# Chapter 1

# MAT 3MI: Supplement on Construction of Measures

**Note.** The present text is intended as a complete guide to some central results on construction of measures. The pace is more leisurely than in my Mat 3MI notes (section 4.6), the coverage less comprehensive (with some exceptions). Definitions and results (theorems and propositions) are stated in full. Proofs are sometimes referred to section 4.6 or to other parts of my basic 3MI text.

## 1.1 Tight set functions

**Definition 1** A *non-negative set function* $\varphi = (X, \mathcal{E}, \varphi)$ is a paved space $(X, \mathcal{E})$ with $\emptyset \in \mathcal{E}$ and a map $\varphi : \mathcal{E} \to [0, \infty]$ with $\varphi(\emptyset) = 0$. (If, occasionally, $\emptyset \in \mathcal{E}$ and $\varphi(\emptyset) = 0$ is not assumed, this must be emphasized). The set function is *finite* if $\varphi E < \infty$ for all $E \in \mathcal{E}$, *bounded* if $\sup\{\varphi E | E \in \mathcal{E}\} < \infty$ and *finitely founded* if, for every set $E \in \mathcal{E}$ with $\varphi(E) = \infty$ and every $0 \le t < \infty$ there exists $F \in \mathcal{E}$ with $F \subseteq E$ and $t < \varphi(F) < \infty$. Furthermore, $\varphi$ is *monotone* if $E_1 \subseteq E_2$ implies $\varphi(E_1) \le \varphi(E_2)$ (here, and often in the sequel, we use the notational strategy that sets denoted by $E_1, E_2$ etc. are automatically assumed to be members of $\mathcal{E}$).

*Example* 1 (part 1). Let $\lambda = (\mathbb{R}, \mathcal{J}, \lambda)$ be the set function "length" on the paving $\mathcal{J}$ of semi-closed intervals, i.e. let the basis set be $X = \mathbb{R}$, let $\mathcal{J}$ be the paving of all intervals of the form $]a, b]$ with $-\infty < a \le b < \infty$ and let $\lambda$ be defined by $\lambda(]a, b]) = b - a$.

Note that $\emptyset \in \mathcal{J}$ (take $a = b$) and that $\lambda(\emptyset) = 0$. Clearly, $\lambda$ is a finite and monotone non-negative set function. As we know from the discussion in 3MI 1.2, it is a key problem to extend $\lambda$ to a measure. We thus seek a measure $\mu = (\mathbb{R}, \mathcal{B}, \mu)$ such that $\mathcal{B} \supseteq \mathcal{J}$ and such that $\mu|\mathcal{J}$, the restriction of $\mu$ to $\mathcal{J}$, coincides with $\lambda$. As $\mathcal{B}$ is required to be a $\sigma$-field containing $\mathcal{J}$, and as $\sigma(\mathcal{J}) = \mathcal{B}(\mathbb{R})$, the natural Borel structure on $\mathbb{R}$, $\mathcal{B} \supseteq \mathcal{B}(\mathbb{R})$ must hold. We will be satisfyed to obtain $\mathcal{B} = \mathcal{B}(\mathbb{R})$ but will, in fact, also look at certain larger Borel structures.

Now, let us expand the example by taking as point of departure a non-decreasing function $F : \mathbb{R} \to \mathbb{R}$ (i.e. $x \le y \Rightarrow F(x) \le F(y)$). Define the $F$-*variation* on the semi-closed intervals to be the set function $\lambda_F = (\mathbb{R}, \mathcal{J}, \lambda_F)$ with the paving $(\mathbb{R}, \mathcal{J})$ as above and $\lambda_F$ defined by $\lambda_F(]a, b]) = F(b) - F(a); -\infty < a \le b < \infty$.

Clearly, $\lambda_F$ too is a finite and monotone non-negative set function, and $\lambda_F$ is bounded if and only if $F$ is bounded. If $F$ is the identity map we obtain the length function we started off with.

Again, the key problem we shall look into is to find, indeed to construct, an extension of $\lambda_F$ to a measure $\mu_F$, say defined on $\mathcal{B}(\mathbb{R})$.

We note that in order for this to be feasible, $F$ must be right-continuous ($x_1 \ge x_2 \ge \cdots, x_n \to x \Rightarrow F(x_n) \to F(x)$). To see this note that if $x_n \downarrow x$, then $]x, x_n] \downarrow \emptyset$ thus, by exterior approximation (Theorem 3MI 4.3.1, (vi)) we find that $\mu_F(]x, x_n]) \downarrow 0$ for any measure extension $\mu_F$ of $\lambda_F$, i.e. $\lambda_F(]x, x_n]) \downarrow 0$ and $F(x_n) \downarrow F(x)$ follows. $\square$

**Definition 2** Let $\varphi = (X, \mathcal{E}, \varphi)$ be a non-negative set function. By $\varphi_*$ we denote the *variation* of $\varphi$, which is the set function on the full power set $\mathcal{P}(X)$ given by

$$\varphi_*(A) = \sup\{\sum_{i \in I} \varphi(E_i) | \sum_{i \in I} E_i \subseteq A\}; \quad A \subseteq X.$$

If $\varphi_*(X) < \infty$, $\varphi$ is of *bounded variation.*

Note that "$\sum E_i \subseteq A$" signals that the sets $E_i$; $i \in I$ are pairwise disjoint (and stones, i.e. members, of the paving $\mathcal{E}$). Also note that we need only consider finite (but arbitrarily large) index sets. Thus $\varphi_*(A)$ is the supremum of all sums of the form $\sum_1^n \varphi(E_k)$ with $n \in \mathbb{N}$ and with $E_1, E_2, \cdots, E_n$ pairwise disjoint sets in $\mathcal{E}$.

The variation $\varphi_*$ has a basic property of "superadditivity":

**Proposition 1** *Let $\varphi = (X, \mathcal{E}, \varphi)$ be a non-negative set function. Then $\varphi_* | \mathcal{E} \ge \varphi$ (i.e. $\varphi_* E \ge \varphi E$ for all $E \in \mathcal{E}$), $\varphi_*$ is monotone, and, finally, $\varphi_*$ is* superadditive *by which we mean that if $A$ and $B$ are any disjoint subsets of $X$, then*

$$\varphi_*(A \cup B) \ge \varphi_*(A) + \varphi_*(B).$$

**Proof** Easy exercise. $\square$

Note that superadditivity automatically extends to arbitrary families $(A_i)_{i \in I}$ of pairwise disjoint subsets of $X$. For such families

$$\varphi_* \left( \bigcup_{i \in I} A_i \right) \ge \sum_{i \in I} \varphi_*(A_i)$$

holds.

*Exercise 1.* To a given non-negative set function $\varphi$, there may be many set functions defined on the full power set $\mathcal{P}(X)$ that satisfy the three conditions of Proposition 1. For instance, the three properties hold for the "extreme" set function that sends every set, except the empty set, into $\infty$. Find a characterization which identifies $\varphi_*$ among all possible set functions satisfying the three conditions in question. $\square$

In general, the variation $\varphi_*$ does not extend $\varphi$:

*Exercise 2.* Find an example to illustrate this point (e.g. by considering certain packing numbers, cf. Øvelse 2, 3MI, p. 58). $\square$

For many of the set functions we shall consider, $\varphi_*$ does in fact extend $\varphi$. In particular, this holds for the "tight" set functions:

**Definition 3** A non-negative set function $\lambda = (X, \mathcal{E}, \lambda)$ defined on a $(\emptyset, \cap f)$-paving $\mathcal{E}$ is *tight* if the requirement

$$\lambda(E_1) + \lambda_*(E_2 \setminus E_1) = \lambda(E_2)$$

is satisfyed for every pair $E_1, E_2$ of sets in $\mathcal{E}$ with $E_1 \subseteq E_2$.

*Remarks.* For most purposes this definition will only be applyed to finite set functions. However, from time to time it is convenient to allow for infinite values. For instance, this is the case for the Exercises 10 and 11 in the next section. The defining relation is formulated so that it always makes sense. In case $\lambda(E_1) < \infty$ it can be put in the form $\lambda_*(E_2 \setminus E_1) = \lambda(E_2) - \lambda(E_1)$.

The definition is basic for our treatment. Our key concern will be to extend tight set functions to set functions with further properties, eventually to measures. For such endeavours, the most modest starting point seems to be a set function defined on a paving which is closed under finite intersections, exactly as required in the definition. It is possible to introduce a more general notion, dropping the requirement that $\mathcal{E}$ be $(\cap f)$-closed and taking as defining relation the requirement that

$$\lambda_*(E_1 \cap E_2) + \lambda_*(E_2 \setminus E_1) = \lambda(E_2)$$

for any pair $E_1, E_2$ of sets in $\mathcal{E}$ (the interested reader is invited to show that this definition is consistent with the one above). However, we believe we have struck the right level of generality in Definition 3.

*Example* 1 (part 2). The set function "length" on the semi closed intervals is tight. More generally, for $F : \mathbb{R} \to \mathbb{R}$ non-decreasing, the $F$-variation $\lambda_F = (\mathbb{R}, \mathcal{J}, \lambda_F)$ is tight. To see this first notice that if $I$ and $J$ are two intervals in $\mathcal{J}$ with $I \subseteq J$, then $J \setminus I$ is the disjoint union of two intervals, $I_1$ and $I_2$, say, in $\mathcal{J}$. Therefore,

$$\lambda_{F*}(J \setminus I) \geq \lambda_F(I_1) + \lambda_F(I_2).$$

Now,

$$\lambda_F(I_1) + \lambda_F(I) + \lambda_F(I_2) = \lambda_F(J)$$

(draw!) and $\lambda_{F*}(J \setminus I) \geq \lambda_F(J) - \lambda_F(I)$ follows. To prove the reverse inequality, consider any finite family $I_1, \cdots, I_n$ of pairwise disjoint intervals, all contained in $J \setminus I$. Then, by enlarging the intervals (draw!), we can find another such family, say $I'_1, I'_2 \cdots, I'_n$, such that $(I'_1, \cdots, I'_n)$ is a decomposition of $J \setminus I$ and, as indicated, such that $I'_1 \supseteq I_1, \cdots, I'_n \supseteq I_n$. Then $\lambda_F(I'_1) + \cdots + \lambda_F(I'_n) + \lambda_F(I) = \lambda_F(J)$, (look at your drawing!), thus $\sum_1^n \lambda_F(I_k) \leq \sum_1^n \lambda_F(I'_k) = \lambda_F(J) - \lambda_F(I)$. As this holds for any family of intervals $(I_k)_{k \leq n}$ of the type considered, $\lambda_{F*}(J \setminus I) \leq \lambda_F(J) - \lambda_F(I)$ follows. We now conclude that $\lambda_F$ is tight. (My notes, 3MI p. 61, contain more details. Instead of enlarging the intervals $I_1, \ldots, I_n$ you could also have argued by supplementing the intervals with extra intervals in order to "fill the gaps"). $\square$

In general, tight set functions have nice properties and constitute a natural starting point for the construction of measures (finitely additive ones as well as the much demanded ordinary, countably additive ones).

For sone of the basic properties which we shall establish of tight set functions, it is convenient to introduce the following purely set theoretical notion.

**Definition 4** Let $(X, \mathcal{E})$ be a paved set. A set $A \subseteq X$ is said to have a *smooth trace* in another set $B \subseteq X$ (w.r.t. $\mathcal{E}$) if there exists $E \in \mathcal{E}$ such that $A \subseteq E$ and such that $A$ and $E$ have the same trace in $B$, i.e. $\mathrm{tr_B(E)} = \mathrm{tr_B(A)}$. If $A$ has a smooth trace in $B$ and if $A$ is a subset of $B$, we say that $A$ is *smooth in* $B$ (w.r.t. $\mathcal{E}$).

Regarding the notion of trace, recall that $\mathrm{tr_B(A)}$ stands for the set $B \cap A$.

By definition, a set $A$ is smooth in $B$ if there exists $E \in \mathcal{E}$ with $E \supseteq A$ and $E \cap B = A$.

The condition that $A$ have a smooth trace in $B$ may also be expressed as the requirement, that $A$ can be extended to a set in $\mathcal{E}$ by adding only points outside $B$.

Figure 1 summarizes the content of the definition ("smooth" sets symbolize sets in $\mathcal{E}$, "irregular" sets symbolize general subsets of $X$).

Figure 1

It is perhaps worth while to point out the following simple properties in relation to the notion of smoothness:

**Proposition 2** *Let $(X, \mathcal{E})$ be a paved set and $A$ and $B$ two subsets of $X$. Then:*

**(i)** *If $A$ has a smooth trace in $B$ then $A \cap B$ is smooth in $B$,*

**(ii)** *If $A$ has a smooth trace in $B$, $A$ has a smooth trace in any subset of $B$,*

**(iii)** *If $A$ is smooth in $B$, then $A \cap M$ is smooth in $B \cap M$ for any set $M$.*

**Proof** Easy exercise. $\square$

**Proposition 3** (basic properties of tight set functions). *Let $\lambda = (X, \mathcal{E}, \lambda)$ be a tight set function on the $(\emptyset, \cap f)$ paving $\mathcal{E}$. Then*

**(i)** (extension) *$\lambda_*$ extends $\lambda$ and is supermodular,*

**(ii)** (monotonicity) *$\lambda$ is monotone,*

**(iii)** (tightness property, generalized) *If $A$ has a smooth trace in $B$, then*

$$\lambda_*(A \cup B) = \lambda_*(A) + \lambda_*(B \setminus A), \tag{1.1}$$

**(iv)** (modularity) *Again, if $A$ has a smooth trace in $B$, the following* modularity *property holds:*

$$\lambda_*(A \cup B) + \lambda_*(A \cap B) = \lambda_*(A) + \lambda_*(B), \tag{1.2}$$

**(v)** (disjointification) *Let $B$ be any subset of $X$ and assume that $A_1, A_2, \cdots, A_n$ (with $n \geq 1$) are subsets of $B$ which are all smooth in $B$. Then $\lambda_*(B)$ can be calculated by disjointification, i.e.*

$$\lambda_*(B) = \sum_{i=1}^{n} \lambda_*(A_i \setminus \bigcup_{j<i} A_j) + \lambda_*(B \setminus \bigcup_{i=1}^{n} A_i).$$

**Proof** (i): Recall that $\lambda(\emptyset) = 0$ is always assumed, cf. Definition 1. If $E \in \mathcal{E}$, tightness gives $\lambda_* E = \lambda_*(E \setminus \emptyset) = \lambda(E) - \lambda(\emptyset) = \lambda(E)$. Thus $\lambda_*$ extends $\lambda$.

(ii): Follows from (i) (or directly from Definition 3).

(iii): By superadditivity, "$\geq$" holds in (1.1). To establish the reverse inequality, first choose $E \in \mathcal{E}$ with $E \supseteq A$ and such that $\mathrm{tr_B(E)} = \mathrm{tr_B(A)}$. Then consider finitely many pairwise disjoint sets in $\mathcal{E}$, say $E_1, E_2, \cdots, E_n$, all contained in $A \cup B$. Then

$$\begin{aligned}
\sum_{i=1}^{n} \lambda(E_i) &= \sum_{i=1}^{n} (\lambda(E_i \cap E) + \lambda_*(E_i \setminus E)) \\
&= \sum_{i=1}^{n} \lambda_*(E_i \cap A) + \sum_{i=1}^{n} \lambda_*(E_i \setminus A) \\
&\leq \lambda_* \left( \bigcup_{i=1}^{n} (E_i \cap A) \right) + \lambda_* \left( \bigcup_{i=1}^{n} (E_i \setminus A) \right) \\
&\leq \lambda_*(A) + \lambda_*(B \setminus A).
\end{aligned}$$

Here it may be a help to consult figure 2

Figure 2

As the above inequality ($\sum \lambda(E_i) \leq \lambda_*(A) + \lambda_*(B \setminus A)$) holds for any finite sequence $E_1, \cdots, E_n$ of the type considered, "$\leq$" must hold in (1.1).

(iv): If $A$ has a smooth trace in $B$ then, firstly, (1.1) holds and, secondly, as $A \cap B$ is smooth in $B$,

$$\lambda_*(B) = \lambda_*(A \cap B) + \lambda_*(B \setminus A).$$

It follows that

$$\lambda_*(A \cup B) + \lambda_*(A \cap B) + \lambda_*(B \setminus A) = \lambda_*(A) + \lambda_*(B \setminus A) + \lambda_*(B).$$

If $\lambda_*(B \setminus A) < \infty$ this implies (1.2), and if $\lambda_*(B \setminus A) = \infty$, (1.2) is trivial, anyhow.

(v): We prove this by induction in $n$. The start ($n = 1$) is secured by (iii). To go from $n$ to $n+1$, consider $n+1$ smooth subsets of $B : A_1, \cdots, A_{n+1}$. From the induction hypothesis we get

$$\lambda_*(B) = \sum_{i=1}^{n} \lambda_*(A_i \setminus \bigcup_{j<i} A_j) + \lambda_*(B \setminus \bigcup_{i=1}^{n} A_i).$$

Noting that $A_{n+1} \setminus \bigcup_1^n A_i$ is smooth in $B \setminus \bigcup_1^n A_i$ (by Proposition 2, (iii)), we find that

$$\lambda_*(B \setminus \bigcup_{i=1}^{n} A_i) = \lambda_*(A_{n+1} \setminus \bigcup_{i=1}^{n} A_i) + \lambda_*(B \setminus \bigcup_{i=1}^{n+1} A_i).$$

Putting things together, we have

$$\lambda_*(B) = \sum_{i=1}^{n+1} \lambda_*(A_i \setminus \bigcup_{j<i} A_j) + \lambda_*(B \setminus \bigcup_{i=1}^{n+1} A_i).$$

We have thus completed the induction step and can now conclude the general validity of the statement in (v). $\square$

*Remark.* The attention of the interested reader is drawn to Exercise 6 of the next section, where another general property for tight set functions related to modularity is pointed out.

*Remark* (added in the last minute!). I now realize that an improved exposition of the subsequent material is possible by turning at this point to the idea of splitting (Section 3). The starting point is the observation that by property (iii), every set $E \in \mathcal{E}$ splits any subset $T \subseteq X$ properly in the sense that $\lambda_*(T) = \lambda_*(T \cap E) + \lambda_*(T \setminus E)$ holds. The next edition of these notes will be somewhat improved by making the change indicated.

It is awkward to work directly with a tight set function $\lambda$ on a paving which is only $(\emptyset, \cap f)$-closed. We must extend $\lambda$ to (at least) the *lattice generated* by $\mathcal{E}$ (lattice = da.: "gitter"). By definition, this is the smallest $(\cup f, \cap f)$-closed paving containing $\mathcal{E}$. Recall (Opgave 3, 3MI p. 31 or (1.2), 3MI, Appendix p. 3), that the lattice generated by $\mathcal{E}$ consists of all finite unions of sets in $\mathcal{E}$ (as $\mathcal{E}$ is assumed to be $(\cap f)$-closed). As a corollary to the proposition just established, we can now prove an appropriate extension result of the desired type.

**Proposition 4** *Let $\lambda$ be a tight set function on the $(\emptyset, \cap f)$ paving $\mathcal{E}$. Denote by $\mathcal{E}^f$ the lattice generated by $\mathcal{E}$, and by $\lambda^f$ the restriction of $\lambda_*$ to $\mathcal{E}^f$.*

*Then the variation of $\lambda^f$ coincides with the variation of $\lambda$ ($\lambda^f_* = \lambda_*$), and $\lambda^f$ is a tight set function on $\mathcal{E}^f$ which extends $\lambda$.*

**Proof** We already know that $\lambda^f$ extends $\lambda$ (Proposition 3). It follows that, for any subset $A$ of $X$, $\lambda^f_*(A) \geq \lambda_*(A)$. To prove the reverse inequality, let $E_1, \cdots, E_n$ be pairwise disjoint sets in $\mathcal{E}^f$, all contained in $A$. Then, by superadditivity and monotonicity of $\lambda_*$, we have

$$\sum_i \lambda^f(E_i) = \sum_i \lambda_*(E_i) \leq \lambda_*(\bigcup_i E_i) \leq \lambda_*(A)$$

and we readily conclude that $\lambda^f_*(A) \leq \lambda_*(A)$.

Finally, to see that $\lambda^f$ is tight, consider $A^f$, $B^f$ in $\mathcal{E}^f$ with $A^f \subseteq B^f$. Let $E_1, \cdots, E_n$ be sets in $\mathcal{E}$ such that $A^f = E_1 \cup \cdots \cup E_n$. Applying Proposition 3, (v) twice, we find

$$\begin{aligned}
&\lambda^f(B^f) = \lambda_*(B^f) \\
&= \lambda_*(E_1 \cup \cdots \cup E_n \cup B^f) \\
&= \sum_{i=1}^{n} \lambda_*(E_i \setminus \bigcup_{j<i} E_j) + \lambda_*(B^f \setminus \bigcup_{i=1}^{n} E_i) \\
&= \lambda_*(A^f) + \lambda_*(B^f \setminus A^f) \\
&= \lambda^f(A^f) + \lambda^f_*(B^f \setminus A^f),
\end{aligned}$$

which is the desired tightness property. $\square$

**Definition 5** Let $\lambda = (X, \mathcal{E}, \lambda)$ be a tight set function on the $(\emptyset, \cap f)$-paving $\mathcal{E}$. The set function $\lambda^f = (X, \mathcal{E}^f, \lambda^f)$ introduced in Proposition 4 is called the $(\cup f, \cap f)$-*extension* of $\lambda$ or the $f$-*extension* of $\lambda$.

*Exercise 3.* Formulate and prove an appropriate uniqueness assertion in relation to Proposition 4.

The tightness condition takes a simpler form if the paving $\mathcal{E}$, the domain of definition of the set function considered, is a lattice:

**Proposition 5** *Let $\lambda : \mathcal{E} \to [0, \infty]$ be a finite non-negative set function on the $(\emptyset, \cup f, \cap f)$-closed paving $\mathcal{E}$. Then the following conditions are equivalent:*

**(i)** *$\lambda$ is tight,*

**(ii)** *for all $E_1, E_2 \in \mathcal{E}$ with $E_1 \subseteq E_2$,*

$$\lambda(E_1) + \sup\{\lambda(E) | E \in \mathcal{E}, E \subseteq E_2 \setminus E_1\} = \lambda(E_2) \tag{1.3}$$

*holds,*

**(iii)** *$\lambda$ is monotone and* superadditive, *i.e. for any pair $E_1, E_2$ of disjoint sets in $\mathcal{E}$, $\lambda(E_1 \cup E_2) \geq \lambda(E_1) + \lambda(E_2)$ holds, and, furthermore, for every pair $E_1, E_2 \in \mathcal{E}$ with $E_1 \subseteq E_2$, and every $t < \lambda(E_2)$, there exists $E \in \mathcal{E}$ with $E \subseteq E_2 \setminus E_1$, such that $\lambda E_1 + \lambda E \geq t$.*

*Furthermore, if these conditions are fulfilled then $\lambda_* : \mathcal{P}(X) \to [0, \infty]$ is given by*

$$\lambda_*(B) = \sup_{E \in \mathcal{E}, E \subseteq B} \lambda E\,; \quad B \subseteq X \tag{1.4}$$

**Proof** (i) $\Rightarrow$ (iii): If $\lambda$ is tight, $\lambda$ is monotone and also superadditive; $\lambda$ is even additive ($\lambda(E_1 \cup E_2) = \lambda(E_1) + \lambda(E_2)$ for disjoint sets in $\mathcal{E}$) as is easily seen from Proposition 3 (e.g. (i) and (iii)). It then follows that (1.4) holds, and the last property in (iii) follows readily.

(iii) $\Rightarrow$ (ii): Under the conditions of (iii) we find, for $E_1, E_2 \in \mathcal{E}$ with $E_1 \subseteq E_2$, that, firstly $\lambda E_1 + \sup\{\lambda E | E \in \mathcal{E}, E \subseteq E_2 \setminus E_1\} \leq \lambda E_2$ (apply monotonicity and superadditivity), and then, the last property of (iii) shows that here, "=" must hold.

(ii) $\Rightarrow$ (i): So assume that (ii) holds. Clearly, $\lambda$ must be monotone. Also then, $\lambda$ must be additive (if $E_1, E_2 \in \mathcal{E}$ are disjoint, apply (1.3) with $E_1 := E_1$, $E_2 := E_1 \cup E_2$). It follows that (1.4) holds, thus (1.3) shows that the usual defining relation for tightness is satisfied. $\square$

*Exercise 4.* Let $\lambda$ be a set function on an algebra. Prove that $\lambda$ is tight if and only if $\lambda$ is a finitely additive measure.

## 1.2 Approximation and continuity

The idea of approximation is natural and essential here as well as in any other part of analysis.

**Definition 6** Let $\varphi = (X, \mathcal{E}, \varphi)$ be a monotone, non-negative set function. The paving $\mathcal{K}$ on $X$ is said to be an *inner approximating paving* (or just an *approximating paving*) for $\varphi$ if, to any $E \in \mathcal{E}$ and any $0 \le t < \varphi(E)$, there exist sets $F, K$ with $F \in \mathcal{E}$, $K \in \mathcal{K}$, $F \subseteq K \subseteq E$ and $\varphi(F) \ge t$.

The defining requirement can be put in the form

$$\varphi(E) = \sup\{\varphi(F) | \exists K : F \subseteq K \subseteq E\},$$

it being understood that $F \in \mathcal{E}$ and $K \in \mathcal{K}$. We remark that the above suppremum could be over the empty set (resulting by definition in the value 0). Note that in case $\mathcal{K} \subseteq \mathcal{E}$, the above requirement reduces to the requirement

$$\varphi(E) = \sup_{K \subseteq E} \varphi(K).$$

This situation is often important, so we introduce a special terminology:

**Definition 7** Let $\varphi = (X, \mathcal{E}, \varphi)$ be a monotone, non-negative set function and let $\mathcal{K} \subseteq \mathcal{E}$ be a subpaving of $\mathcal{E}$. We say that $\varphi$ is $\mathcal{K}$-*inner regular* (or just $\mathcal{K}$-regular) if $\varphi(E) = \sup\{\varphi K | K \subseteq E, K \in \mathcal{K}\}$ for all $E \in \mathcal{E}$.

Analogous notions of *outer approximating pavings* and *outer regularity* are introduced in the obvious way.

Below we suggest one lemma which is intended to cover all approximation arguments of a certain type which we shall need, either for the study of given measures or for our main endeavour here, the actual construction of measures. Apparently, such arguments require certain modularity type properties of the set functions under consideration. We shall state the lemma under the condition of modularity as such.

**Definition 8** Let $(X, \mathcal{E})$ be a *lattice*, i.e. $\mathcal{E}$ is a $(\cup f, \cap f)$-paving on $X$. A set function $\kappa = (X, \mathcal{E}, \kappa)$ on $\mathcal{E}$ is *modular* if

$$\kappa(E \cup F) + \kappa(E \cap F) = \kappa(E) + \kappa(F)$$

for every pair of sets $E, F$ in $\mathcal{E}$.

*Exercise 5.* Illustrate by a trivial example that a modular set function on a lattice need not be tight. *Note:* As is apparent, we have focused on the notion of tightness in our attempt to extend set functions to measures (finitely additive and countably additive ones). This is a wise strategy. However, in a sense, we loose something as there are set functions on lattices, which have extensions to measures that are not tight (easy: consider your example!)

In this connection we mention the following result: A set function (say bounded) on a lattice has an extension to a finitely additive measure if and only if the set function is modular. This innocent looking result is surprisingly difficult to prove (try it!) but throws some light on the condition of modularity.

**Proposition 6** (the squeezing inbetween lemma). *Let $\kappa = (X, \mathcal{E}, \kappa)$ be a non-negative, monotone and modular set function on the lattice $\mathcal{E}$, and assume that $\mathcal{K}$ is an inner approximating paving for $\kappa$.*

*If $E_1 \supseteq E_2 \supseteq \cdots$ is any decreasing sequence of sets in $\mathcal{E}$ with $\kappa(E_1) < \infty$ and $t$ any real number with $t < \lim \kappa(E_n)$, then there exist sequences $(\widetilde{E}_n) \subseteq \mathcal{E}$ and $(K_n) \subseteq \mathcal{K}$ such that the inclusions in the following diagram hold:*

$$\begin{array}{ccccccccccc} E_1 & & \supseteq & & E_2 & & \supseteq & & E_3 & & \supseteq & & \cdots \\ \cup\!\!| & & & & \cup\!\!| & & & & \cup\!\!| & & & & \\ K_1 & \supseteq & \widetilde{E}_1 & \supseteq & K_2 & \supseteq & \widetilde{E}_2 & \supseteq & K_3 & \supseteq & \widetilde{E}_3 & \supseteq & \cdots \end{array}$$

*and such that*

$$\lim_{n\to\infty} \kappa(\widetilde{E}_n) > t.$$

**Proof** See 3MI, Theorem 4.6.1. □

*Remark* The proof shows that it is enough to assume *supermodularity* instead of modularity, i.e. we need only require that

$$\kappa(E \cup F) + \kappa(E \cap F) \geq \kappa(E) + \kappa(F)$$

holds for $E, F \in \mathcal{E}$.

*Exercise 6.* Prove that if $\lambda = (X, \mathcal{E}, \lambda)$ is a tight set function on the $(\emptyset, \cap f)$-closed paving $\mathcal{E}$, then $\lambda_*$ is supermodular, i.e. for any subsets $A$ and $B$ of $X$, $\lambda_*(A \cup B) + \lambda_*(A \cap B) \geq \lambda_* A + \lambda_* B$. Show that $\lambda_*$ need not be modular, indeed, $\lambda_*$ need not be additive.

Of particular importance is the situation when a set function has a semi-compact approximating paving. The following will be our key result on construction of measures:

**Theorem 1.2.1** *(*Main theorem on construction of measures*). A finite tight set function which has a semi-compact approximating paving can be extended to a measure.*

The proof is technical and cannot be accomplished until we have been through further preparations. In order to throw some light on the result already now we indicate a (much!) weaker result (in the form of the exercise below) and then we continue to demonstrate the ease with which the theorem applies to our key example.

*Exercise 7.* Prove that a finitely additive measure defined on a $\sigma$-algebra which has a semicompact approximating paving is already itself a measure. Hint: Apply Proposition 6 and use the result of Opgave 11, 3 MI p. 95. □

*Example* 1 (part 3). Consider again the set function $\lambda_F = (\mathbb{R}, \mathcal{J}, \lambda_F)$ with $F$ a right continuous non-decreasing real function on $\mathbb{R}$. Recall that we showed (in part 2 of our example) that right continuity of $F$ is essential if we want to maintain the hope to extend $\lambda_F$ to a measure.

By $\mathcal{K}$ we denote the paving consisting of all closed intervals $[a,b]$ with $-\infty < a \leq b < \infty$, and of $\emptyset$.

Then clearly, by right continuity of $F$, $\mathcal{K}$ is an approximating paving for $\lambda_F$. We already noted that $\lambda_F$ is tight. It remains to note that $\mathcal{K}$ is semi-compact before the main theorem can be applyed. However, this is really trivial! As this property is the essential property needed, we write down in detail what is involved. As $\mathcal{K}$ is $(\emptyset, \cap f)$-closed, all we have to prove is that if $K_n \downarrow (K_1 \supseteq K_2 \supseteq \cdots)$ and if all the $K_n$ are non-empty, then $\bigcap_1^\infty K_n \neq \emptyset$. Let $K_n = [a_n, b_n]$. Then $-\infty \leq a_n \leq b_n < \infty$ and $a_1 \leq a_2 \leq \cdots$ and $b_1 \geq b_2 \geq \cdots$. In particular, $a_n \leq b_1$ for all $n$. As $(a_n)$ is also nondecreasing, we conclude that $(a_n)$ converges, say $a_n \to x$. Clearly, $x \in \bigcap_1^\infty K_n$. Thus $\bigcap_1^\infty K_n \neq \emptyset$ as requested.

In anticipation, we now apply the main theorem and readily conclude that $\lambda_F$ has an extension to a measure. If $F$ is the identity map, we arrive at the much sought *Lebesgue measure* which extends the length function. In the general case, the measure we arrive at is the *Lebesgue-Stieltjes* measure with *distribution* function $F$ which may be taken to be defined on the natural Borel structure $\mathcal{B}(\mathbb{R})$.

For the easy question about uniqueness, please consult 3MI, Sætning 4.1.2. □

A comment may be in order. Noone have ever constructed a non-trivial measure without recourse to some kind of compactness property. The way this was done for the Lebesgue-Stieltjes measures is crystal clear (modulo the proof, yet to come, of the abstract main theorem). The essential property turned out to be the intuitively convincing, yet delicate fact that a bounded non-decreasing sequence on $\mathbb{R}$ converges. In a certain sense, the rest is trivial!

There are important relations between approximation with semi-compact pavings and continuity.

**Definition 9** Let $\varphi = (X, \mathcal{E}, \varphi)$ be a monotone non-negative set function. We say that $\varphi$ is *continuous* at $\emptyset$ if $\varphi(E_n) \to 0$ for any sequence $(E_n)_{n\geq 1}$ of sets in $\mathcal{E}$ with $E_n \downarrow \emptyset$ for which $\varphi(E_1) < \infty$. And we say that $\varphi$ is *continuous* if $\varphi(E_n) \to \varphi(E)$ for any sequence $(E_n)_{n\geq 1}$ of sets in $\mathcal{E}$ for which $E_n \downarrow E$ with $E$ a set in $\mathcal{E}$ and with $\varphi(E_1) < \infty$.

*Remarks.* The notions introduced could be called *continuity from above* or *right continuity.* The dual notion of *continuity from below* or *left continuity* is introduced in the natural way (note, however, that for continuity from below ($E_n \uparrow E \Rightarrow \varphi(E_n) \to \varphi(E)$, no condition of finiteness is required).

**Proposition 7** *Let $\lambda = (X, \mathcal{E}, \lambda)$ be a tight set function on a $(\emptyset, \cap f)$-paving and assume that $\lambda$ has a semi-compact approximating paving. Denote by $\lambda^f = (X, \mathcal{E}^f, \lambda^f)$ the $f$-extension of $\lambda$.*

*Then $\lambda^f$ (hence also $\lambda$) is continuous at $\emptyset$.*

**Proof** Let $\mathcal{K}$ be a semi-compact approximating paving for $\lambda$. Denote by $\mathcal{K}^f$ the paving of finite unions of sets in $\mathcal{K}$. Then $\mathcal{K}^f$ is also semi compact (3 MI, Sætning 2.5.3). We shall show that $\mathcal{K}^f$ is an approximating paving for $\lambda^f$ (in fact then also for $\lambda^f_*$). Indeed, let $A$ be a set in $\mathcal{E}^f$ and let $\varepsilon > 0$ be given. Choose finitely many pairwise disjoint sets in $\mathcal{E}$, say $E_1, \cdots, E_n$, such that

$$\lambda^f(A) = \lambda_*(A) < \lambda(E_1) + \cdots + \lambda(E_n) + \varepsilon.$$

Then choose $F_i \subseteq K_i \subseteq E_i$; $i \leq n$ with the $F_i$ in $\mathcal{E}$ and the $K_i$ in $\mathcal{K}$ such that

$$\begin{aligned} \lambda^f(A) &< \lambda(F_1) + \cdots + \lambda(F_n) + \varepsilon = \lambda_*(F_1 \cup \cdots \cup F_n) + \varepsilon \\ &= \lambda^f(F_1 \cup \cdots \cup F_n) + \varepsilon. \end{aligned}$$

As $F_1 \cup \cdots \cup F_n \subseteq K_1 \cup \cdots \cup K_n \subseteq A$ we see from this (and the arbitrary nature of $A$ and $\varepsilon$) that $\mathcal{K}^f$ is an approximating paving for $\lambda^f$.

A straight forward application of the squeezing in between lemma, Proposition 6, finishes the proof. In detail: If $E_n \downarrow \emptyset$ with the $E_n$ in $\mathcal{E}^f$, we choose $\widetilde{E}_n$-sets in $\mathcal{E}^f$ and $K_n$-sets in $\mathcal{K}^f$ so that the inclusions of Proposition 6 hold and such that $\lim \lambda^f(\widetilde{E}_n) > \lim \lambda^f(E_n) - \varepsilon$ ($\varepsilon$ a given positive number). As $E_n \downarrow \emptyset$, $K_n \downarrow \emptyset$. By compactness $K_n = \emptyset$, eventually, hence $\widetilde{E}_n = \emptyset$, eventually. Therefore, $\lim \lambda^f(E_n) < \lim \lambda^f(\widetilde{E}_n) + \varepsilon = \varepsilon$. As $\varepsilon$ was arbitrary, $\lambda^f(E_n) \to 0$ follows. $\square$

**Proposition 8** *Let $\lambda$ be a tight set function on $\mathcal{E}$ and assume that $\mathcal{E}$ is $(\emptyset, \cup f, \cap f)$-closed. Then, if $\lambda$ is continuous at $\emptyset$, $\lambda$ is continuous.*

**Proof** Let $(E_n)$ be a sequence of sets in $\mathcal{E}$, let $E \in \mathcal{E}$ and assume that $E_n \downarrow E$ and that $\lambda(E_1) < \infty$. To $\varepsilon > 0$ choose $F \in \mathcal{E}$ such that $F \subseteq E_1 \setminus E$ and $\lambda E + \lambda F > \lambda(E_1) - \varepsilon$ (cf. Proposition 5). Then $\lambda_*(E_1 \setminus (E \cup F)) < \varepsilon$. As $E_n \cap F \downarrow \emptyset$, $\lambda(E_n \cap F) < \varepsilon$ holds for some $n$. Then (e.g. by disjointification),

$$\begin{aligned} \lambda(E_n) &= \lambda(E) + \lambda(E_n \cap F) + \lambda_*(E_n \setminus (E \cup F)) \\ &< \lambda(E) + \varepsilon + \lambda_*(E_1 \setminus (E \cup F)) \\ &< \lambda(E) + 2\varepsilon. \end{aligned}$$

This argument tells us that $\lim \lambda(E_n) \leq \lambda(E)$. The reverse inequality is trivial. $\square$

Combining the two last propositions we find

**Proposition 9** *The $f$-extension of a tight set function with a semicompact approximating paving is continuous.*

Comparing with Proposition 8 it lies nearly to ask if this result can be strengthened, replacing the assumption that $\lambda$ has a semi-compact approximating paving with the weaker assumption that $\lambda$ be continuous at $\emptyset$. It is important to understand that this is not the case. There is an interesting obstacle which teaches us something about the essential structure of the continuum, i.e. of $\mathbb{R}$. The fact is that if we destroy the continuum by pinching too many holes in it, e.g. so that we arrive at $\mathbb{Q}$ as in the exercise below, continuity obstacles appear which prevent the construction of non-trivial measures. The reader is invited to discover this himself through the following exercise:

*Exercise 8.* Let $\lambda = (\mathbb{R}, \mathcal{J}, \lambda)$ be the usual length function on the semi closed intervals. Denote by $\lambda_{\mathbb{Q}} = (\mathbb{Q}, \mathcal{E}_{\mathbb{Q}}, \lambda_{\mathbb{Q}})$ the trace of $\lambda$ in $\mathbb{Q}$, i.e. $\mathcal{E}_{\mathbb{Q}}$ is the paving on $\mathbb{Q}$ of sets of the form $]a,b] \cap \mathbb{Q}$ with $-\infty < a \le b < \infty$, and $\lambda_{\mathbb{Q}}$ is the set function given by $\lambda_{\mathbb{Q}}(\,]a,b] \cap \mathbb{Q}) = \lambda(\,]a,b]) = b - a$; $-\infty < a \le b < \infty$.

(i) Show that $\lambda_{\mathbb{Q}}$ is tight and continuous at $\emptyset$ (indeed, continuous at any set in $\mathcal{E}_{\mathbb{Q}}$).

(ii) Let $\mathcal{K}$ be the usual semicompact paving in $\mathbb{R}$ which approximates $\lambda$ (cf. Example 1, part 3) and denote by $\mathcal{K}_{\mathbb{Q}}$ the trace of $\mathcal{K}$ in $\mathbb{Q}$. Show that $\mathcal{K}_{\mathbb{Q}}$ is an approximating paving for $\lambda_{\mathbb{Q}}$. Is $\mathcal{K}_{\mathbb{Q}}$ semi-compact?

(iii) Let $\lambda_{\mathbb{Q}}^f$ be the $f$-extension of $\lambda_{\mathbb{Q}}$. Show that $\lambda_{\mathbb{Q}}^f$ is *not* continuous and conclude that $\lambda$ cannot be extended to a measure.

(iv) Can $\lambda_{\mathbb{Q}}$ have any semi-compact approximating paving?

(v) If, instead of "length", we use $F$-variation $\lambda_F$ on $\mathcal{J}$ as the starting point, and consider the trace $\mathrm{tr}_{\mathbb{Q}}(\lambda_{\mathrm{F}})$ of $\lambda_F$ on $\mathbb{Q}$ (defined in the natural way), discover the condition $F$ must satisfy in order that $\mathrm{tr}_{\mathbb{Q}}(\lambda_{\mathrm{F}})$ can be extended to a measure.

A further approximation procedure is needed on our way to extend a given set function to a measure. The foregoing leads us, typically via Proposition 9, to a finite, tight and continuous set function defined on a lattice. A key result is that this suffices for our purposes, indeed such a set function can always be extended to a measure (Theorem 1.3.1 below). Still, the extension is not straight forward. Consider, for instance a countable intersection $A = \bigcap_1^\infty E_n$ of sets in the lattice. How should the measure of $A$ be defined? Up to now we have been guided by approximation "from inside" (via tightness and the notion of approximating pavings). We have to do something else now. For instance the set $A = \bigcap_1^\infty E_n$ may be "thin" and it could be impossible to find even a single of the "fat" sets in the original lattice inside $A$. It may be worth while to consider a concrete example:

*Exercise 9.* Let $\lambda = (X, \mathcal{J}, \lambda)$ be the usual length function on semiclosed intervals, but restricted to $]0,1]$, i.e. $X = ]0,1], \mathcal{J} = \{]a,b] | 0 \le a \le b \le 1\}$, and $\lambda$ is defined by $\lambda(]a,b]) = b - a$.

Denote by $\lambda^f = (X, \mathcal{J}^f, \lambda^f)$ the $f$-extension of $\lambda$.

Show that, for every $0 \le t < 1$, there exist sets $E_n \downarrow A$ with the $E_n$ in $\mathcal{J}^f$ such that $\lambda^f(E_n) \to t$, but $A$ contains no subsets which belong to $\mathcal{J}^f$ other than the empty set, hence $\lambda_*(A) = \lambda_*^f(A) = 0$ (so that, unless $t = 0$, $\lambda_*(A)$ is *not* the right value for the length of $A$). *Hint.* For $t = 0$, the Cantor set constructions will do, for other values of $t$, that construction can be modified in a suitable way.

It lies nearby to attempt an approximation procedure "from outside" at this stage. Indeed, this is all we have to do in order, finally, to arrive at an appropriate point of departure for the construction of a measure.

**Definition 10** Let $\lambda = (X, \mathcal{E}, \lambda)$ be a non-negative, monotone set function defined on the $(\emptyset, \cup f, \cap f)$ paving $\mathcal{E}$. By the $(\cup f, \cap c)$-*extension* or the $\delta$-*extension* $\lambda^\delta = (X, \mathcal{E}^\delta, \lambda^\delta)$ of $\lambda$ we understand the extension for which $\mathcal{E}^\delta$ is the $(\cup f, \cap c)$-closure of $\mathcal{E}$ and $\lambda^\delta$ is given by *approximation from outside,* i.e..

$$\lambda^\delta(A) = \inf\{\lambda E | E \supseteq A, E \in \mathcal{E}\}; \quad A \in \mathcal{E}^\delta.$$

We note that by 3MI, (1.2), Appendix p. 3 the paving $\mathcal{E}^\delta$ is easy to characterize:

**Proposition 10** *Let $\mathcal{E}$ be a $(\cup f, \cap f)$-closed paving (a lattice). Then the $(\cup f, \cap c)$-closure of $\mathcal{E}$ is the paving $\mathcal{E}^\delta$ of all countable intersections of sets in $\mathcal{E}$.*

The extension of a continuous tight set function is well behaved:

**Proposition 11** *Let $\lambda = (X, \mathcal{E}, \lambda)$ be a finite continuous and tight set function on the $(\emptyset, \cup f, \cap f)$ paving $\mathcal{E}$, and denote by $\lambda^\delta = (X, \mathcal{E}^\delta, \lambda^\delta)$ the $\delta$-extension of $\lambda$. Then $\lambda^\delta$ is likewise a finite continuous and tight set function, and $\lambda^\delta$ extends $\lambda$.*

**Proof** cf. Mat 3 MI, Sætning 4.6.8 p. 85. □

**Proposition 12** *Let $\lambda = (X, \mathcal{E}, \lambda)$ be a tight and continuous set function on the $(\emptyset, \cup f, \cap c)$-closed paving $\mathcal{E}$ (for instance, $\lambda$ could be a $\delta$-extension, cf. Proposition 11). Then $\lambda_*$ is continuous.*

**Proof** Note that $\lambda_* = (X, \mathcal{P}(X), \lambda_*)$ has $\mathcal{E}$ as a approximating paving and apply the squeezing-inbetween lemma, Proposition 6 (note that $\lambda_*$ is supermodular).. □

A finite, tight and continuous set function on a $(\emptyset, \cup f, \cap c)$-paving is just the right starting point for the construction of a measure. In fact, we shall see that by restricting the variation we obtain the desired extension to a measure. In order for this to function smoothly, we need one more idea which we introduce in the next section. The point is that, as we have discussed before, the $\sigma$-algebra generated by the $(\emptyset, \cup f, \cap c)$-paving in question is hard to get your hands at. We could attempt a step by step extension through the Baire hierarki, and this is indeed possible (cf. Exercise 11 below). However, if we do not want to rely on familiarity with transfinite recursion, another route must be followed. The extra idea needed will be developed in the next section.

*Exercise 10.* Let $\lambda = (X, \mathcal{E}, \lambda)$ be a tight, finitely founded and continuous set function on the $(\emptyset, \cup f, \cap f)$-closed paving $\mathcal{E}$. Denote by $\lambda^\sigma = (X, \mathcal{E}^\sigma, \lambda^\sigma)$ the $\sigma$-*extension* of $\lambda$, i.e. $\mathcal{E}^\sigma =$ the $(\emptyset, \cup c, \cap f)$-closure of $\mathcal{E}$ and $\lambda^\sigma = \lambda_* | \mathcal{E}_\sigma$. Prove that $\lambda^\sigma$ is a tight, finitely founded and continuous extension of $\lambda$. *Hint:* Note that $\lambda_*$ is $\mathcal{E}$-regular. Apply Proposition 6 (several times). It is convenient to apply the result of Exercise 4. However, less will do (inspection of the proof of Proposition 6 shows that it suffices to use the fact that $\lambda_*(E \cup B) + \lambda_*(E \cap B) = \lambda E + \lambda_* B$ for $E \in \mathcal{E}$).

*Exercise 11* (for readers who master transfinite induction and recursion). Prove that a finite, tight and continuous set function on a lattice $\mathcal{E}$ for which co$\mathcal{E} \subseteq \mathrm{Bor}(\mathcal{E})$ has a unique extension to a measure $\mu$ on $\sigma(\mathcal{E})$. *Hint.* By transfinite recursion define set functions $\lambda_\alpha = (X, \mathcal{E}_\alpha, \lambda_\alpha)$; $\lambda \leq \omega_1$ ($\omega_1$ denoting the first uncountable ordinal) such that $\lambda_0$ is the set function given, $\lambda_1$ the $\delta$-extension of $\lambda_0$, $\lambda_2$ the $\sigma$-extension of $\lambda_1$ (cf. Exercise 10) and so on in a natural manner. For limit ordinals, just collect what you have constructed so far. Prove that all the $\lambda_\alpha$ are finitely founded, tight and continuous. As for continuity, note that you only have to worry about continuity at $\emptyset$; for this it is convenient to observe that $\mathcal{E}_1$ is an approximating paving for all the $\lambda_\alpha$'s. Note that if co$\mathcal{E} \subseteq \mathrm{Bor}(\mathcal{E})$, $\mathcal{E}_{\omega_1} = \sigma(\mathcal{E})$. Regarding the notation $\mathrm{Bor}(\mathcal{E})$ for the $(\cup c, \cap c)$-closure of $\mathcal{E}$, see Mat 3MI, Opgave II.6, p. 31. Using techniques of the next section, $\lambda_{\omega_1}$ can always be extended to a measure on $\sigma(\mathcal{E})$.

Deduce the announced main theorem as a corollary: A finite tight set function on a $(\emptyset, \cap f)$-closed paving $\mathcal{E}$ with co$\mathcal{E} \subseteq \mathrm{Bor}(\mathcal{E})$ which has a semi-compact approximating paving can be extended to a measure. Again, the restriction co$\mathcal{E} \subseteq \mathrm{Bor}(\mathcal{E})$ can be removed by techniques of the next section.

*Exercise 12.* A tight and continuous set function on a lattice which is "too infinite" (especially it should not be finitely founded) need not have an extension to a measure. Find an example to illustrate this point. *Hint.* One idea is to find an appropriate set function defined on the full powet set $\mathcal{P}(X)$ (note, itself a $\sigma$-algebra!) which only assumes two values.

## 1.3 Splitting

It is possible to avoid constructions involving the ordinals and in one step go from a suitable set function, which must have some continuity condition built into it, and to the desired extension to a measure. The starting point for such constructions will always be a set function defined on the full power set $\mathcal{P}(X)$. The trick then, introduced by Carathéodory in 19??, is to find some descriptive property of subsets of the basic set $X$ which picks out a $\sigma$-algebra such that the restriction of the set function to this $\sigma$-algebra is a measure. The relevant property is one of "splitting". We shall apply Carathéodory's idea to the variation $\lambda_*$ of a suitable tight set function. As other important applications of the idea are possible we start by proving a rather abstract result which exhibits the main features of the approach, and then continue with specific applications.

**Definition 11** Let $\kappa : \mathcal{P}(X) \to [0, \infty]$ be a non-negative set function on the full power set $\mathcal{P}(X)$. As usual, $\kappa(\emptyset) = 0$ is assumed. The set $A \subseteq X$ *splits* the set $T \subseteq X$ w.r.t. $\kappa$ (or we say that $A$ *splits* $T$ *properly*) if

$$\kappa(T) = \kappa(T \cap A) + \kappa(T \setminus A). \tag{1.5}$$

A subset $A \subseteq X$ is $\kappa$-*measurable*, if $A$ splits every subset $T$ of $X$ properly. The paving of all $\kappa$-measurable sets is denoted $\mathfrak{M}_\kappa$:

$$\mathfrak{M}_\kappa = \{A \mid \forall T : \kappa(T) = \kappa(T \cap A) + \kappa(T \setminus A)\},$$

and the restriction $\kappa|\mathfrak{M}_\kappa$ is called the *Caracthéodory restriction of* $\kappa$.

The sets $T$ that occur in (1.5) we may think about as "test sets". Only if we get a positive result for every test set, can we conclude that $A$ is $\kappa$-measurable. Note that the condition of $\kappa$-measurability of $A$ may be reformulated as the requirement that for every subset $R$ of $A$ and for every subset $S$ of $\complement A$, the additivity relation

$$\kappa(R \cup S) = \kappa(R) + \kappa(S)$$

holds.

For the result below we need the notion of completeness:

**Definition 12** A finitely additive measure $\mu = (X, \mathcal{E}, \mu)$ is *complete* if every set $A \subseteq X$ such that, to $\varepsilon > 0$ there exist sets $E_\varepsilon, F_\varepsilon$ in $\mathcal{E}$ with $E_\varepsilon \subseteq A \subseteq F_\varepsilon$ and $\mu(F_\varepsilon \setminus E_\varepsilon) < \varepsilon$, is a member of $\mathcal{E}$.

**Proposition 13** (the splitting lemma). *Let $\kappa : \mathcal{P}(X) \to [0, \infty]$ be a set function on the full power set $\mathcal{P}(X)$ and denote by $\mu = (X, \mathfrak{M}_\kappa, \mu)$ the Carathéodory restriction of $\kappa$. Then:*

**(i)** *$\mu$ is a finitely additive measure,*

**(ii)** $A \in \mathfrak{M}_\kappa,\ B \subseteq X \Rightarrow \kappa(A \cup B) + \kappa(A \cap B) = \kappa(A) + \kappa(B)$,

**(iii)** *If $A_1, \cdots, A_n$ can be separated by $\kappa$-measurable sets (i.e. $\exists (\widetilde{A}_i)_{i \le n}$ pairwise disjoint sets in $\mathfrak{M}_\kappa$ s.t. $A_i \subseteq \widetilde{A}_i$; $i \le n$), then $\kappa(\bigcup_1^n A_k) = \sum_1^n \kappa(A_k)$,*

**(iv)** *If $\kappa$ is monotone, $\mu$ is complete,*

**(v)** *If the following four conditions are fulfilled:*

*a) $\kappa$ is monotone,*

*b) $(A_n) \subseteq \mathfrak{M}_\kappa, A_n \uparrow A \Rightarrow \forall T \subseteq X : \kappa(T \cap A_n) \to \kappa(T \cap A)$,*

*c) $(A_n) \subseteq \mathfrak{M}_\kappa$, $A_n \downarrow A \Rightarrow \forall T \subseteq X$ with $\kappa(T) < \infty : \kappa(T \cap A_n) \to \kappa(T \cap A)$,*

*d) $\kappa$ is either finitely founded, or the implication $\kappa(R \cup S) = \infty \Rightarrow \kappa(R) = \infty \vee \kappa(S) = \infty$ holds,* *then $\mu$ is a measure, in particular $\mathfrak{M}_\kappa$ is a $\sigma$-algebra, and for every sequence $(A_n)_{n\geq 1}$ which can be separated by $\kappa$-measurable sets (i.e. $\exists(\widetilde{A}_n)_{n\geq 1}$ pairwise disjoint sets in $\mathfrak{M}_\kappa$ s.t. $A_n \subseteq \widetilde{A}_n$; $n \geq 1$) the additivity relation $\kappa(\bigcup_1^\infty A_n) = \sum_1^\infty \kappa(A_n)$ holds.*

**Proof** We refer to Mat 3MI, p. 78–79. □

We can now prove two main theorems:

**Theorem 1.3.1** *A finite, tight and continuous set function on a lattice can be extended to a measure.*

**Proof** With Propositions 11, 12 and 13 at hand, this is straight forward (cf. also the proof of Mat 3MI, Korollar 4.6.1.) □

By Proposition 9, we then obtain the announced main theorem (Theorem 1.2.1) as a corollary:

**Theorem 1.3.2** *A finite and tight set function defined on a $(\emptyset, \cap f)$-closed paving which has an approximating semi-compact paving can be extended to a measure.*

Further special cases and extensions are referred to my 3MI-notes (topological measure theory, Hausdorff measures etc.).

## 1.4 Notes

*Note* 1 (Overview). The approach which we followed in order to construct a measure was to make sucessive extensions of a given set function so that the defining pavings have more and more structure, basicly so that they undergo the following transformations:

$$(\cap f) \rightsquigarrow (\cup f, \cap f) \rightsquigarrow (\cup f, \cap c) \rightsquigarrow (\complement, \cup c),$$

the final structure being that of a $\sigma$-algebra. We realize that three main ideas were involved in this process: **tightness, approximation** and **splitting.** We can also point out which are the key auxillary results, viz. the following five: Propositions 3 (properties of tightness), 6 (squeezing inbetween), 7 (on semi-compactness and continuity), 11 ($\delta$-extension) and 13 (splitting).

*Note* 2 (on Proposition 6, the squeezing inbetween lemma).

The special case when $\mathcal{K} \subseteq \mathcal{E}$ is worth while pointing out explicitly: *Let* $\kappa = (X, \mathcal{E}, \kappa)$ *be a finite set functions defined on the lattice* $\mathcal{E}$, *let* $\mathcal{K} \subseteq \mathcal{E}$ *be a subpaving of* $\mathcal{E}$ *and assume that* $\kappa$ *is* $\mathcal{K}$*-regular and that* $\kappa$ *satisfies the following condition of supermodularity: For* $K \in \mathcal{K}$ *and* $E \in \mathcal{E}$, $\kappa(E \cup K) + \kappa(E \cap K) \geq \kappa(E) + \kappa(K)$. *Then, for every decreasing sequence of sets in* $\mathcal{E}$, *say* $E_n \downarrow$, *and every* $\varepsilon > 0$ *there exists a decreasing sequence of sets in* $\mathcal{K}$, *say* $K_n \downarrow$, *such that* $K_n \subseteq E_n$; $n \geq 1$ *and such that* $\lim \kappa(K_n) \geq \lim \kappa(E_n) - \varepsilon$. This is seen by inspection of the proof (in Mat 3MI, Theorem 4.6.1). Incidentally, we note that it is only formally that the stated supermodularity condition is weaker than "full" supermodularity of $\kappa$ (under $\mathcal{K}$-regularity, the latter condition follows from the former). This comment, in connection with 1.3) of Proposition 3, shows that $\lambda_*$ is supermodular if $\lambda$ is a tight content on a lattice (special case of the result claimed in Exercise 6).

As a last comment on Proposition 6, we point out that a dual result on "outer approximation" is also important, cf. Mat 3MI, Theorem 4.6.1.

*Note* 3 (on Proposition 7 and on semicompactness in Theorem 1.3.2). Note the essential role played by König' lemma. If we adopt the thesis that there is a basic combinatorial argument underlying every important theorem of mathematics, this is it for the key extension theorem, Theorem 1.2.1 (= Theorem 1.3.2). In Proposition 7 we did not, in the formulation of the result, keep trach of the approximating semi compact paving. We did not even state, as we perhaps should have, that $\lambda^f$ of Proposition 7 has an approximating semi compact paving. If we look at the further extension to a measure (Theorem 1.3.2), we obtain an interesting result which, among other things, implies that a measure which has a semi compact approximating paving consisting of measurable sets. The result hinted at we state in the form of an exercise:

*Exercise* 13. Let $\lambda$ be finite and tight on a $(\emptyset, \cap f)$-closed paving, and assume that $\mathcal{K}$ is an approximating semi compact paving. Prove that the extension $\mu$ to a measure constructed in (the proof of) Theorem 1.3.2 has a semi compact approximating paving consisting of measurable sets. *Hint.* Consider the paving of measurable sets in the $(\emptyset, \cup f, \cap c)$-closure of $\mathcal{K}$. Use Proposition 6. *Comment.* A natural question arises: Can every measure be obtained from Theorem 1.3.2, in principle? In other words, does every measure have a semi compact approximating paving? This is not the case, but examples to illustrate this are not so simple to construct.

*Note* 4 (on Proposition 11, the $\delta$-extension). As this is one of the key auxillary results and as the proof indicated (in Mat 3MI, p. 85) may be hard to read we shall here give a full proof. First we note that $\lambda^\delta$ is finite and that $\lambda^\delta$ extends $\lambda$. Then we establish the following property:

(i): $(E_n) \subseteq \mathcal{E}$, $E_n \downarrow A \Rightarrow \lambda(E_n) \to \lambda^\delta(A)$.

As $\lambda$ is assumed to be finite, we need not assume $\lambda(E_1) < \infty$ as this is automatic. Let $\varepsilon > 0$ be given and choose $E \in \mathcal{E}$ with $E \supseteq A$ and $\lambda(E) \leq \lambda^\delta(A) + \varepsilon$. Then, as $E_n \cup E \downarrow E$, we can find $n$ such that $\lambda(E_n \cup E) \leq \lambda(E) + \varepsilon$, and it follows that $\lambda(E_n) \leq \lambda(E_n \cup E) \leq \lambda^\delta(A) + 2\varepsilon$. Now (i) follows readily.

Then we note that

(ii): $\lambda^\delta$ is modular: $\forall A \in \mathcal{E}^\delta \forall B \in \mathcal{E}^\delta : \lambda^\delta(A \cup B) + \lambda^\delta(A \cap B) = \lambda^\delta(A) + \lambda^\delta(B)$.

Choose $(E_n) \subseteq \mathcal{E}$ and $(E_n) \subseteq \mathcal{E}$ such that $E_n \downarrow A$ and $F_n \downarrow B$. As $E_n \cup F_n \downarrow A \cup B$ and $E_n \cap E_n \downarrow A \cap B$, we see that the modularity property of (ii) holds by appealing to (i) and using the fact that $\lambda$ is modular.

The nest step is to show that:

(iii): $\lambda^\delta$ is continuous.

To see this, assume that $(A_n) \subseteq \mathcal{E}^\delta$ and that $A_n \downarrow A$. Choose, for each $n \geq 1$, $(E_{\cap k}) \subseteq \mathcal{E}$ such that $E_{\cap k} \downarrow A_n$. For each $\nu \geq 1$, put $E_\nu = \bigcap\{E_{\cap k} \mid n \leq \nu, k \leq \nu\}$. Then $(E_\nu) \subseteq \mathcal{E}$ and $E_\nu \downarrow A$. Choose by (i), to a given $\varepsilon > 0$, $\nu \geq 1$ such that $\lambda(E_\nu) \leq \lambda^\delta(A) + \varepsilon$. As $E_\nu \supseteq A_\nu$, $\lambda(E_\nu) \geq \lambda^\delta(A_\nu)$. Thus $\lambda^\delta(A_\nu) \leq \lambda^\delta(A) + \varepsilon$. Continuity of $\lambda^\delta$ is now clear.

Finally we prove that

(iv): $\lambda^\delta$ is tight.

As $\lambda^\delta$ is monotone and superadditive (even modular, cf. (ii)), we only have to worry about the essential requirement of (iii), Proposition 5. So let $A_1 \in \mathcal{E}^\delta$ and $A_2 \in \mathcal{E}^\delta$ with $A_1 \subseteq A_2$ be given together with a positive $\varepsilon$. Choose $E_1 \in \mathcal{E}$ with $E_1 \supseteq A_1$ such that $\lambda(E_1) \leq \lambda^\delta(A_1) + \varepsilon$. Let $E_2$ be any set in $\mathcal{E}$ which contains $E_1 \cup A_2$. Choose $F \in \mathcal{E}$ with $F \subseteq E_2 \setminus E_1$ such that $\lambda F + \lambda(E_1) \geq \lambda(E_2) - \varepsilon$. By modularity of $\lambda^\delta$,

$$\begin{aligned} \lambda^\delta(A_2 \cap f) &= \lambda^\delta(A_2) + \lambda^\delta(F) - \lambda^\delta(A_2 \cup F) \geq \lambda^\delta(A_2) + \lambda(F) - \lambda(E_2) \\ &\geq \lambda^\delta(A_2) - \lambda(E_1) - \varepsilon \geq \lambda^\delta(A_2) - \lambda^\delta(A_1) - 2\varepsilon, \end{aligned}$$

and the desired tightness property is verifyed.