1 Renewal processes

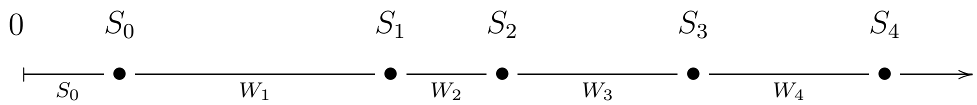
Let $W_1, W_2, \ldots$ be independent identically distributed random variables on $[0;\infty[$ with distribution A. We also use the letter A to denote the distribution function of A i.e.

$$A(x) = \mathbb{P}(W_1 \le x).$$

Let S_0 be another random variable on $[0;\infty[$ with distribution A_0. Suppose that S_0 is independent of $\{W_n\}$. For $i \ge 1$ define

$$S_i = S_0 + W_1 + \ldots + W_i.$$

The process $\mathcal{S} = \{S_n\}_{n \in \mathbb{N}_0}$ is called a *renewal process* with *interarrival* or *waiting-time* distribution A and *delay* distribution A_0.



The interpretation is that $\mathcal{S}$ signifies the occurence times of some specific events or *renewals*. The time between the renewals are assumed to be i.i.d. except that the distribution of the first renewal is allowed to follow some other distribution A_0.

1.1 The Renewal Measure

For any $t \ge 0$ denote by $N(t)$ the number of renewals in the interval $[0;t]$. Define the *renewal function* H by

$$H(t) = \mathbb{E}N(t), \quad t \ge 0,$$

as the expected number of renewals in the interval $[0;t]$. As an increasing function H can be extended to a measure, the *renewal measure*, on the Borel sets in $[0;\infty[$. Thus $H(B)$ is the expected number of renewals in the set B.

The renewal function can be expressed as

$$\begin{aligned} H(t) &= \sum_{k=1}^{\infty} \mathbb{P}(N(t) \ge k) \\ &= A_0(t) + A_0 * A(t) + A_0 * A_*^2(t) + \ldots \\ &= \sum_{k=0}^{\infty} A_0 * A_*^k(t). \end{aligned} \tag{1}$$

Here $A_0 * A$ is the distribution of $S_1 = S_0 + W_1$, $A_0 * A_*^2$ is the distribution of $S_2 = S_0 + W_1 + W_2$, and so on. Further, we use the convention that $A_*^0 = \delta_0$. The convolution operator, $*$, is discussed in detail in appendix A

We use the special notation H^0, N^0, etc. for the renewal process corresponding to $A_0 = \delta_0$, the probability with mass 1 in 0. This corresponds to the case, where $\mathcal{S}$ has a renewal at time 0 with probability 1.

Theorem 1.1 *The renewal function is finite for all $t \geq 0$ i.e. $H(t) < \infty$.*

Proof: Using lemma $A.13$ there is a $\beta > 0$ such that for any $t \geq 0$ and all $k \in \mathbb{N}$

$$A_*^k(t) \leq \exp(\beta t)\left(\hat{F}(\beta)\right)^k.$$

Hence, since $\hat{F}(\beta) < 1$ we get that

$$\sum_{k=0}^{\infty} A_*^k(t) \leq \exp(\beta t)\frac{1}{1-\hat{F}(\beta)} < \infty.$$

As $\sum_{k=0}^{\infty} A_*^k$ is bounded on finite intervals we get by combining (1) and corollary $A.11$ that $H(t) = A_0 * \left(\sum_{k=0}^{\infty} A_*^k\right)(t) < \infty$. $\square$

In the following we derive an integral equation for the renewal function H^0 in the zero-delayed case. For the zero-delayed process there is a renewal at time 0 with probability one. Conditioning on the time of the next renewal, S_1, we get

$$\begin{aligned}
H^0(t) &= \mathbb{E}(N^0(t)) \\
&= 1 \cdot \mathbb{P}(S_1 > t) + \int_0^t (1 + \mathbb{E}(N^0(t-u)))\mathbb{P}(S_1 \in du) \\
&= \mathbb{P}(S_1 > t) + \mathbb{P}(S_1 \leq t) + \int_0^t H^0(t-u)A(du) \\
&= 1 + \int_0^t H^0(t-u)A(du).
\end{aligned}$$

1.2 Forwards and Backwards Recurrence Time

Given a renewal process, $\mathcal{S}$, we introduce the *forwards recurrence* time process, $\{R(t)\}$, and the *backward recurrence* time process, $\{B(t)\}$. For any $t \geq 0$ $R(t)$ measures the time until next renewal after time t. Similarly, at time t $B(t)$

denotes the time elapsed since last renewal before t. Expressed in terms of $\mathcal{S}$ and $\{N(t)\}$ we get that

$$R(t) = S_{N(t)} - t, \quad B(t) = t - S_{N(t)-1}.$$

For some $x \geq 0$ let $Z_R(t) = \mathbb{P}(R^0(t) \leq x)$, the probability that the waiting time from time t until next renewal is less than x. Conditioning on the value of S_1 we find that

$$\begin{aligned} Z_R(t) &= \mathbb{P}(t < S_1 \leq t + x) + \int_0^t \mathbb{P}(R^0(t-u) \leq x)\mathbb{P}(S_1 \in du) \\ &= A(t+x) - A(t) + \int_0^t Z_R(t-u)A(du). \end{aligned}$$

Here we use that if Z_1 happens at time $u \leq t$ then the process $\{R^0(t)\}$ restarts.

1.3 The Elementary Renewal Theorem

The Elementary Renewal Theorem determines the asymptotic behaviour of the renewal function, H, for large t. Assuming that the inter-arrival times have finite expectation, i.e.

$$\mu = \int_{[0;\infty[} uA(du) < \infty,$$

then the expected number of renewals, $H(t)$, in $[0;t]$ is asymptotically equal to $\frac{t}{\mu}$. The important thing is that the result is independent of the delay A_0. Let's try to look at an example, where we can write down an explicit expression for H.

Example 1.2 *Consider the renewal process with $A \sim Exp(1)$. Then the distribution, A_*^k, of $W_1 + \ldots + W_k$ is a $\Gamma-$distribution with shape parameter k hence*

$$A_*^k(t) = \sum_{i=k}^{\infty} \frac{t^i}{i!} \exp(-t).$$

Summing over k yields

$$H^0(t) = \sum_{k=0}^{\infty} A_*^k(t) = 1 + t,$$

and we get that

$$\begin{aligned} H(t) &= \sum_{k=0}^{\infty} A_0 * A_*^k(t) \\ &= A_0 * \left(\sum_{k=0}^{\infty} A_*^k(t) \right) \\ &= A_0 * H^0(t) = H^0 * A_0(t) \\ &= \int_{[0;t]} H^0(t-u) A_0(du) \\ &= \int_{[0;t]} (1+t-u) A_0(du). \end{aligned}$$

If A_0 is the uniform distribution on $[0;1]$ we get that

$$H(t) = 0.5 * \min(t^2, 1) + t, \quad t \geq 0,$$

and if A_0 has a Pareto-like distribution with distribution function

$$A_0(t) = 1 - \frac{1}{1+t}$$

we get that

$$H(t) = 1 + t - \log(1+t) - \frac{1}{1+t}, \quad t \geq 0.$$

In both examples (and in the zero-delayed case) we see that

$$\lim_{t \to \infty} \frac{H(t)}{t} = 1 = \frac{1}{\int_{[0;\infty[} u A(du)}.$$

$\square$

2 Excessive and Defective Renewal Equations

The Key Renewal Theorem describes the asymptotic behaviour as $t \to \infty$ for solutions to the renewal equation

$$Z = z + Z * A, \tag{2}$$

where A is a probability on $[0; \infty[$. However, it turns out that our insight obtained so far can be used to say something in the general case where A is just a finite measure on $[0; \infty[$. (The material of this section can be found in S. Asmussen: Applied Probability and Queues)

For a general finite measure A on $[0; \infty[$ we may define a renewal function

$$H^0(t) = \sum_{k=0}^{\infty} A_*^k(t),$$

that can be shown to be finite for all $t \geq 0$. Inspecting the proof of [KA] lemma 4.5 we see that, provided z is bounded on finite intervals, $Z = z * H^0$ is the unique solution to (2) that is bounded on finite intervals. But we can not directly apply the Key Renewal Theorem to determine the asymptotic behaviour of Z as $t \to \infty$.

Now suppose that there is a solution β to the *Lundberg* equation

$$\hat{A}(\beta) = \int_0^{\infty} \exp(\beta u) A(du) = 1. \tag{3}$$

Let $\tilde{A}$ be the measure on $[0; \infty[$ that has density $u \to \exp(\beta u)$ w.r.t. A, i.e.

$$\tilde{A}(du) = \exp(\beta u) A(du).$$

By (3) $\tilde{A}$ has total mass 1 hence is a probability. If Z satisfies the renewal equation (2) the function $\tilde{Z}(t) = \exp(\beta t) Z(t)$ satisfies the renewal equation

$$\tilde{Z}(t) = \exp(\beta t) z(t) + \tilde{Z} * \tilde{A}(t). \tag{4}$$

Assuming that

$$\tilde{\mu} = \int_0^{\infty} u \tilde{A}(du) < \infty$$

and that $\tilde{z}$ defined by

$$\tilde{z}(t) = \exp(\beta t) z(t)$$

is directly Riemann integrable the Key Renewal Theorem yields that

$$\tilde{Z}(t) = \exp(\beta t) Z(t) \to \frac{1}{\tilde{\mu}} \int_0^{\infty} \tilde{z}(u) du, \quad t \to \infty.$$

2.1 The Excessive Case

We consider in this section the *excessive* case where the total mass of A is greater than 1. The excessive case is easy to handle due to the following theorem.

Theorem 2.1 *In the excessive case there is always a solution β to the Lundberg equation and furthermore*

$$\tilde{\mu} = \int_0^\infty u\tilde{A}(du) = \int_0^\infty u\exp(\beta u)A(du) < \infty.$$

Note that in the excessive case we always have that the solution $\beta < 0$.

Example 2.2 *(Lotka's Integral Equation) We consider in this example a model from population theory connected to the names Lotka and Sharpe. As usual we consider only the female part of the population. Introduce*

$\lambda(a)$	*the female birth-rate for women aged a*
${}_tp_a$	*the probability of surviving until age $a+t$ for a woman aged a*
f_0	*the density for the age-distribution of women at time* 0
$Z(t)$	*the* total *birth-rate in the population at time t*
$z(t)$	*the total birth-rate at time t for women that were alive at time* 0
$Z_0(t)$	*the total birth-rate at time t for women that have been born after time* 0

We find that

$$\begin{aligned} z(t) &= \int_0^\infty \lambda(a+t)\,{}_tp_a f_0(a)da \\ Z_0(t) &= \int_0^t Z(t-u)\,{}_up_0\lambda(u)du \end{aligned}$$

hence

$$\begin{aligned} Z(t) &= z(t) + Z_0(t) \\ &= z(t) + \int Z(t-u)\,\underbrace{{}_up_0\lambda(u)du}_{A(du)} \\ &= z(t) + Z * A(t). \end{aligned}$$

Note that the total mass of A is

$$||A|| = \int_0^\infty {}_up_0\lambda(u)du,$$

the average number of daughters of a woman. Assuming that $||A|| > 1$ *let* β *be the solution to the Lundberg equation and define as usual*

$$\tilde{\mu} = \int_0^\infty u \exp(\beta u) A(du) < \infty.$$

From the discussion of section 2 *we conclude, that if* $\tilde{z}(t) = \exp(\beta t) z(t)$ *is d.R.i. then*

$$Z(t) \sim C \exp(-\beta t),$$

with

$$C = \frac{1}{\tilde{\mu}} \int_0^\infty \tilde{z}(u) du = \frac{1}{\tilde{\mu}} \int_0^\infty \exp(\beta u) \int_0^\infty \lambda(a+u)_u p_a f_0(a) da \ du.$$

Sufficient conditions for $\tilde{z}$ *to be d.R.i. are that* λ *is bounded and continuous and that* ${}_t p_u$ *is continuous.*

Other interesting characteristics of the population can be expressed in terms of Z*. For instance the expected population size,* $N(t)$*, at time* t *is given by*

$$N(t) = \int_0^t Z(t-u)_u p_0 du + \int_0^t f_0(u)_t p_u du.$$

□

2.2 The Defective Case

Consider now the renewal equation

$$Z = z + Z * A,$$

in the case where the total mass, $||A||$, of A is < 1. We know that if z is bounded on finite intervals then $Z = z * H^0$ is the unique solution that is bounded on finite intervals.

Contrary to the excessive case existence of a solution to the Lundberg equation in the defective case requires exponential moments of the distribution A. A sufficient condition for existence of a solution is that $1 < \hat{A}(\delta) < \infty$ for some $\delta > 0$. This further implies that $\tilde{\mu} < \infty$. In this case we can restate the result of section 2 to get that

$$\exp(\beta t) Z(t) \to \frac{1}{\tilde{\mu}} \int_0^\infty \exp(\beta u) z(u) du.$$

Without assuming exponential moments of A one can prove that

Theorem 2.3 *If z is bounded and if $\lim_{t\to\infty} z(t)$ exists, then*

$$\lim_{t\to\infty} Z(t) = \frac{\lim_{t\to\infty} z(t)}{1 - ||A||} := Z(\infty) < \infty.$$

To obtain further information about the rate of convergence of $Z(t)$ to $Z(\infty)$, one can try to explore further the approach of section 2.

Theorem 2.4 *Assume that a solution β to the Lundberg equation exists and that $\tilde{\mu} < \infty$. If $\lim_{t\to\infty} z(t) := z(\infty)$ exists and if $\exp(\beta t)(z(t) - z(\infty))$ is d.R.i. it holds that*

$$Z(t) - Z(\infty) \sim \exp(-\beta t) \frac{1}{\tilde{\mu}} \left(\int_0^\infty \exp(\beta u)(z(u) - z(\infty)) du - \frac{z(\infty)}{\beta} \right).$$

A Convolutions

Let F be a measure on $[0;\infty[$ that is bounded on finite intervals. Further, let G be a real-valued measurable function that is bounded on finite intervals.

Definition A.1 *The* convolution *of G with respect to F is the function, $G*F$, on $[0;\infty[$ given by*

$$G * F(t) = \int_{[0;t]} G(t-u)F(du), \quad t \geq 0.$$

From the boundedness of F and G it follows that $G * F$ is bounded on finite intervals.

Remark A.2 *For any measure, F, on $[0;\infty[$ that is bounded on finite intervals we can consider its distribution function, $F(t) = \int_{[0;t]} F(du)$. This is a non-decreasing function on $[0;\infty[$ which is finite on finite intervals. Conversely, any non-decreasing non-negative function, F, on $[0;\infty[$ can be considered as a measure on the Borel sets of $[0;\infty[$ defined by $F([0;t]) = F(t)$. Obviously, this measure is bounded on finite intervals. In the applications it should be clear whether F should be treated as a measure or a non-decreasing function.*

Example A.3 *Let F and G be probabilities on $[0;\infty[$. In previous courses we have defined a probability $G * F$ on $[0;\infty[$ by*

$$G * F([0;t]) = \int_{[0;t]} G([0;t-u])F(du), \quad t \geq 0.$$

With $G(t) = G([0;t])$ the distribution function of G we recognise this integral as the convolution, $G * F$, of G w.r.t. F. Remember that if X and Y are independent random variables with distribution F and G the convolution $G*F$ is the distribution of the sum $X + Y$. □

Example A.4 *Let G be some measurable function on $[0;\infty[$ that is bounded on finite intervals, and denote by λ the Lebesgue measure on $[0;\infty[$. Then we find that*

$$G * \lambda(t) = \int_0^t G(t-u)du = \int_0^t G(v)dv.$$

From remark A.2 we see that if G is non-negative then $G * \lambda$ is the measure on $[0;\infty[$ with density G w.r.t. λ. □

In the following we derive some important properties about convolutions that will be used during the course. G, G_1, G_2 are anywhere assumed to be real-valued measurable functions on $[0;\infty[$ that are bounded on finite intervals.

Theorem A.5 *With δ_0 the measure with mass one at zero it holds that*

$$G * \delta_0 = G.$$

Proof: For all $t \geq 0$ it holds that

$$G * \delta_0(t) = \int_{[0;t]} G(t-u)\delta_0(du) = G(t).$$

□

Theorem A.6 *For a measure, F, bounded on finite intervals, and I the constant function with value 1 it holds that*

$$I * F = F.$$

Proof: For the identity to make sense the right hand side must be interpreted as the distribution function of F, c.f. remark A.2. For any $t \geq 0$ we get that

$$I * F(t) = \int_{[0;t]} I(t-u)F(du) = \int_{[0;t]} F(du) = F(t).$$

□

As the domain of definition of G and F in definition A.1 are not the same the convolution operator is clearly not commutative. However, commutativity holds if both arguments are measures.

Theorem A.7 *For measures F_1, F_2, that are bounded on finite intervals it holds that*

$$F_1 * F_2 = F_2 * F_1$$

Proof: For any $t \geq 0$ the function

$$(u, v) \to 1_{[0;t]}(u+v)$$

is measurable. Hence by Tonellis theorem we get

$$\begin{aligned} F_1 * F_2(t) &= \int_{[0;t]} F_1(t-u)F_2(du) \\ &= \int_{[0;t]} \int_{[0;t-u]} F_1(dv)F_2(du) \\ &= \int_{[0;t]} \int_{[0;t-v]} F_2(du)F_1(dv) \\ &= \int_{[0;t]} F_2(t-v)F_1(dv) = F_2 * F_1(t). \end{aligned}$$

□

From the linearity of integration we get that

Theorem A.8 *For $\alpha_1, \alpha_2 \in \mathbb{R}$ it holds that*

$$(\alpha_1 G_1 + \alpha_2 G_2) * F = \alpha_1 (G_1 * F) + \alpha_2 (G_2 * F).$$

By dominated convergence the result may be extended to

Theorem A.9 *Let $G_1, G_2, \ldots$ be measurable functions that are bounded on finite intervals with a pointwise limit G that is bounded on finite intervals, then it holds that*

$$\lim_{n\to\infty} G_n * F = G * F.$$

Convolution is also linear in the second argument

Theorem A.10 *Let $F_1, F_2,$ be measures that are bounded on finite intervals. For $\alpha_1, \alpha_2 > 0$ it holds that*

$$G * (\alpha_1 F_1 + \alpha_2 F_2) = \alpha_1 (G * F_1) + \alpha_2 (G * F_2).$$

The result is easily extended to a countable number of measures

Theorem A.11 *Let $F_1, F_2, \ldots$ be measures on $[0;\infty[$ that are bounded on finite intervals. If $\sum_{n=1}^{\infty} F_n$ is bounded on finite intervals it holds that*

$$G * (\sum_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} G * F_n.$$

$\square$

Theorem A.12 *For measures F_1, F_2, that are bounded on finite intervals it holds that*

$$G * (F_1 * F_2) = (G * F_1) * F_2$$

Proof: For $v \geq 0$ let $G = 1_{[0;v)}$. Noting that $F_1 * F_2$ is the measure on $[0;\infty[$ defined by $(F_1 * F_2)([0;t]) = \int_{[0;t]} F_1(t-u)F_2(du)$ the left hand side becomes

$$\begin{aligned} G * (F_1 * F_2)(t) &= \int_{[0;t]} 1_{[0;v)}(t-u) F_1 * F_2(du) \\ &= \int_{[0;t]} 1_{(t-v;t]}(u) F_1 * F_2(du) \\ &= F_1 * F_2(t) - F_1 * F_2(t-v). \end{aligned}$$

Calculating the right hand side we get

$$\begin{aligned} (G * F_1) * F_2(t) &= \int_{[0;t]} (G * F_1)(t-u) F_2(du) \\ &= \int_{[0;t]} \int_{[0;t-u]} 1_{[0;v)}(t-u-u') F_1(du') F_2(du) \\ &= \int_{[0;t]} \int_{[0;t-u]} 1_{(t-u-v;t-u])}(u') F_1(du') F_2(du) \\ &= \int_{[0;t]} \left(F_1(t-u) - F_1(t-u-v)\right) F_2(du) \\ &= F_1 * F_2(t) - \int_{[0;t-v]} F_1(t-v-u) F_2(du) \\ &= F_1 * F_2(t) - F_1 * F_2(t-v), \end{aligned}$$

hence we have proved the result for $G = 1_{[0;v)}$. By theorem A.8 the result also holds for G being simple and using standard extension techniques together with theorem A.9 the result follows for general G. $\square$

For a measure F on $[0;\infty[$ that is bounded on finite intervals we define recursively $F_*^2 = F * F$ and in general

$$F_*^{n+1} = F_*^n * F, \quad n \geq 2.$$

When F is a probability we know that F_*^n has a total mass of 1. In this case we can prove the following inequality

Lemma A.13 *Let $F \neq \delta_0$ be a probability on $[0;\infty[$. For any $t \geq 0$ there exist a constants $C_t > 0$ and $0 < \delta < 1$ such that for all $n \in \mathbb{N}$*

$$F_*^n(t) \leq C_t \delta^n.$$

Proof: Define the Laplace transform $\hat{F} : [0;\infty[\to]0;\infty[$ of F by

$$\hat{F}(\beta) = \int_0^\infty \exp(-\beta u) F(du), \quad \beta > 0.$$

By induction one proves that

$$\left(\hat{F}(\beta)\right)^n = \int_0^\infty \exp(-\beta u) F_*^n(du).$$

Since F does not have mass 1 in zero there is a $\beta > 0$ such that $0 < \hat{F}(\beta) < 1$. Now, for all $t \geq 0$ we get

$$\begin{aligned}
\exp(-\beta t) F_*^n(t) &= \int_{[0;t]} \exp(-\beta t) F_x^n(du) \\
&\leq \int_{[0;t]} \exp(-\beta u) F_*^n(du) \\
&\leq \int_0^\infty \exp(-\beta u) F_*^n(du) \\
&= \left(\hat{F}(\beta)\right)^n,
\end{aligned}$$

and the result follows by using $C_t = \exp(\beta t)$ and $\delta = \hat{F}(\beta)$. $\square$