory

4.0 Chow groups and rings. We fix a base scheme X over which there is a bivariant intersection theory as described in Fulton [F1, Chapters 17, 20] or Kleiman [K, Section 3, p. 332–338]; see Thorup [T1] for a theory over a general noetherian base scheme. So, in the category of schemes essentially of finite type over X, there is associated to any map $P \to Q$ a bivariant group $\mathrm{A}(P/Q)$ with natural functorial properties with respect to proper maps, flat maps, and regular embeddings. The elements of $\mathrm{A}(P/Q)$, called bivariant classes, act as operators: they associate with every integral scheme V/Q an element in the Chow group $\mathrm{A}(P \times_Q V)$. The group associated with the identity of P has a natural structure as a is a ring, and it will be denoted $\mathrm{A}^*(P)$. It is the analogue of the Chow ring of a variety. The group $\mathrm{A}(P/X)$ is the analogue of the Chow group; it is a module over the bivariant Chow ring $\mathrm{A}^*(P)$.

4.1 Setup. Fix over the base scheme X a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank $n \geq 1$. In addition, fix a positive integer $d \leq n$. Let

$$F := \mathrm{Flag}^d(\mathcal{E}) \quad \text{and} \quad \pi_F\colon F \to X$$

be the d'th *partial flag scheme* and its structure map, parametrizing flags of corank d in $\mathcal{E}$. On F there is a universal flag of submodules,

$$\mathcal{E}_d \subset \cdots \subset \mathcal{E}_1 \subset \mathcal{E}_0 = \mathcal{E}_F, \tag{4.1.1}$$

such that the successive quotients $\mathcal{L}_i := \mathcal{E}_{i-1}/\mathcal{E}_i$ for $i = 1, \ldots, d$ are locally free of rank 1. Let

$$G := \mathrm{Grass}^d(\mathcal{E}) \quad \text{and} \quad \pi_G\colon G \to X$$

be the d'th Grassmannian and its structure map, parametrizing rank-d quotients of $\mathcal{E}$. On G there is a universal rank-d quotient $\mathcal{Q}$ of $\mathcal{E}_G$. It defines a short exact sequence,

$$0 \to \mathcal{R} \to \mathcal{E}_G \to \mathcal{Q} \to 0 \tag{4.1.2}$$

with $\mathcal{R}$ locally free of corank d. On the flag scheme F, the quotient $\mathcal{E}_F \to \mathcal{E}_F/\mathcal{E}_d$ defines an X-morphism $F \to G$ under which the submodule $\mathcal{E}_d$ of $\mathcal{E}_F$ is the pull-back of the submodule $\mathcal{R}$ of $\mathcal{E}_G$. In fact, it follows from the functorial properties of flags that the map $F \to G$ identifies F and the complete flag scheme of $\mathcal{Q}$ over G:

$$\mathrm{Flag}^d_X(\mathcal{E}) = \mathrm{Flag}^d_G(\mathcal{Q}). \tag{4.1.3}$$

Under this identification, we have the equality $\mathcal{Q}_F = \mathcal{E}_F/\mathcal{E}_d$ and the universal flag of submodules of $\mathcal{Q}_F$ are the submodules $\mathcal{E}_i/\mathcal{E}_d$ for $i = 0, \ldots, d$. In particular, the rank-1 modules $\mathcal{L}_i$ are the successive quotients of the universal flag in $\mathcal{Q}_F$.

In the case $d = 1$, the two schemes F and G reduce to projective space: Let

$$P := \mathbb{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}\,\mathcal{E}) \quad \text{and} \quad \pi\colon P \to X$$

be the projective bundle and its structure map. On P there is an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E}_P \to \mathcal{O}_P(1) \to 0. \tag{4.1.4}$$

Clearly, the flag schemes may be defined inductively as a chain of schemes and maps,

$$F_d \to \cdots \to F_2 \to F_1 \to X,$$

with $F_1 := \mathbb{P}(\mathcal{E})$, and, inductively, if F_d is given with the flag (4.1.1) for $F = F_d$, then $F_{d+1} = \mathbb{P}(\mathcal{E}_d)$, $\mathcal{L}_{d+1} := \mathcal{O}_{F_{d+1}}(1)$, and $\mathcal{E}_{d+1}$ is the kernel of the surjection $(\mathcal{E}_d)_{F_{d+1}} \to \mathcal{L}_{d+1}$.

4.2 Theorem. *Set $P := \mathbb{P}(\mathcal{E})$, and let $x := c_1\mathcal{O}_P(1) \in \mathrm{A}^1(P)$ be the first Chern class of the canonical quotient $\mathcal{O}_P(1)$. Then:*

(1) (see [F1, Theorem 3.3(b), p. 64]) *Every class β in $\mathrm{A}(P)$ has an expansion,*

$$\beta = \sum_{i=0}^{n-1} x^i \cap \pi^*(\alpha_i), \tag{4.2.1}$$

with uniquely determined classes $\alpha_i \in \mathrm{A}(X)$.

(2) (cf. [F1, Example 3.3.3, p. 67]) *For any class $\alpha \in \mathrm{A}(X)$, we have the equation,*

$$\pi_*(x^i \cap \pi^*\alpha) = \begin{cases} 0 & \text{for} \quad 0 \le i < n-1, \\ \alpha & \text{for} \quad i = n-1. \end{cases} \tag{4.2.2}$$

The uniqueness in (1) is an easy consequence of (2), see Formula (4.6.2) below.

4.3 Corollary. *The powers $1, x, \dots, x^{n-1}$ form an $\mathrm{A}^*(X)$-basis of $\mathrm{A}^*(P)$. In other words, every bivariant class $h \in \mathrm{A}^*(P)$ has an expansion,*

$$h = \pi^*(h_0) + \pi^*(h_1)x + \cdots + \pi^*(h_{n-1})x^{n-1}, \tag{4.3.1}$$

with uniquely determined coefficients $h_i \in \mathrm{A}^*(X)$. Moreover, the natural $\mathrm{A}^*(P)$-linear map induced by π^* is an isomorphism,

$$\mathrm{A}^*(P) \otimes_{\mathrm{A}^*(X)} \mathrm{A}(X) \xrightarrow{\sim} \mathrm{A}(P). \tag{4.3.2}$$

Proof. Consider a bivariant class $h \in \mathrm{A}^*(P)$. If h has an expansion (4.3.1), then the bivariant classes h_i are uniquely determined. Indeed, if α is a class over X, then evaluation of (4.3.1) on a $\pi^*(\alpha)$ yields the equation,

$$h \cap \pi^*(\alpha) = \pi^*(h_0 \cap \alpha) + x \cap \pi^*(h_1 \cap \alpha) + \cdots + x^{n-1} \cap \pi^*(h_{n-1} \cap \alpha), \tag{4.3.3}$$

and this equation determines the values $h_i \cap \alpha$ by the uniqueness in Theorem 4.2(1). Conversely, again by Theorem 4.2(1), we may define bivariant classes h_i by the equation (4.3.3). Then, by construction, the two sides of (4.3.1) are equal on classes of the form $\pi^*(\alpha)$. Therefore, since the Chern class x commutes with the bivariant classes h and $\pi^*(h_i)$, the two sides are equal on a class of the form $x^i \cap \pi^*(\alpha)$. Therefore, by Theorem 4.2 (1), the two sides are equal on any class in $\mathrm{A}(P)$. By functoriality they are also equal on classes in $\mathrm{A}(P')$ for all morphisms $P' \to P$, that is, the two sides are equal bivariant classes in $A^*(P)$. So uniqueness and existence of the expansion (4.3.1) has been proved. Clearly, the isomorphism (4.3.2) follows from Theorem 4.2 (1). □

From the Corollary it follows in particular that the restriction $\pi^*\colon \mathrm{A}^*(X) \to \mathrm{A}^*(P)$ is injective. As is customary, if $h \in \mathrm{A}^*(X)$ we will often write h for $\pi^*(h)$.

4.4 Chern classes. We will write the expansion (4.3.1) of the bivariant class x^n as an equation in $\mathrm{A}^*(P)$ of the following form:

$$x^n - c_1 x^{n-1} + \cdots + (-1)^n c_n = 0, \tag{4.4.1}$$

with bivariant classes $c_i \in \mathrm{A}^*(X)$. By definition, the *i'th Chern class* of $\mathcal{E}$ is $c_i = c_i(\mathcal{E})$, and $c_0 := 1$. The polynomial in $\mathrm{A}^*(X)[T]$,

$$C_{\mathcal{E}}(T) := T^n - c_1 T^{n-1} + \cdots + (-1)^n c_n,$$

will be called the *Chern polynomial* of $\mathcal{E}$. By construction, $C_{\mathcal{E}}(T)$ is the unique monic polynomial of degree n in $\mathrm{A}^*(X)[T]$ having the root x in $\mathrm{A}^*(P)$.

4.5 Segre classes. We will define the Segre classes of $\mathcal{E}$ in terms of the following Laurent series in $\mathrm{A}^*(X)[T][[T^{-1}]]$,

$$S_{\mathcal{E}}(T) := \pi_*\big(\frac{1}{T-x} \cap \pi^*\big).$$

As $1/(T-x) = \sum_{j\geq 1} x^{j-1}T^{-j}$, the coefficient to T^i in $S_{\mathcal{E}}(T)$ vanishes when $i \geq 0$ and the coefficient to T^{-j}, evaluated at a class α, is $\pi_*(x^{j-1} \cap \pi^*\alpha)$ when $j \geq 1$. It follows from Theorem 4.2 (2) that the coefficient to T^{-j} vanishes when $0 < j < n$ and is equal to the identity when $j = n$. So the Laurent series is of degree $-n$, and we may write it in the following form:

$$S_{\mathcal{E}}(T) = \cdots + s_2 T^{-n-2} + s_1 T^{-n-1} + T^{-n}.$$

By definition, the bivariant class $s_i \in \mathrm{A}^*(X)$ is the *i'th Segre class* $s_i = s_i(\mathcal{E})$ of $\mathcal{E}$; we let $s_0 := 1$.

4.6 Formulas. (1) *The Segre series is the inverse of the Chern polynomial:*

$$C_{\mathcal{E}}(T) S_{\mathcal{E}}(T) = 1. \tag{4.6.1}$$

(2) *For any class $\beta \in \mathrm{A}(P)$, the coefficients α_i in the expansion* (4.2.1) *are determined by the following equation in the* $\mathrm{A}^*(X)[T][[T^{-1}]]$*-module* $\mathrm{A}(X)[T][[T^{-1}]]$*:*

$$C_{\mathcal{E}}(T) \cap \pi_*\big(\frac{1}{T-x} \cap \beta\big) = \alpha_0 + \alpha_1 T + \cdots + \alpha_{n-1} T^{n-1}. \tag{4.6.2}$$

(3) *For any bivariant class $h \in \mathrm{A}^*(P)$, the coefficients h_i in the expansion* (4.3.1) *are determined by the following equation in the ring* $\mathrm{A}^*(X)[T][[T^{-1}]]$*:*

$$C_{\mathcal{E}}(T) \pi_*\big(\frac{1}{T-x}\, h \cap \pi^*\big) = h_0 + h_1 T + \cdots + h_{n-1} T^{n-1}. \tag{4.6.3}$$

Proof. It follows from Theorem 4.2 (2) that we have the following equations for $i = 0, \ldots, n$:

$$\pi_*\big(\frac{1}{T-x}\, x^i \cap \pi^*\big) = \begin{cases} T^i S_{\mathcal{E}}(T) & \text{for } 0 \leq i \leq n-1, \\ T^n S_{\mathcal{E}}(T) - 1 & \text{for } i = n. \end{cases} \tag{4.6.4}$$

Consequently, by the projection formula, if $P(T)$ is any monic polynomial of degree n in $\mathrm{A}^*(X)[T]$, then

$$\pi_*\big(\frac{1}{T-x}\,P(x)\cap\pi^*\big) = P(T)S_{\mathcal{E}}(T) - 1.$$

Take $P = C_{\mathcal{E}}$. By (4.4.1), we obtain the equation $0 = C_{\mathcal{E}}(T)S_{\mathcal{E}}(T) - 1$; so Equation (4.6.1) holds.

Similarly, if $\beta = \sum_{i=0}^{n-1} x^i \cap \pi^*(\alpha_i)$, then it follows from (4.6.4) that

$$\pi_*\big(\frac{1}{T-x}\cap\beta\big) = S_{\mathcal{E}}(T)(\alpha_0 + \cdots + \alpha_{n-1}T^{n-1}).$$

Now multiply by $C_{\mathcal{E}}(T)$ and use (4.6.1) to obtain (4.6.2).

The proof of Equation (6.3) is similar. □

4.7 Note. If the Chern classes and Segre classes are collected, as usual, as power series $c(\mathcal{E})(T) = 1 + c_1T + c_2T^2 + \cdots$ and $s(\mathcal{E})(T) = 1 + s_1T + s_2T^2 + \cdots$, then equation (4.6.1) takes the form $c(\mathcal{E})(-T)s(\mathcal{E})(T) = 1$.

The Segre class $s_i(\mathcal{E})$ differs by the sign $(-1)^i$ from the i'the Segre class of Fulton [F1]. However, the Chern class $c_i(\mathcal{E})$ coincides with the i'th Chern class defined by Fulton [F1].

4.8 The Whitney Formula. *For an exact sequence of locally free $\mathcal{O}_X$-modules,*

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0,$$

the Chern polynomials are related by the following equation,

$$C_{\mathcal{E}}(T) = C_{\mathcal{F}}(T)C_{\mathcal{K}}(T). \tag{4.8.1}$$

Proof. Consider first the case when $\mathcal{K}$ is locally free of rank 1. Set $y := c_1\mathcal{K}$. We have to verify the equation,

$$C_{\mathcal{E}}(T) = C_{\mathcal{F}}(T)(T-y). \tag{4.8.2}$$

Set $P := \mathbb{P}(\mathcal{E})$, $\mathcal{L} := \mathcal{O}_P(1)$, and $x := c_1\mathcal{L}$. The zero scheme of the composition $\mathcal{K}_P \to \mathcal{E}_P \to \mathcal{L}$ is the 'hyperplane' $Y := \mathbb{P}(\mathcal{F})$ in $\mathbb{P}(\mathcal{E})$. So the first Chern class,

$$c_1(\mathcal{L}\otimes\mathcal{K}_P^{-1}) = x - y,$$

is represented by intersecting with Y. In other words, for any class β over P we have the following equation:

$$(x-y)\cap\beta = j_*(Y\cdot\beta), \tag{4.8.3}$$

where $Y\cdot\beta$ is a class over Y, and $j\colon Y \to P$ is the inclusion. When the bivariant class $x = c_1\mathcal{L}$ is restricted to classes over Y it becomes the first Chern class of $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$. So, by definition of the Chern polynomial, the bivariant class $C_{\mathcal{F}}(x)$ is zero on classes over Y. Therefore, by (4.8.3), the bivariant class $C_{\mathcal{F}}(x)(x-y)$ is the zero operator in $\mathrm{A}^*(\mathbb{P}(\mathcal{E}))$. As $C_{\mathcal{F}}(T)(T-y)$ is a monic polynomial of degree n and with x as a root, it is equal to $C_{\mathcal{E}}(T)$, as noted in Section 4.4. Hence (4.8.2) holds.

So, the Formula holds if $\mathcal{K}$ is of rank 1. Proceed by induction on the rank of $\mathcal{K}$. Assume that there is in $\mathcal{K}$ a locally split submodule $\mathcal{M}$ of rank 1. Then, by induction, the formula holds for the exact sequence,

$$0 \to \mathcal{K}/\mathcal{M} \to \mathcal{E}/\mathcal{M} \to \mathcal{F} \to 0. \tag{4.8.4}$$

Multiply the formula for (4.8.4) by $T - c_1\mathcal{M}$. We obtain, by the rank 1-case, the formula for the original exact sequence.

In the general case, with no assumption on $\mathcal{K}$, a locally split rank 1-submodule of $\mathcal{K}$ exists for the pull-back of $\mathcal{K}$ to $\mathbb{P}(\mathcal{K}^*)$. As the induced map $\mathrm{A}^*(X) \to \mathrm{A}^*(\mathbb{P}(\mathcal{K}^*))$ is injective, it follows that the formula holds in general. □

4.9 Proposition. *Let $F := \mathrm{Flag}^d(\mathcal{E})$ be the d'th partial flag scheme. Then the bivariant Chow ring $\mathrm{A}^*(F)$ is canonically isomorphic to the d'th partial splitting algebra $\mathrm{Split}^d(C_{\mathcal{E}})$ of the Chern polynomial $C_{\mathcal{E}}(T)$ over $\mathrm{A}^*(X)$,*

$$\mathrm{A}^*(\mathrm{Flag}^d(\mathcal{E})) = \mathrm{Split}^d(C_{\mathcal{E}}).$$

Under the isomorphism, the Chern classes $x_i := c_1\mathcal{L}_i$ correspond to the universal roots of $C_{\mathcal{E}}(T)$; in particular, $\mathrm{A}^*(F)$ is generated as an $\mathrm{A}^*(X)$-algebra by the x_i. Moreover, the canonical $\mathrm{A}^*(F)$-linear map induced by $F \to X$ is an isomorphism,

$$\mathrm{A}^*(F) \otimes_{\mathrm{A}^*(X)} \mathrm{A}(X) \xrightarrow{\sim} \mathrm{A}(F). \tag{4.9.1}$$

In particular, the map induced by pull back is an injection $\mathrm{A}^*(X) \to \mathrm{A}^*(F)$.

Proof. The assertions in the special case $d = 1$ follow from Corollary 4.3 and the definition of the Chern polynomial in 4.4: The first flag scheme is $F_1 = \mathbb{P}(\mathcal{E})$ and the first Chern class $x_1 = x \in \mathrm{A}^*(F_1)$ is a root of $C_{\mathcal{E}}(T)$. From the description of the $\mathrm{A}^*(X)$-algebra $\mathrm{A}^*(F_1)$ in Corollary 4.3, it follows that the induced map from the first splitting algebra into $\mathrm{A}^*(F_1)$ is an isomorphism; moreover, the map in (4.9.1) is the isomorphism of (4.3.2). The assertions in the general case follow from the inductive constructions of the splitting algebra in Section 0.5 and of the flag scheme in Section 4.1, once it is noted that the splitting of $C_{\mathcal{E}}(T)$ over $\mathrm{A}^*(F_1)[T]$ is given by the equation,

$$C_{\mathcal{E}}(T) = (T - x_1)C_{\mathcal{E}_1}(T),$$

resulting from (4.1.4) and the Whitney formula. □

4.10 Note. Set $A := \mathrm{A}^*(X)$. Then the A-algebra $\mathrm{A}^*(F)$, as the d'th partial splitting algebra of a polynomial of degree n, is freely generated as an A-module by all monomials

$$x^J = x_1^{j_1} \cdots x_d^{j_d}$$

where $J = (j_1, \dots, j_d)$ and $0 \le j_1 \le n-1, \dots, 0 \le j_d \le n-d$. So, by the isomorphism in (4.9.1), every class $\varphi \in \mathrm{A}(F)$ has a unique expansion in the form,

$$\varphi = \sum\nolimits_J x^J \cap \pi_F^*(\alpha_J),$$

where the sum is over $J = (j_1, \dots, j_d)$ as above, with uniquely determined classes $\alpha_J \in \mathrm{A}(X)$.

4.11 Proposition. *Let $G := \mathrm{Grass}^d(\mathcal{E})$ be the d'th Grassmannian. Then the bivariant Chow ring ring $\mathrm{A}^*(G)$ is canonically isomorphic to the d'th factorization algebra of the Chern polynomial $C_\mathcal{E}(T)$ over $\mathrm{A}^*(X)$,*

$$\mathrm{A}^*(\mathrm{Grass}^d(\mathcal{E})) = \mathrm{Fact}^d(C_\mathcal{E}).$$

Under the isomorphism, the factorization over $\mathrm{A}^*(G)$,

$$C_{\mathcal{E}_G}(T) = C_\mathcal{Q}(T)C_\mathcal{R}(T),$$

corresponds to the universal factorization of $C_\mathcal{E}(T)$ in factors of degree d and $n-d$; in particular, the operator ring $\mathrm{A}^*(G)$ is generated as an $\mathrm{A}^*(X)$-algebra by the Chern classes of $\mathcal{Q}$. Moreover, the canonical $\mathrm{A}^*(G)$-linear map induced by $G \to X$ is an isomorphism,

$$\mathrm{A}^*(G) \otimes_{\mathrm{A}^*(X)} \mathrm{A}(X) \xrightarrow{\sim} \mathrm{A}(G). \tag{4.11.1}$$

In particular, the map induced by pull-back is an injection $\mathrm{A}^*(X) \to \mathrm{A}^*(G)$.

Proof. Let $F = \mathrm{Flag}^d(\mathcal{E})$. As noted in (4.1.3), there is a canonical map $F \to G$ under which F may be identified with the (full) flag scheme of $\mathcal{Q}$ over the Grassmannian. Under this identification, the i'th quotient $\mathcal{L}_i = \mathcal{E}_{i-1}/\mathcal{E}_i$ of the partial flag (4.1.1) in $\mathcal{E}_F$ may be identified with the i'th quotient of the full flag in $\mathcal{Q}_F$. Set $x_i := c_1\mathcal{L}_i$.

By Proposition 4.9 applied to $\mathcal{E}$ on X and to $\mathcal{Q}$ on G, the operator ring $\mathrm{A}^*(F)$ may be identified with the d'th partial splitting algebra of $C_\mathcal{E}(T)$ over $\mathrm{A}^*(X)$ and with the full splitting algebra of $C_\mathcal{Q}(T)$ over $\mathrm{A}^*(G)$. In particular, the induced map $\mathrm{A}^*(G) \to \mathrm{A}^*(F)$ is injective, and $\mathrm{A}^*(F) = \mathrm{A}^*(X)[x_1, \dots, x_d] = \mathrm{A}^*(G)[x_1, \dots, x_d]$. Moreover, under the identifications, the Chern classes x_i are the d first universal roots of $C_\mathcal{E}(T)$ and of $C_\mathcal{Q}(T)$. Whence, the Chern polynomial $C_\mathcal{E}(T)$ factors over $\mathrm{A}^*(F)$ as follows:

$$C_\mathcal{E}(T) = C_\mathcal{Q}(T)C_{\mathcal{E}_d}(T) = (T - x_1)\cdots(T - x_d)C_{\mathcal{E}_d}(T).$$

As shown in Section 1.1, the d'th factorization algebra of a polynomial embeds into the d'the splitting algebra as the subalgebra generated by the elementary symmetric polynomials in the first d universal roots. Consequently, the d'th factorization algebra of $C_\mathcal{E}(T)$ is the is the $\mathrm{A}^*(X)$-subalgebra of $\mathrm{A}^*(F)$ generated by the coefficients of $C_\mathcal{Q}(T)$, that is, it is the subalgebra $\mathrm{A}^*(X)[e_1, \dots, e_d]$ where $e_i = c_i\mathcal{Q}$. Thus we have the inclusion of subalgebras of $\mathrm{A}^*(F)$:

$$\mathrm{A}^*(X)[e_1, \dots, e_d] \subseteq \mathrm{A}^*(G).$$

Moreover, $\mathrm{A}^*(F)$ is the splitting algebra of $C_\mathcal{Q}(T)$ over any of these subalgebras. In particular, the monomials $x_1^{i_1}\cdots x_d^{i_d}$ where $0 \le i_1 \le d-1$, $0 \le i_2 \le d-2, \dots$ form a basis of $\mathrm{A}^*(F)$ over both subalgebras $\mathrm{A}^*(X)[e_1, \dots, e_d]$ and $\mathrm{A}^*(G)$. Clearly, therefore the two subalgebras are equal.

Now, for simplicity, write $A := \mathrm{A}^*(X)$, $A' := \mathrm{A}^*(G)$, and $A'' := \mathrm{A}^*(F)$. In addition, write $C := \mathrm{A}(X)$ (a module over A), $C' := \mathrm{A}(G)$ (a module over A'), and $C'' := \mathrm{A}(F)$ (a module over A''). There are inclusions $A \subseteq A' \subseteq A''$, and flat pull-back of Chow groups along $G \to X$ and along $F \to G$ induce an A-linear map $C \to C'$ and an A'-linear map $C' \to C''$. In particular, we get induced maps,

$$A' \otimes_A C \to C', \quad A'' \otimes_A C \to C'', \quad A'' \otimes_{A'} C' \to C''. \tag{4.11.2}$$

The last two maps are isomorphisms by Proposition 4.9 applied to $\mathcal{E}$ on X and to $\mathcal{Q}$ on G. It remains to prove that the first map is an isomorphism.

Tensor the first map over A' with A''. The result is an isomorphism $A'' \otimes_A C \to A'' \otimes_{A'} C'$; indeed, under the last two isomorphisms of (4.11,2), the result becomes the identity on C''. Therefore, since A'' is free and non-zero over A', the first map is an isomorphism.

Thus the proposition has been proved. □

4.12 Note. Set $A := \mathrm{A}^*(X)$. It follows from Proposition 4.11 and Corollary 0.7 that the A-algebra $\mathrm{A}^*(G)$, as the d'th factorization algebra of a polynomial of degree n, is freely generated as an A-module by all Schur polynomials $s_{h_1,\dots,h_d}$ in the universal roots $x_1, \dots, x_d$, where $n - d \geq h_1 \geq \cdots \geq h_d \geq 0$. The Schur polynomials are given by certain determinants in the elementary symmetric functions of the roots, and hence by explicit expressions in the Chern classes $c_i\mathcal{Q}$. By the isomorphism in (4.11.1), every class $\gamma \in \mathrm{A}(G)$ has a unique expansion in the form,

$$\gamma = \sum\nolimits_h s_h \cap \pi_G^*(\alpha_h),$$

where the sum is over partitions $h = (h_1, \dots, h_d)$ such that $n-d \geq h_1 \geq \cdots \geq h_d \geq 0$, with uniquely determined classes $\alpha_h \in \mathrm{A}(X)$.

5. Connections with Schubert calculus

5.1. Setup. Schubert schemes are subschemes of the Grassmannian $\mathrm{Grass}^d(\mathcal{E})$ defined by a number of conditions of the following form:

$$\dim(Q \cap A) \geq h. \tag{5.1.1}$$

In this inequality, A and Q are *linear* subschemes of $\mathbb{P}(\mathcal{E})$, that is, $A = \mathbb{P}(\mathcal{A})$ with a locally free quotient $\mathcal{E} \twoheadrightarrow \mathcal{A}$, or $A = \mathbb{P}(\mathcal{E}/\mathcal{V})$ with a locally split submodule $\mathcal{V} \subseteq \mathcal{E}$, and, similarly, $Q = \mathbb{P}(\mathcal{Q}) = \mathbb{P}(\mathcal{E}/\mathcal{R})$. The (relative) dimension of the linear subscheme $A = \mathbb{P}(\mathcal{A})$ is determined by the rank, $\dim A = \operatorname{rk} \mathcal{A} - 1$. The intersection $Q \cap A$ is the scheme $\mathbb{P}(\mathcal{E}/(\mathcal{R} + \mathcal{V}))$, which is not necessarily linear. The dimension in (5.1.1) is, by definition, equal to the number $r - 1$, where the number $r := \mathrm{r}(\mathcal{E}/(\mathcal{R} + \mathcal{V}))$, which may be called the *Fitting rank* of the module, is defined as follows: In terms of the presentation,

$$\mathcal{V} \xrightarrow{\varphi} \mathcal{E}/\mathcal{R} \to \mathcal{E}/(\mathcal{R}+\mathcal{V}) \to 0, \tag{5.1.2}$$

$r = \operatorname{rk} \mathcal{E}/\mathcal{R} - \operatorname{rk} \varphi$; hence

$$\dim A \cap Q = \operatorname{rk} \mathcal{E}/\mathcal{R} - \operatorname{rk} \varphi - 1 = \dim Q - \operatorname{rk} \varphi. \tag{5.1.3}$$

More intrinsically, the number r is defined in terms of the Fitting ideals of the module $\mathcal{E}/(\mathcal{R}+\mathcal{V})$: From the presentation (5.1.2), if $d := \operatorname{rk}\mathcal{E}/\mathcal{R}$ then the j'th Fitting ideal $\mathcal{F}itt_j(\mathcal{E}/(\mathcal{R}+\mathcal{V}))$ is generated by the minors of size $d-j$ of φ. Hence, r is the smallest index j such that the j'th Fitting ideal is non-zero:

$$\mathrm{r}(\mathcal{E}/(\mathcal{R}+\mathcal{V})) = \min\{\, j \mid \mathcal{F}itt_j(\mathcal{E}/(\mathcal{R}+\mathcal{V})) \neq 0\,\}. \tag{5.1.4}$$

In particular we have the following biimplication:

$$\dim(A\cap Q) \geq h \iff \mathcal{F}itt_h(\mathcal{E}/(\mathcal{R}+\mathcal{V})) = 0. \tag{5.1.5}$$

5.2. Schubert subschemes. Consider an increasing sequence of d linear subschemes of $I\!P(\mathcal{E})$:

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq \mathrm{A}_d \subseteq I\!P(\mathcal{E}),$$

say $A_i = I\!P(\mathcal{E}/\mathcal{V}_i)$ with a decreasing sequence

$$\mathcal{E} \supseteq \mathcal{V}_1 \supseteq \cdots \supseteq \mathcal{V}_d$$

of d locally split submodules of $\mathcal{E}$. The associated *Schubert subscheme* is the closed subscheme of the Grassmannian,

$$\Omega(A_1,\dots,A_d) =\subseteq \operatorname{Grass}^d_X(\mathcal{E}),$$

defined as follows: Informally, the Grassmannian $\operatorname{Grass}^d(\mathcal{E})$ parametrizes the set of $(d-1)$-planes $Q \subseteq I\!P(\mathcal{E})$, and $\Omega(A_1,\dots,A_d)$ parametrizes the subset of $(d-1)$-planes satisfying the following d conditions:

$$\dim Q\cap A_i \geq i-1 \quad \text{for } i=1,\dots,d. \tag{5.2.1}$$

To be more precise, the Grassmannian represents the functor whose value at a scheme T/X is the set of $(d-1)$-planes $Q \subseteq I\!P(\mathcal{E}_T)$, and the Schubert subscheme $\Omega(A_1,\dots,A_d)$ represents the subfunctor defined by the following d conditions on the $(d-1)$-plane Q:

$$\dim Q\cap A_{i,T} \geq i-1 \quad \text{for } i=1,\dots,d.$$

It follows from the discussion in 5.1 that the subfunctor is representable by a closed subscheme of the Grassmannian $G = \operatorname{Grass}^d(\mathcal{E})$: The ideal of the Schubert subscheme $\Omega(A_1,\dots,A_d)$ is the sum of the d Fitting ideals corresponding to the d inequalities for the universal $(d-1)$-plane in $I\!P(\mathcal{E}_G)$, as in (5.1.5).

Note that the Schubert subscheme is empty if one of the inequalities $\dim A_i \geq i-1$ is violated. Note also that some of the inequalities may be redundant: If for some j,k we have $\dim A_{j+k} \leq \dim A_j + k$, then the inequalities (5.2.1) for $j \leq i < j+k$ are consequences of the inequality for $i = j+k$; and if $\dim A_d = n-1$, then $A_d = I\!P(\mathcal{E})$ and the inequality for $i=d$ is redundant.

We will always assume that the sequence of subspaces A_i is strictly increasing and $A_1 \neq \emptyset$. In fact, we will assume that the A_i are obtained from a full complete flag of linear subschemes of $I\!P(\mathcal{E})$:

$$\emptyset = P_{-1} \subset P_0 \subset P_1 \subset \cdots \subset P_{n-1} = I\!P(\mathcal{E}), \qquad \dim P_k = k,$$

by specifying the dimensions, $A_i = P_{a_i}$, for a sequence (a_i) with

$$0 \le a_1 < a_2 < \cdots < a_d \le n-1.$$

Then, by arguing as in the proof of [L, Theorem 1, p. 7], it follows that the Schubert subscheme is flat of pure relative dimension equal to

$$\sum_{i=1}^{d} (\dim A_i - i + 1) = \sum_{i=1}^{d} a_i - \tfrac{1}{2} d(d-1). \tag{5.2.2}$$

5.3. Conclusion. Let $G := \mathrm{Grass}^d(\mathcal{E})$ be the Grassmannian, with the universal rank-d quotient $\mathcal{Q}$ of $\mathcal{E}_G$. The structure map $G \to X$ is smooth of pure dimension; denote by $[G/X]$ its fundamental class in the bivariant group $\mathrm{A}(G/X)$, given by flat pull back. Then, as is well known, the composition $\alpha \mapsto \alpha[G/X]$ is an isomorphism of $\mathrm{A}^*(G)$-modules,

$$\mathrm{A}^*(G) \simeq \mathrm{A}(G/X).$$

Consider the Schubert subscheme $\Omega(A_1, \ldots, A_d)$ determined as in 5.2 by the linear subschemes $A_i = I\!P(\mathcal{E}/\mathcal{V}_i)$. Let $[\Omega(A_1, \ldots, A_d)]$ be the class of the Schubert subscheme in $\mathrm{A}(G/X)$. Then the determinantal formula of Schubert calculus, see [KL, Theorem 5, p. 158], may be rewritten as follows:

$$\big[\Omega(A_1, \ldots, A_d)\big] = \mathrm{Res}\Big(\frac{C_{\mathcal{V}_1}}{C_{\mathcal{Q}}}, \ldots, \frac{C_{\mathcal{V}_d}}{C_{\mathcal{Q}}}\Big)[G/X]. \tag{5.3.2}$$

Let $A := \mathrm{A}^*(X)$ be the bivariant Chow ring of the base scheme X. Consider the Chern polynomial $C_{\mathcal{E}} \in A[X]$ of $\mathcal{E}$, and let $A[\xi] = \mathrm{Split}^1(C_{\mathcal{E}})$ be its first splitting algebra and $\mathrm{Fact}^d(C_{\mathcal{E}})$ its d'th factorization algebra. By the Main Theorem, the exterior power $\bigwedge_A^d A[\xi]$ has a natural structure as a module over $\mathrm{Fact}^d(C_{\mathcal{E}})$, and it is free of rank 1 generated by the d-vector $\xi^{d-1} \wedge \cdots \wedge \xi^0$.

Theorem. *Under the identification of rings* $\mathrm{Fact}^d(C_{\mathcal{E}}) = A^*(G)$, *see Proposition* (4.11), *there is a unique linear isomorphism of rank-*1 *modules,*

$$\bigwedge_A^d A[\xi] \xrightarrow{\sim} \mathrm{A}(G/X), \tag{5.3.3}$$

such that $\xi^{d-1} \wedge \cdots \xi \wedge 1$ *is mapped to* $[G/X]$. *Moreover, under this isomorphism, for any flag* (5.3.1) *we have that*

$$C_{\mathcal{V}_1}(\xi) \wedge \cdots \wedge C_{\mathcal{V}_d}(\xi) \qquad \mapsto \qquad [\Omega(A_1, \ldots, A_d)]. \tag{5.3.4}$$

Proof. The existence and uniqueness of the isomorphism is immediate since the two sides of (5.3.3) are free of rank 1 and the requirement is that the basis element is mapped to the basis element.

The assertion (5,5.4) follows from the linearity of the map, using The Determinantal Formula (0.6.1) on the left side and Formula (5.3.2) on the right side.

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