Norm maps in Milnor K-theory

Lars Hesselholt

The purpose of this note is to give a detailed exposition of the construction of norm maps in Milnor K-theory following the original papers of Bass and Tate [**1**] and Kato [**4**]. Needless to say that we make no claim of originality.

The Milnor K-theory of a field k is defined to be the graded ring

$$K_*^M(k) = T_{\mathbb{Z}}(k^*)/(x \otimes (1-x) \mid x \in k \smallsetminus \{0,1\})$$

and the class of $x_1 \otimes \cdots \otimes x_n$ is denoted by $\{x_1, \dots, x_n\}$ and called a symbol. We derive some immediate consequences of the relation that $\{x, 1-x\} = 0$. First, since we can write $-x = (1-x)/(1-x^{-1})$, we have

$$\{x, -x\} = \{x, 1-x\} + \{x^{-1}, 1-x^{-1}\} = 0.$$

This shows

$$\{x,y\} + \{y,x\} = \{x,-x\} + \{x,y\} + \{y,x\} + \{y,-y\} = \{xy, -xy\} = 0$$

so the Milnor ring is anti-symmetric. However, we have

$$\{x,x\} = \{x, -(-x)\} = \{x,-1\} + \{x,-x\} = \{x,-1\},$$

which is generally non-zero, so the Milnor ring is generally not alternating.

Proposition 1. *Let K be a field, and let v be a normalized discrete valuation on K. Let $\mathcal{O}_v \subset K$ be the valuation ring, let $\mathfrak{m}_v \subset \mathcal{O}_v$ be the maximal ideal, and let $k(v) = \mathcal{O}_v/\mathfrak{m}_v$ be the residue field. Then there is a unique homomorphism*

$$\partial_v \colon K_n^M(K) \to K_{n-1}^M(k(v))$$

such that for all $u_1, \dots, u_{n-1} \in \mathcal{O}_v^*$ and $x \in K^*$,

$$\partial_v(\{u_1, \dots, u_{n-1}, x\}) = v(f)\{\bar{u}_1, \dots, \bar{u}_{n-1}\},$$

where $\bar{u}_i$ is the class of u_i in $k(v)^*$.

Proof. The uniqueness is clear since the symbols $\{u_1, \dots, u_{n-1}, x\}$ generate $K_n^M(K)$ as a abelian group. To prove the existence, we choose a generator $\pi \in \mathfrak{m}_v$ and show that there is map of graded rings

$$\theta_\pi \colon K_*^M(K) \to K_*^M(k(v))[\varepsilon]/(\varepsilon^2 - \{-1\}\varepsilon)$$

that to $\{\pi^i u\}$ with $u \in \mathcal{O}_v^*$ assigns $\{\bar{u}\} + i\varepsilon$. An easy calculation then shows that the homomorphism $\partial_v \colon K_n^M(K) \to K_{n-1}^M(k(v))$ defined by the formula

$$\theta_\pi(z) = \psi_\pi(z) + \partial_v(z)\varepsilon$$

maps $\{u_1, \ldots, u_{n-1}, x\}$ to $v(x)\{\bar{u}_1, \ldots, \bar{u}_{n-1}\}$ as desired. We have

$$\theta_\pi(\{\pi^{i_1}u_i, \pi^{i_2}u_2\}) = \{\bar{u}_1, \bar{u}_2\} + (i_2\{\bar{u}_1\} - i_1\{\bar{u}_2\} + i_1 i_2\{-1\})\varepsilon$$

and must show this expression is zero whenever $\pi^{i_1}u_i + \pi^{i_2}u_2 = 1$. There are four cases to consider. If $i_1 > 0$, then $i_2 = 0$ and $\bar{u}_1 = 1$. So $\theta_\pi(\{\pi^{i_1}u_1, \pi^{i_2}u_2\}) = 0$. Similarly, if $i_1 = 0$, and $i_2 > 0$, we have $\theta_\pi(\pi^{i_1}u_1, \pi^{i_2}u_2\}) = 0$. If $i_1 = i_2 = 0$, then $\bar{u}_1 + \bar{u}_2 = 1$, so $\theta_\pi(\{\pi^{i_1}u_1, \pi^{i_2}u_2\}) = 0$. Finally, if $i_1 < 0$, then $i_2 = i_1$ and $\bar{u}_1 + \bar{u}_2 = 0$. In this case, we have

$$\begin{aligned}
\theta_\pi(\{\pi^{i_1}u_1, \pi^{i_2}u_2\}) &= \{\bar{u}_i, -\bar{u}_1\} + (i_1\{\bar{u}_1\} - i_1\{-\bar{u}_1\} + i_1^2\{-1\})\varepsilon \\
&= 0 + (i_1\{\bar{u}_1\} + i_1\{-1\} - i_1\{\bar{u}_1\} + i_1^2\{-1\})\varepsilon \\
&= i_1(i_1+1)\{-1\}\varepsilon
\end{aligned}$$

which is zero, since $i_1(i_1+1)$ is even. This proves the claim. It is now an easy calculation to see that the map ∂_v given by the formula

$$\theta_\pi(x) = \psi_\pi(x) + \partial_v(x)\varepsilon$$

is given by the stated formula and, in particular, is independent of the choice of generator $\pi \in \mathfrak{m}_v$. □

By definition, $\partial_v \colon K_1^M(K) \to K_0^M(k(v))$ takes $\{x\}$ to $v(f)$. It is also not difficult to see that $\partial_v \colon K_2^M(K) \to K_1(k(v))$ takes $\{x, y\}$ to $\{(x, y)_v\}$, where

$$(x, y)_v = (-1)^{v(x)v(y)} y^{v(x)} x^{-v(y)}$$

is the tame symbol.

Lemma 2. *Let K be a field, and let v be a discrete valuation on K. Let L be a finite extension field K, and let w be a discrete valuation on L that extends v. Suppose that $\mathfrak{m}_v\mathcal{O}_w = \mathfrak{m}_w^{e_{w/v}}$. Then the following diagram commutes:*

$$\begin{array}{ccc}
K_n^M(L) & \xrightarrow{\partial_w} & K_{n-1}^M(k(w)) \\
\uparrow{\scriptstyle j_{L/K*}} & & \uparrow{\scriptstyle e_{w/v} j_{k(w)/k(v)*}} \\
K_n^M(K) & \xrightarrow{\partial_v} & K_{n-1}^M(k(v)).
\end{array}$$

Proof. Indeed, if $u_1, \ldots, u_{n-1} \in \mathcal{O}_v^*$ and $x \in K^*$, then

$$\partial_w(\{u_1, \ldots, u_{n-1}, x\}) = w(x)\{\bar{u}_1, \ldots, \bar{u}_{n-1}\} = e_{w/v}v(x)\{\bar{u}_1, \ldots, \bar{u}_{n-1}\}$$

as stated. □

We shall now state the theorem of Kato that characterizes the norm homomorphisms associated with a finite field extension; the proof occupies the rest of this note. Let $k(t)$ be the field of rational functions in one variable over a field k. Then

$$v_\infty(f) = -\deg(f)$$

is a discrete valuation on $k(t)$ that is trivial on k and for which t^{-1} is a generator of $\mathfrak{m}_{v_\infty}$. Every other discrete valution v on $k(t)$ that is trivial on k determines and is determined by a monic irreducible polynomial $\pi_v \in k[t]$ that is a generator of $\mathfrak{m}_v$, and the residue field $k(v)$ is $k[t]/(\pi_v)$.

THEOREM 3. *There exists a unique family of natural homomorphisms*

$$N_{k'/k}\colon K_n^M(k') \to K_n^M(k)$$

associated with finite field extensions k'/k such that $N_{k/k} = \mathrm{id}$ and such that the reciprocity formula holds: Let $k(t)$ be the field of rational functions in one variable over a field k. Then, for all $x \in K_*^M(k(t))$, the sum $\sum_v N_{k(v)/k}(\partial_v(x))$ that ranges over all discrete valuations v of $k(t)$ that are trivial on k is equal to zero.

REMARK 4. We first note that for $n = 0$, we must define $N_{k'/k}$ to be multiplication by the index $[k' : k]$. Indeed, this is the statement that for every $f \in k(t)^*$,

$$\sum_v [k(v) : k] v(f) = 0.$$

To see this, we recall that $k[t]$ is a unique factorization domain with quotient field $k(t)$. Hence, for every $f \in k(t)^*$, we have

$$f = \mathrm{lead}(f) \prod_{v \neq v_\infty} \pi_v^{v(f)}$$

where $\mathrm{lead}(f) \in k$ is the leading coefficient of f. Hence,

$$\sum_{v \neq v_\infty} [k(v) : k] \cdot v(f) = \sum_{v \neq v_\infty} \deg(\pi_v) \cdot v(f) = \deg(f),$$

and since $v_\infty(f) = -\deg(f)$, the statement follows.

We also note that for $n = 1$, we must define $N_{k'/k}(\{x\}) = \{N_{k'/k}(x)\}$ where on the right-hand side $N_{k'/k}$ is the usual norm that to $x \in k'^*$ assigns the determinant of the endomorphism of the k-vector space k' that is given by multiplication by x. Indeed, if v is a valuation on $k(t)$ that is trivial on k, then

$$\partial_v(\{f, g\}) = \{(f, g)_v\},$$

where $(f, g)_v$ is the tame symbol, and hence, the statement is equivalent to the Weil reciprocity formula

$$\prod_v N_{k(v)/k}((f, g)_v) = 1.$$

A proof is given in [**1**, Thm. 5.6].

We now begin the construction of the norm maps in general following Bass and Tate [**1**]. The starting point is the following theorem of Milnor and Tate.

THEOREM 5. *There is an exact sequence of graded $K_*^M(k)$-modules*

$$0 \to K_*^M(k) \xrightarrow{j_{k(t)/k*}} K_*^M(k(t)) \xrightarrow{(\partial_v)} \bigoplus_{v \neq v_\infty} K_{*-1}^M(k(v)) \to 0$$

where, on the right-hand side, the sum ranges over all discrete valuations v on $k(t)$ that are trivial on k and that are different from v_∞.

PROOF. We first note that the map

$$\psi_{t^{-1}}\colon K_*^M(k(t)) \to K_*^M(k)$$

that takes $\{f_1, \dots, f_r\}$ to $\{\mathrm{lead}(f_1), \dots, \mathrm{lead}(f_r)\}$ defines a retraction of the left-hand map of the statement. Now, let d be a non-negative integer, and let

$$\mathrm{Fil}_d\, K_*^M(k(t)) \subset K_*^M(k(t))$$

be the subring generated by the symbols $\{f\} \in K_1^M(k(t))$ such that $f \in k[t] \cap k(t)^*$ and $\deg(f) \leqslant d$. The subring $\mathrm{Fil}_0 K_*^M(k(t))$ is identified with the image of the map

$$j_{k(t)/k*} \colon K_*^M(k) \to K_*^M(k(t)),$$

which is split injective. We claim that, for d positive, $\mathrm{Fil}_d K_*^M(k(t))$ is generated as a left $K_*^M(k)$-module by the symbols $\{\pi_1, \dots, \pi_r\}$, where $\pi_1, \dots, \pi_r$ are monic irreducible polynomials and $0 < \deg(\pi_1) < \cdots < \deg(\pi_r) \leqslant d$. Granting this for the moment, we see that the maps ∂_v induce an isomorphism

$$\mathrm{gr}_d K_*^M(k(t)) \xrightarrow{\sim} \bigoplus K_*^M(k(v))$$

onto the sum the $K_*^M(k(v))$ such that $v \neq v_\infty$ and such that $[k(v) : k] = d$. Indeed, if $x = \{\pi_1, \dots, \pi_r\}$, where $\pi_1, \dots, \pi_r \in k[t]$ are monic irreducible polynomials and $0 < \deg(\pi_1) < \cdots < \deg(\pi_r) \leqslant d$, then $\partial_v(x)$ is non-zero if and only if $\pi_r = \pi_v$, and in this case, $\partial_v(x) = \{\bar{\pi}_1, \dots, \bar{\pi}_{r-1}\}$.

We prove the claim by induction on d starting from the case $d = 1$ which is trivial. To prove the induction step, it suffices to show that if $\pi, \pi' \in k[t]$ are two irreducible monic polynomials of degree d, then

$$\mathrm{Fil}_{d-1} K_*^M(k(t)) \cdot \{\pi, \pi'\} \subset \mathrm{Fil}_{d-1} K_*^M(k(t)) \cdot \{\pi\} + \mathrm{Fil}_{d-1} K_*^M(k(t)) \cdot \{\pi'\}.$$

To this end, we write $\pi = \pi' + f$ where $f \in k[t]$ and $\deg(f) < d$. If $f = 0$, then we have $\{\pi, \pi'\} = \{\pi, \pi\} = \{-1, \pi\}$. And if $f \neq 0$, then $(\pi'/\pi) + (f/\pi) = 1$, so

$$(\{f\} - \{\pi\})(\{\pi'\} - \{\pi\}) = \{\frac{f}{\pi}, \frac{\pi'}{\pi}\} = 0,$$

and hence,

$$\{\pi, \pi'\} = \{f, \pi'\} - \{f, \pi\} + \{-1, \pi\}.$$

This completes the proof. □

Addendum 6. *Let k be a field with the property that the degree of every finite extension of k is a power of a fixed prime p, and let k' be a finite extension of k of degree p. Then $K_n^M(k')$ is generated by symbols of the form $\{x, y_2, \dots, y_n\}$ where $x \in k'^*$ and $y_2, \dots, y_n \in k^*$.*

Proof. In general, an extension k'/k is generated by a single element $a \in k'$ if and only if the set of intermediate extensions $k \subset L \subset k'$ is finite. In the case at hand, there are no non-trivial intermediate extensions, since $[k' : k]$ is a prime, and hence $k' = k(a)$, for some $a \in k'$. Let π be the minimal polynomial of a, and let v be the discrete valuation on $k(t)/k$ with $\pi_v = \pi$. Hence, the proof of Thm. 5 shows that, as a $K_*^M(k)$-module, $K_*^M(k')$ is generated by symbols of the form $\{\pi_1(a), \dots, \pi_r(a)\}$, where $\pi_1, \dots, \pi_{r-1} \in k[t]$ are irreducible monic polynomials and $0 < \deg(\pi_1) < \cdots < \deg(\pi_{r-1}) < p$. Since there are no finite extensions of k of degree prime to p, we have $r-1 = 1$ and $\deg(\pi_{r-1}) = 1$. The statement follows. □

It follows from Thm. 5 that there are unique homomorphisms

$$N_v \colon K_{n-1}^M(k(v)) \to K_{n-1}^M(k)$$

such that $N_{v_\infty} = \mathrm{id}$ and such that the composite map

$$K_n^M(k(t)) \xrightarrow{(\partial_v)} \bigoplus_v K_{n-1}^M(k(v)) \xrightarrow{\sum N_v} K_{n-1}^M(k)$$

is equal to zero.

Definition 7. Let k be a field, and let $k' = k(a)$ be a finite simple extension with minimal polynomial π. Let v be the unique discrete valuation on $k(t)$ such that $\mathfrak{m}_v \subset k[t]$ is generated by π, and let $j_{k'/k(v)} \colon k(v) \to k'$ be the isomorphism that maps the class of t to a. Then the norm map

$$N_{a/k} \colon K_n^M(k') \to K_n^M(k)$$

is defined to be the composition of $j_{k'/k(v)*}^{-1}$ and N_v.

Lemma 8 (Projection formula). *Let k be a field, and let $k' = k(a)$ be a finite simple extension. Then for all $x \in K_*^M(k')$ and $y \in K_*^M(k)$,*

$$N_{a/k}(x \cdot j_{k'/k*}(y)) = N_{a/k}(x) \cdot y.$$

In particular, the composite $N_{a/k} \circ j_{k'/k*}$ is multiplication by $[k' : k']$.

Proof. The projection formula is a reformulation of the fact that the norm maps N_v are $K_*^M(k)$-linear. The projection formula shows in particular that the composite $N_{a/k} \circ j_{k'/k*}$ is multiplication by $N_{a/k}(1) \in K_0^M(k)$, and Rem. 4 shows that $N_{a/k}(1) = [k' : k]$. $\square$

Corollary 9. *If $k' = k(a) = k$, then $N_{a/k}$ is the identity map.*

Proof. Indeed, the map $j_{k/k*}$ and the composite $N_{a/k} \circ j_{k/k*}$ both are the identity map of $K_*^N(k)$. $\square$

We use the projection formula to prove the following result. I thank Tyler Lawson for help with the proof.

Lemma 10. *Let k be a field, and let p be a prime. Then there exists an algebraic extension L of k such that every finite extension of L has order a power of p and such that the map $j_{L/k*} \colon K_n^M(k)_{(p)} \to K_n^M(L)_{(p)}$ is injective.*

Proof. We let k^a be an algebraic closure of k and consider the partially ordered set S defined as follows. An element of S is a pair $(\alpha, \{L_\beta \mid \beta \leqslant \alpha\})$ of an ordinal α and, for every ordinal $\beta \leqslant \alpha$, an extension field $k \subset L_\beta \subset k^a$ such that $L_0 = k$, such that for every $\beta < \alpha$, $L_{\beta+1}$ is a non-trivial finite extension of L_β of degree prime to p, and such that for every limit ordinal $\gamma \leqslant \alpha$, L_γ is the union of the fields L_β, where $\beta < \gamma$. Since the cardinality of the ordinal α is necessarily less than or equal to the cardinality of k^a, S is indeed a set. We define

$$(\alpha, \{L_\beta \mid \beta \leqslant \alpha\}) \leqslant (\alpha', \{L'_{\beta'} \mid \beta' \leqslant \alpha'\})$$

to mean that $\alpha \leqslant \alpha'$ and that, for all $\beta \leqslant \alpha$, $L_\beta = L'_\beta$. The set S is non-empty since $(0, \{k\})$ is an element. We use Zorn's lemma to show that S has a maximal element. We must show that every non-empty totally ordered subset

$$T = \{(\alpha(i), \{L_\beta(i) \mid \beta \leqslant \alpha(i)\}) \mid i \in I\} \subset S$$

has an upper bound $(\alpha, \{L_\beta \mid \beta \leqslant \alpha\})$. We define α to be the smallest ordinal such that, for all $i \in I$, $\alpha(i) \leqslant \alpha$, and we define L_β, for $\beta \leqslant \alpha$, to be the union of all $L_{\beta(i)}$ with $\beta(i) \leqslant \beta$. Then $(\alpha, \{L_\beta \mid \beta \leqslant \alpha\})$ is an upper bound of T in S. By Zorn's lemma, the partially ordered set S has a maximal element $(\alpha, \{L_\beta \mid \beta \leqslant \alpha\})$. It is then clear that the field $L = L_\alpha$ does not have any finite extensions $L \subset L' \subset k^a$ of degree prime to p. We show by transfinite induction that the map

$$j_{L/k*}\colon K_n^M(k)_{(p)} \to K_n^M(L)_{(p)}$$

is injective. The composite

$$K_n^M(L_\beta)_{(p)} \xrightarrow{j_{L_{\beta+1}/L_\beta *}} K_n^M(L_{\beta+1})_{(p)} \xrightarrow{N_{L_{\beta+1}/L_\beta}} K_n^M(L_\beta)_{(p)}$$

is given by multiplication by the index $[L_{\beta+1}, L_\beta]$, and therefore, is an isomorphism. Hence, the left-hand map is injective. If $\gamma \leqslant \alpha$ is a limit ordinal, then the map

$$\operatorname*{colim}_{\beta<\gamma} K_n^M(L_\beta)_{(p)} \to K_n^M(L_\gamma)_{(p)}$$

induced by the maps $j_{L_\gamma/L_\beta *}$ is an isomorphism, and the canonical map

$$K_n^M(L_\beta)_{(p)} \to \operatorname*{colim}_{\beta<\gamma} K_n^M(L_\beta)_{(p)}$$

is injective since the limit system is filtered and since the structure maps in the limit system are injective. □

LEMMA 11. *Let $k' = k(a)$ be a finite extension of k, and let $\pi \in k[t]$ be the minimal polynomial of a. Let L be an extension of the field k, and let*

$$\pi = \prod_i \pi_i^{e_i}$$

be the decomposition of π into a product of irreducible monic polynomials in $L[t]$. Let $L_i' = L[t]/(\pi_i)$, let $a_i \in L_i'$ be the class of t, and and let $j_{L_i'/k}$ be the embedding of k in L_i' that maps a to a_i and that maps k to L by the embedding $j_{L/k}$. Then the following diagram commutes:

$$\begin{array}{ccc} K_n^M(k') & \xrightarrow{(e_i j_{L_i'/k*})} & \bigoplus_i K_n^M(L_i') \\ \big\downarrow N_{a/k} & & \big\downarrow \sum N_{a_i/L} \\ K_n^M(k) & \xrightarrow{j_{L/k*}} & K_n^M(L). \end{array}$$

PROOF. Let v be a discrete valuation on $k(t)$ that is trivial on k, and let w range over all extension of v to a valuation on $L(t)$ that is trivial on L. If $v = v_\infty$, then $w = w_\infty$ is the only extension, and t^{-1} is a uniformizer for both v_∞ and w_∞. If $v \neq v_\infty$, then the the monic irreducible polynomial π_v that generates the kernel of the canonical projection $k[t] \to k(v)$ decomposes in $L[t]$ as a product of monic irreducible polynomials

$$\pi_v = \prod_{w/v} \pi_w^{e_{w/v}}.$$

The map $k[t]/(\pi_v) \to L[t]/(\pi_w)$ that maps t to t and k to L by the embedding $j_{L/k}$ defines an embedding $j_{k(w)/k(v)}$ of $k(v)$ in $k(w)$. We prove that the following diagram commutes:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
K_n^M(k) & \xrightarrow{j_{L/k*}} & K_n^M(L) \\
\downarrow & & \downarrow \\
K_n^M(k(t)) & \xrightarrow{j_{L(t)/k(t)*}} & K_n^M(L(t)) \\
\downarrow{\scriptstyle (\partial_v)} & & \downarrow{\scriptstyle (\partial_w)} \\
\bigoplus_v K_{n-1}^M(k(v)) & \xrightarrow{\bigoplus_v (e_{w/v} j_{k(w)/k(v)*})} & \bigoplus_v \bigoplus_{w/v} K_{n-1}^M(k(w)) \\
\downarrow{\scriptstyle \sum N_v} & & \downarrow{\scriptstyle \sum N_w} \\
K_{n-1}^M(k) & \xrightarrow{j_{L/k*}} & K_{n-1}^M(L) \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}
$$

The top square commutes since Milnor K-theory is a functor, and one immediately verifies from the definitions that the middle square commutes. Since the columns in the diagram are exact, it follows that there exists a unique map

$$h\colon K_{n-1}^M(k) \to K_{n-1}^M(L)$$

that makes the lower square commutes. We must show that $h = j_{L/k*}$. In particular, the following square commutes:

$$
\begin{array}{ccc}
K_{n-1}^M(k(v_\infty)) & \xrightarrow{j_{k(w_\infty)/k(v_\infty)*}} & K_{n-1}^M(k(w_\infty)) \\
\downarrow{\scriptstyle N_{v_\infty}} & & \downarrow{\scriptstyle N_{w_\infty}} \\
K_{n-1}^M(k) & \xrightarrow{h} & K_{n-1}^M(L).
\end{array}
$$

But $j_{k(w_\infty)/k(v_\infty)} = j_{L/k}$ and N_{v_∞} and N_{w_∞} are the respective identity maps. Hence, the diagram commutes as stated. The lemma follows. □

Corollary 12. *Let k be a field, and let $k' = k(a)$ be a finite extension.*

(i) If k'/k is Galois, then $j_{k'/k*} \circ N_{a/k} = \sum_{\sigma \in G_{k'/k}} \sigma_*$.

(ii) If k'/k is purely inseparable, then $j_{k'/k*} \circ N_{a/k} = [k' : k]$.

Proof. If k'/k is Galois, then the minimal polynomial $\pi \in k[t]$ of $a \in k'$ decomposes in $k'[t]$ as the product

$$\pi = \prod_{\sigma \in G_{k'/k}} (t - \sigma(a)).$$

Hence, Lemma 11 gives a commutative square

$$\begin{array}{ccc} K_n^M(k') & \xrightarrow{(\sigma_* \circ j_{k'/k*})} & \bigoplus_{\sigma \in G_{k'/k}} K_n^M(k') \\ \Big\downarrow{\scriptstyle N_{a/k}} & & \Big\downarrow{\scriptstyle \sum N_{\sigma(a)/k'}} \\ K_n^M(k) & \xrightarrow{j_{k'/k*}} & K_n^M(k'). \end{array}$$

Since $j_{k'/k'}$ is the identity map of k', the projection formula shows that $N_{\sigma(a)/k'}$ is the identity map of $K_n^M(k')$. The statement (i) follows. Similarly, if k'/k is a purely inseparable extension, then k and k' are both of positive characteristic p and $[k' : k] = p^r$, for some $r \geqslant 0$. The minimal polynomial $\pi \in k[t]$ of $a \in k'$ is $\pi = t^{p^r} - \alpha$, where $\alpha = a^{p^r} \in k$. It decomposes as the product

$$\pi = t^{p^r} - \alpha = t^{p^r} - a^{p^r} = (t - a)^{p^r}$$

in $k'[t]$. Hence, Lemma 11 gives a commutative square

$$\begin{array}{ccc} K_n^M(k') & \xrightarrow{p^r} & K_n^M(k') \\ \Big\downarrow{\scriptstyle N_{a/k}} & & \Big\downarrow{\scriptstyle N_{a/k'}} \\ K_n^M(k) & \xrightarrow{j_{k'/k*}} & K_n^M(k'). \end{array}$$

Again, $N_{a/k'}$ is the identity map, and the statement (ii) follows. □

Proposition 13. *Let k be a field, let k' be a finite extension of prime degree, and let $a \in k'$ be an element such that $k' = k(a)$. Then the map*

$$N_{a/k} \colon K_n^M(k') \to K_n^M(k)$$

is independent of the choice of generator $a \in k'$.

Proof. Since $[k' : k] = p$ is a prime, there exists $a \in k'$ such that $k' = k(a)$. Suppose first that, for some prime p, all finite extensions of k have degree a power of p. Then Addendum 6 shows that the abelian group $K_n^M(k')$ is generated by symbols of the form $\{x, y_2, \dots, y_n\}$, where $x \in k'^*$ and $y_2, \dots, y_n \in k^*$, and the projection formula and the Weil reciprocity formula show that

$$N_{a/L}(\{x, y_2, \dots, y_n\}) = \{N_{L'/L}(x), y_2, \dots, y_n\}.$$

It follows that, in this case, the norm map $N_{k'/k} = N_{a/k}$ is independent of the choice of generator $a \in k'$.

Let k be any field. It will suffice to show that, for every prime p, the map

$$N_{a/k} \colon K_n^M(k')_{(p)} \to K_n^M(k)_{(p)}$$

does not depend on a. By Lemma 10, there exists an extension L of k such that every finite extension of L has order a power of p and such that the map

$$j_{L/k*} \colon K_n^M(k)_{(p)} \to K_n^M(L)_{(p)}$$

is injective. Hence, it suffices to show that the composite map

$$K_n^M(k')_{(p)} \xrightarrow{N_{a/k}} K_n^M(k)_{(p)} \xrightarrow{j_{L/k*}} K_n^M(L)_{(p)}$$

is independent of a. Since $[k' : k]$ is a prime, the extension k'/k is either separable or purely inseparable.

Suppose first that k' is separable over k. Then the ring $L \otimes_k k'$ is a product of fields, and since $[k' : k]$ is a prime, this ring is either a field L' or product of copies of L. If $L \otimes_k k' = L'$ is a field, then $[k' : k] = p$, since otherwise, L'/L would be a finite extension of degree prime to p. By Lemma 11, there is a commutative diagram

$$\begin{array}{ccc} K_n^M(k') & \xrightarrow{j_{L'/k'*}} & K_n^M(L') \\ \downarrow{\scriptstyle N_{a/k}} & & \downarrow{\scriptstyle N_{L'/L}} \\ K_n^M(k) & \xrightarrow{j_{L/k*}} & K_n^M(L), \end{array}$$

and hence, the composite $j_{L/k*} \circ N_{a/k}$ is independent of a. If $L \otimes_k k'$ is a product of copies of L, then Lemma 11 gives a commutative diagram

$$\begin{array}{ccc} K_n^M(k') & \xrightarrow{(\sigma_*)} & \bigoplus_\sigma K_n^M(L) \\ \downarrow{\scriptstyle N_{a/k}} & & \downarrow{\scriptstyle \sum N_{L/L}} \\ K_n^M(k) & \xrightarrow{j_{L/k*}} & K_n^M(L), \end{array}$$

where the sum in the upper right-hand term ranges over the possible embeddings σ of k' in L [**3**, Chap. V, §2, Prop. 4]. So $j_{L/k*} \circ N_{a/k}$ is independent of a.

Suppose next that k' is a purely inseparable extension of k. Then k has positive characteristic ℓ, and $k' = k[t]/(t^\ell - a)$. If $a \notin L^\ell$, then $L \otimes_k k' = L'$ is a purely inseparable extension of L of degree ℓ, and if $a \in L^\ell$, then $L \otimes_k k'$ is a product of copies of L. In the former case, Lemma 11 gives a commutative diagram

$$\begin{array}{ccc} K_n^M(k') & \xrightarrow{j_{L'/k'*}} & K_n^M(L') \\ \downarrow{\scriptstyle N_{a/k}} & & \downarrow{\scriptstyle N_{L'/L}} \\ K_n^M(k) & \xrightarrow{j_{L/k*}} & K_n^M(L) \end{array}$$

which shows that $j_{L/k*} \circ N_{a/k}$ is independent of a. In the latter case, we have

$$L \otimes_k k' = L[t]/(t^\ell - a) = L[t]/((t - \alpha)^\ell),$$

where $\alpha \in L$ is the unique ℓth root of a. Hence, in this case, Lemma 11 gives a commutative diagram

$$\begin{array}{ccc} K_n^M(k') & \xrightarrow{\ell \cdot j_{L/k'*}} & K_n^M(L) \\ \downarrow{\scriptstyle N_{a/k}} & & \downarrow{\scriptstyle N_{L/L}} \\ K_n^M(k) & \xrightarrow{j_{L/k*}} & K_n^M(L) \end{array}$$

which shows that, also in this case, $j_{L/k*} \circ N_{a/k}$ is independent of a. This completes the proof. □

It follows from Prop. 13 that we have well-defined norm map

$$N_{k'/k} = N_{a/k} \colon K_n^M(k') \to K_n^M(k),$$

for every finite extension $k' = k(a)$ of k of prime degree. Before we state the next result, we recall from [**5**, Chap. II, §2] that, if K is a complete discrete valution field, and if L is a finite extension of K, then the discrete valuation on K extends uniquely to a discrete valuation on L and L is complete with respect to this valuation. Moreover, if k_L/k_K is the extension of residue fields, then

$$[L:K] = e_{L/K} \cdot [k_L : k_K],$$

where $e_{L/K}$ is the ramification index defined by $\mathfrak{m}_K \mathcal{O}_L = \mathfrak{m}_L^{e_{L/K}}$.

LEMMA 14. *Let K be a complete discrete valuation field, and let K' be a finite normal extension of K of prime degree. Let k and k' be the residue fields of K and K', respectively. Then the following diagram commutes:*

$$\begin{array}{ccc} K_n^M(K') & \xrightarrow{\partial_{K'}} & K_{n-1}^M(k') \\ \Big\downarrow{\scriptstyle N_{K'/K}} & & \Big\downarrow{\scriptstyle N_{k'/k}} \\ K_n^M(K) & \xrightarrow{\partial_K} & K_{n-1}^M(k). \end{array}$$

PROOF. We wish to show that the map

$$\delta_{K'/K} = \partial_K \circ N_{K'/K} - N_{k'/k} \circ \partial_{K'}$$

is equal to zero. We first show that $p\delta_{K'/K}$ is zero. We consider several cases. Suppose first that K'/K is unramified. If K'/K is separable, then the extesion k'/k is normal [**5**, Chap. I, §7, Prop. 20]. If, in addition, k'/k is separable, then the Galois groups $G_{K'/K}$ and $G_{k'/k}$ are canonically isomorphic, and we have

$$\begin{aligned} j_{k'/k*}(\delta_{K'/K}(z)) &= j_{k'/k*}(\partial_K(N_{K'/K}(z))) - j_{k'/k*}(N_{k'/k}(\partial_{K'}(z))) \\ &= \partial_{K'}(j_{K'/K*}(N_{K'/K}(z))) - j_{k'/k*}(N_{k'/k}(\partial_{K'}(z))) \\ &= \sum_{\sigma \in G_{K'/K}} \partial_{K'}(\sigma_*(z)) - \sum_{\bar{\sigma} \in G_{k'/k}} \bar{\sigma}_*(\partial_{K'}(z)) \end{aligned}$$

which is equal to zero, since $\partial_{K'}$ is a natural homomorphism. Here the second equality uses Lemma 2, and the third equality uses Cor. 12(i). If, instead, k'/k is a purely inseparable extension, we find as above that

$$j_{k'/k*}(\delta_{K'/K}(z)) = \sum_{\sigma \in G_{K'/K}} \partial_{K'}(\sigma_*(z)) - p\partial_{K'}(z).$$

But $\sigma \in G_{K'/K}$ induces the identity map of k' [**3**, Chap. V, §6, Prop. 3], so this expression is equal to zero. If K'/K is purely inserable, then k'/k is also purely inseparable [**5**, Chap. I, §6, Prop. 16], so

$$j_{k'/k*}(\delta_{K'/K}(z)) = \partial_{K'}(pz) - p\partial_{K'}(z) = 0.$$

Suppose next that K'/K is totally ramified. If K'/K is Galois, then

$$\begin{aligned} p\delta_{K'/K}(z) &= p\partial_K(N_{K'/K}(z)) - p\partial_{K'}(z) \\ &= \partial_{K'}(j_{K'/K*}(N_{K'/K}(z))) - p\partial_{K'}(z) \\ &= \sum_{\sigma \in G_{K'/K}} \partial_{K'}(\sigma_*(z)) - p\partial_{K'}(z) \end{aligned}$$

which is zero, since $G_{K'/K}$ acts trivially on $k' = k$. Finally, if K'/K is purely inseparable, then

$$\begin{aligned} p\delta_{K'/K}(z) &= \partial_{K'}(j_{K'/K*}(N_{K'/K}(z))) - p\partial_{K'}(z) \\ &= \partial_{K'}(pz) - p\partial_{K'}(z) \end{aligned}$$

which again is zero. We conclude from the above that $p\delta_{K'/K}(z)$ is zero. Therefore, if suffices to show that, for every $z \in K_n^M(K')$, there exists an integer m prime to p such that $m\delta_{K'/K}(z)$ is zero.

Suppose that L is an extension of K of degree prime to p, and let L' be the compositum of L and K' in an algebraic closure of K. Since K'/K is a normal extension whose degree is prime to the degree of the extension L/K, the extension L'/L is again normal of degree p. By Lemma 11, the diagram

$$\begin{array}{ccc} K_n^M(K') & \xrightarrow{\;j_{L'/K'*}\;} & K_n^M(L') \\ \big\downarrow{\scriptstyle N_{K'/K}} & & \big\downarrow{\scriptstyle N_{L'/L}} \\ K_n^M(K) & \xrightarrow{\;j_{L/K*}\;} & K_n^M(L) \end{array}$$

commutes. Moreover, the discussion before the statement shows that the ramification indices $e_{L'/L}$ and $e_{K'/K}$ are equal and that the canonical map from $k' \otimes_k k_L$ to $k_{L'}$ is an isomorphism. Therefore, we conclude from Lemma 11 that there is a commutative diagram

$$\begin{array}{ccc} K_n^M(k') & \xrightarrow{\;j_{k_{L'}/k'*}\;} & K_n^M(k_{L'}) \\ \big\downarrow{\scriptstyle N_{k'/k}} & & \big\downarrow{\scriptstyle N_{k_{L'}/k_L}} \\ K_n^M(k) & \xrightarrow{\;j_{k_L/k*}\;} & K_n^M(k_L), \end{array}$$

and from Lemma 2 that there is a pair of commutative diagrams

$$\begin{array}{ccc} K_n^M(L) & \xrightarrow{\;\partial_L\;} & K_n^M(k_L) \\ \big\uparrow{\scriptstyle j_{L/K*}} & & \big\uparrow{\scriptstyle e_{L/K}j_{k_L/k*}} \\ K_n^M(K) & \xrightarrow{\;\partial_K\;} & K_n^M(k) \end{array} \qquad \begin{array}{ccc} K_n^M(L') & \xrightarrow{\;\partial_{L'}\;} & K_n^M(k_{L'}) \\ \big\uparrow{\scriptstyle j_{L'/K'*}} & & \big\uparrow{\scriptstyle e_{L/K}j_{k_{L'}/k'*}} \\ K_n^M(K') & \xrightarrow{\;\partial_{K'}\;} & K_n^M(k'). \end{array}$$

We now fix an element $z \in K_n^M(K')$ and consider the difference

$$\delta_{K'/K}(z) = \partial_K(N_{K'/K}(z)) - N_{k'/k}(\partial_{K'}(z))$$

which we wish to show is zero. Then the diagrams above show that

$$e_{L/K}j_{k_L/k*}\delta_{K'/K}(z) = \delta_{L'/L}(j_{L'/K'*}(z)),$$

where $\delta_{L'/L}(w)$ is defined similarly. We claim that for a given $z \in K_n^M(K')$, there exists L/K such that the right-hand side is zero. The projection formula then shows that $[L : K] \cdot \delta_{K'/K}(z)$ is zero which complete the proof of the proposition.

It remains to prove the claim. It follows from Lemma 10 that there exists a finite extension L/K of degree prime to p such that $j_{L'/K'*}(z) \in K_n^M(L')$ is a sum of symbols of the form $\{x, y_2, \dots, y_n\}$, where $x \in L'^*$ and $y_2, \dots, y_n \in L^*$. Hence, we may assume that z is a sum of symbols $\{x, y_2, \dots, y_n\}$ with $x \in K'^*$ and $y_2, \dots, y_n \in K^*$ and show that $\delta_{K'/K}(z)$ is zero in this case. This, in turn, is proved by direct calculation. The extension K'/K is either unramified or totally ramified. We consider the latter case and leave the former to the reader.

Let $\pi_{K'}$ and π_K be uniformizers of K' and K, respectively. Then

$$\pi_{K'}^p + \theta_{K'/K}(\pi_{K'})\pi_K = 0$$

where $\theta_{K'/K}(X) \in \mathcal{O}_K[X]$ is a polynomial of degree at most $p-1$ such that $\theta_{K'/K}(0) \in \mathcal{O}_K^*$. Since $\mathcal{O}'_K$ is $\mathfrak{m}'$-adically complete, it follows that $\theta_{K'/K}(\pi_{K'}) \in \mathcal{O}_{K'}^*$ such that

$$\pi_K = -\theta_{K'/K}(\pi_{K'})^{-1}\pi_{K'}^p.$$

By using the K-basis $1, \pi_{K'}, \dots, \pi_{K'}^{p-1}$ of K', one shows that

$$N_{K'/K}(\pi_{K'}) = (-1)^p\theta_{K'/K}(0)\pi.$$

We show that, if $x \in K'^*$ and $y \in K^*$, then

$$\partial_{K'}(\{x,y\}) = \partial_K(N_{K'/K}(\{x,y\}));$$

the general case is only notationally more complicated. We write $x = \pi_{K'}^i u$ with $u \in \mathcal{O}_{K'}^*$ and $y = \pi_K^j v$ with $v \in \mathcal{O}_K^*$. Then

$$\begin{aligned}\{x,y\} &= \{\pi_{K'}^i u, \pi_K^j v\} = \{\pi_{K'}^i u, (-\theta_{K'/K}(\pi_{K'})^{-1}\pi_{K'}^p)^j v\} \\ &= ij\{\pi_{K'}, -1\} - ij\{\pi_{K'}, \theta_{K'/K}(\pi_{K'})\} + pij\{\pi_{K'}, \pi_{K'}\} + i\{\pi_{K'}, v\} \\ &\quad + j\{u,-1\} - j\{u, \theta_{K'/K}(\pi_{K'})\} + pj\{u, \pi_{K'}\} + \{u,v\},\end{aligned}$$

and hence,

$$\partial_{K'}(\{x,y\}) = (p+1)ij\{-1\} + ij\{\overline{\theta_{K'/K}(\pi_{K'})}\} - i\{\bar{v}\} + pj\{\bar{u}\}.$$

On the other hand,

$$\begin{aligned}N_{K'/K}(\{x,y\}) &= \{N_{K'/K}(x), y\} = \{((-1)^p\theta_{K'/K}(0)\pi_K)^i N_{K'/K}(u), \pi_K^j v\} \\ &= pij\{-1, \pi_K\} + ij\{\theta_{K'/K}(0), \pi\} + ij\{\pi_K, \pi_K\} + j\{N_{K'/K}(u), \pi\} \\ &\quad + pi\{-1, v\} + i\{\theta_{K'/K}(0), v\} + i\{\pi_K, v\} + \{N_{K'/K}(u), v\},\end{aligned}$$

and hence,

$$\partial_K(N_{K'/K}(\{x,y\})) = (p+1)ij\{-1\} + ij\{\overline{\theta_{K'/K}(0)}\} + j\{\overline{N_{K'/K}(u)}\} - i\{\bar{v}\}.$$

To finish the proof we must show that

$$\overline{N_{K'/K}(u)} = (\bar{u})^p.$$

But this is easily seen by using $1, \pi_{K'}, \dots, \pi_{K'}^{p-1}$ as a K-basis of K'. □

Proposition 15. *Let k be a field, and let k' be a finite normal extension of k of prime degree p. Let $F = k(a)$ be a finite extension, and suppose that $F' = k'(a)$ is a field. Then the following diagram commutes:*

$$\begin{array}{ccc} K_n^M(F') & \xrightarrow{N_{a/k'}} & K_n^M(k') \\ \downarrow{\scriptstyle N_{F'/F}} & & \downarrow{\scriptstyle N_{k'/k}} \\ K_n^M(F) & \xrightarrow{N_{a/k}} & K_n^M(k). \end{array}$$

Proof. Let v be a discrete valuation on $k(t)/k$, and let $k(t)_v$ be the completion of $k(t)$ at v. Since $k(t)_v$ is a separable extension of $k(t)$, the minimal polynomial $\Pi \in k(t)[x]$ of a generator α of $k'(t)/k(t)$ decomposes as a product

$$\Pi = \prod_{w/v} \Pi_{w/v}$$

where $\Pi_{w/v} \in k(t)_v[x]$ are distinct monic irreducible polynomials, and where w ranges over the possible extensions of v to a discrete valuation on $k'(t)/k'$. We then consider the following diagram:

$$\begin{array}{ccccc}
K^M_{n+1}(k'(t)) & \xrightarrow{(j_{k'(t)_w/k'(t)*})} & \bigoplus_{w/v} K^M_{n+1}(k'(t)_w) & \xrightarrow{\bigoplus \partial_w} & \bigoplus_{w/v} K^M_n(k(w)) \\
\downarrow N_{k'(t)/k(t)} & & \downarrow \sum N_{k'(t)_w/k(t)_v} & & \downarrow \sum N_{k(w)/k(v)} \\
K^M_{n+1}(k(t)) & \xrightarrow{j_{k(t)_v/k(t)*}} & K^M_{n+1}(k(t)_v) & \xrightarrow{\partial_v} & K^M_n(k(v)).
\end{array}$$

It follows from Lemma 2 that the left-hand square commutes and from Lemma 14 that the right-hand square commutes. Now let $\pi \in k[t]$ and $\pi' \in k'[t]$ be the minimal polynomials of a over k and k', respectively. Given $x' \in K^M_n(F')$, Thm. 5 shows that there exists $y' \in K^M_{n+1}(k'(t))$ such that $\partial_{w_{\pi'}}(y') = x'$, and such that $\partial_w(y') = 0$, if $w \neq w_{\pi'}$ and $w \neq w_\infty$. Then, by definition,

$$N_{a/k'}(x') = -\partial_{w_\infty}(y').$$

We define $x = N_{F'/F}(x')$ and $y = N_{k'(t)/k(t)}(y')$. The diagram above then shows that $\partial_{v_\pi}(y) = x$ and that $\partial_v(y) = 0$, if $v \neq v_\pi$ and $v \neq v_\infty$, such that

$$N_{a/k}(x) = -\partial_{v_\infty}(y).$$

Applying the diagram again for $v = v_\infty$ gives

$$\partial_{v_\infty}(N_{k'(t)/k(t)}(y')) = N_{k'/k}(\partial_{w_\infty}(x'))$$

which shows that $N_{a/k}(N_{F'/F}(x')) = N_{k'/k}(N_{a/k'}(x'))$ as desired. ☐

Definition 16. Let k be a field, and let $k' = k(a_1, \dots, a_r)$ be a finite field extension of k. Then the norm map

$$N_{a_1,\dots,a_r/k} \colon K^M_n(k') \to K^M_n(k)$$

is defined to be the composite map

$$N_{a_1,\dots,a_r/k} = N_{a_1/k} \circ N_{a_2/k(a_1)} \circ \cdots \circ N_{a_r/k(a_1,\dots,a_{r-1})}.$$

Proof of Thm. 3. Suppose $k' = k(a_1, \dots, a_n)$. We show that the map

$$N_{a_1,\dots,a_r/k} \colon K^M_n(k') \to K^M_n(k)$$

does not depend on the choice of generators $a_1, \dots, a_n \in k'$. The proof is by induction on the degree $d = [k' : k]$. The case $d = 1$ was proved in Cor. 9. So we assume that the statement has been proved for all finite extensions of degree strictly smaller than d and let k'/k be a finite extension of degree d. It will suffice to show that, for every prime number p, the map

$$N_{a_1,\dots,a_r/k} \colon K^M_n(k')_{(p)} \to K^M_n(k)_{(p)}$$

does not depend on the choice of generators $a_1, \dots, a_n \in k'$. By Lemma 10, there exists an extension L of k such that every finite extension of L has degree a power of p and such that the map

$$j_{L/k*} \colon K_n^M(k)_{(p)} \to K_n^M(L)_{(p)}$$

is injective. Hence, it suffices to show that the composite map

$$K_n^M(k')_{(p)} \xrightarrow{N_{a_1,\dots,a_r/k}} K_n^M(k)_{(p)} \xrightarrow{j_{L/k*}} K_n^M(L)_{(p)}$$

is independent of $a_1, \dots, a_r \in k'$.

The ring $L' = L \otimes_k k'$ is an artinian L-algebra. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the minimal prime ideals of L', let $L'_i = L'_{\mathfrak{p}_i}/\mathfrak{p}_i L'_{\mathfrak{p}_i}$ be the residue field at $\mathfrak{p}_i$, and let

$$e_i = \operatorname{length}_{L'_{\mathfrak{p}_i}}(L'_{\mathfrak{p}_i}).$$

Suppose that L' is not a field. Then $[L'_i : L] < d$, for all $1 \leqslant i \leqslant m$, and hence, we have a well-defined norm map $N_{L'_i/L} \colon K_n^M(L'_i) \to K_n^M(L)$. Moreover, iterated use of Lemma 11 shows that the following diagram commutes:

$$\begin{array}{ccc} K_n^M(k') & \xrightarrow{(e_i j_{L'_i/k'*})} & \bigoplus_{1 \leqslant i \leqslant m} K_n^M(L'_i) \\ \big\downarrow{\scriptstyle N_{a_1,\dots,a_r/k}} & & \big\downarrow{\scriptstyle \sum N_{L'_i/L}} \\ K_n^M(k) & \xrightarrow{j_{L/k*}} & K_n^M(L). \end{array}$$

Hence, the composite $j_{L/k*} \circ N_{a_1,\dots,a_r/k}$ does not depend on $a_1, \dots, a_r \in k'$ in this case. It remains to consider the case where L' is a field. We claim that, in this case, there exists a sequence of field extensions

$$L = E_0 \subset E_1 \subset \cdots \subset E_m = L'$$

such that, for all $1 \leqslant i \leqslant m$, E_i is a finite normal extension of E_{i-1} of degree p. Given this, iterated use of Prop. 15 shows that

$$N_{a_1/L} \circ \cdots \circ N_{a_r/L(a_1,\dots,a_{r-1})} = N_{E_1/E_0} \circ \cdots \circ N_{E_m/E_{m-1}},$$

and the right-hand side is independent of the choice of $a_1, \dots, a_r \in k'$. This proves the induction step and hence the theorem.

It remains to prove the claim. We first decompose the extension L'/L as a purely inseparable extension F/L followed by a separable extension L'/F. The extension F/L is simple and the minimal polynomial $\pi \in L[t]$ of a generator a takes the form

$$\pi = t^{p^s} - \alpha,$$

where $\alpha = a^{p^s} \in L$. We define E_i, $0 \leqslant i \leqslant s$, to be the subfield $L(a^{p^{s-i}})$ of F. Then E_i is a normal extension of E_{i-1} of degree p for $0 \leqslant i \leqslant s$. We next choose a finite Galois extension M of F that contains L' as a subfield. Then L' is the subfield of M fixed by the subgroup $G_{M/L'}$ of the Galois group $G_{M/F}$. Since every finite extension of L, and hence F, has degree a power of p, the Galois group $G_{M/L}$ is a finite p-group. We recall from [**2**, Chap. I, §6, Prop. 12] that every proper subgroup of a finite p-group is contained in a normal subgroup of index p. It follows that there exists a sequence of subgroups

$$G_{M/F} = G^s \supset G^{s+1} \supset \cdots \supset G^m = G_{M/L'}$$

such that G^i is a normal subgroup of G^{i-1} of index p, $s+1 \leqslant i \leqslant m$. We define E_i to be the subfield of M fixed by the subgroup $G^i \subset G_{M/L}$. Then E_i is a normal extension of E_{i-1} of degree p, for all $s+1 \leqslant i \leqslant m$, as desired. □

References

[1] H. Bass and J. Tate, *The Milnor ring of a global field*, Algebraic K-theory II (Battelle Memorial Inst., Seattle, Washington, 1972), Lecture Notes in Math., vol. 342, Springer-Verlag, New York, 1973.

[2] N. Bourbaki, *Algebra I. Chapters 1–3. Translated from the French. Reprint of the 1990 English translation.*, Elements of Mathematics, Springer-Verlag, Berlin, 1998.

[3] ———, *Algebra II. Chapters 4–7. Translated from the French. Reprint of the 1990 English translation.*, Elements of Mathematics, Springer-Verlag, Berlin, 2003.

[4] K. Kato, *A generalization of local class field theory by using K-groups. II*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 603–683.

[5] J.-P. Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York, 1979.

Massachusetts Institute of Technology, Cambridge, Massachusetts

E-mail address: larsh@math.mit.edu

Nagoya University, Nagoya, Japan

E-mail address: larsh@math.nagoya-u.ac.jp