18.022 Solution to problem set 1 (version 2)

1. 1.1.22

$$P := \{(x_1 + sa_1 + tb_1, x_2 + sa_2 + tb_2, x_3 + sa_3 + tb_3)\}$$

$$\text{for } 0 \le s \le 1 \text{ and } 0 \le t \le 1$$

2. 1.1.25

   (a)

$$\mathbf{F}_1 + \mathbf{F}_2 = \langle 5, 5, 4 \rangle$$

   (b)

$$\mathbf{F}_3 = \langle -5, -5, -4 \rangle$$

3. 1.2.4

$$-\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$$

4. 1.2.5

$$2\mathbf{i} + 4\mathbf{j}$$

5. 1.2.6

$$\langle 1, 1, -3 \rangle$$

6. 1.2.7

$$\langle 9, -2, \sqrt{2} \rangle$$

7. 1.2.11

   (a)

$$\langle 3, 1 \rangle = 2\langle 1, 1 \rangle + \langle 1, -1 \rangle$$

   (b)

$$\langle 3, -5 \rangle = -\langle 1, 1 \rangle + 4\langle 1, -1 \rangle$$

   (c)

$$\langle b_1, b_2 \langle = \frac{b_1 + b_2}{2}\langle 1, 1 \rangle + \frac{b_1 - b_2}{2}\langle 1, -1 \rangle$$

8. 1.2.13

$$x = 2 + t$$

$$y = -1 + 3t$$

$$z = 5 - 6t$$

9. 1.2.16

$$x = 2 + t$$
$$y = 1 - 2t$$
$$z = 2 + 3t$$

10. 1.2.17

$$x = 1 + t$$
$$y = 4$$
$$z = 5 - 6t$$

11. 1.2.26

These equations determine a line. Reparametrizing by setting $s = t^3$ we get the linear equations

$$x = 7 + 3s$$
$$y = 2 - s$$
$$z = 1 + 5s$$

12. 1.2.29

$$19 = x + 3y - z = (3t - 5) + 3(2 - t) - 6t = 1 - 6t$$

so $t = -3$, and they interesect at the point

$$(-14, 5, -18)$$

13. 1.2.35

These lines do not intersect. The first line is contained in the plane $2x + 5y + 11z = -9$ the second line is contained in the plane $2x + 5y + 11z = 2 + 25 + 77 \neq -9$. It is not enough to show that there is no t for which the parametric equations match.

14. 1.2.38

Assume that the wheel makes one rotation in 2π units of time. The center of the weel is at the position (at, a) at time t. The vector from the center to the reflector at time t is given by $\langle -b\sin(t), -b\cos(t)\rangle$. Therefore the location of the reflector at time t is given by

$$(at - b\sin(t), a - b\cos(t))$$

15. 1.3.20

$$\mathbf{F}_1 = \frac{(\mathbf{a} \cdot \mathbf{F})}{(\mathbf{a} \cdot \mathbf{a})}\mathbf{a} = \frac{2}{17}\langle 4, 1\rangle$$
$$\mathbf{F}_2 = \mathbf{F} - \mathbf{F}_1 = \langle \frac{9}{17}, -\frac{36}{17}\rangle$$

16. 1.3.24

Opposite sides of this figure are given by vectors

$$\mathbf{M}_1 - \mathbf{M}_2 = \frac{1}{2}(\mathbf{A} + \mathbf{B}) - \frac{1}{2}(\mathbf{B} + \mathbf{C}) = \frac{1}{2}\mathbf{A} - \frac{1}{2}\mathbf{C}$$
$$\mathbf{M}_4 - \mathbf{M}_3 = \frac{1}{2}(\mathbf{D} + \mathbf{A}) - \frac{1}{2}(\mathbf{C} + \mathbf{D}) = \frac{1}{2}\mathbf{A} - \frac{1}{2}\mathbf{C}$$

So this is a parallelogram.

17. 1.3.28

(a) We can expand $(\|\mathbf{b}\|\,\mathbf{a} + \|\mathbf{a}\|\,\mathbf{b}) \cdot (\|\mathbf{b}\|\,\mathbf{a} - \|\mathbf{a}\|\,\mathbf{b})$ as

$$\|\mathbf{b}\|^2\,\mathbf{a} \cdot \mathbf{a} - \|\mathbf{a}\|\,\|\mathbf{b}\|\,\mathbf{a}.\mathbf{b} + \|\mathbf{a}\|\,\|\mathbf{b}\|\,\mathbf{b}.\mathbf{a} - \|\mathbf{a}\|^2\,\mathbf{b}.\mathbf{b}$$

Noting that $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$ and $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ we see that this is 0, so these vectors are perpendicular.

(b) The angle between the vector $\mathbf{a}$ and $\|\mathbf{b}\|\,\mathbf{a} + \|\mathbf{a}\|\,\mathbf{b}$ is given by

$$\cos^{-1}\left(\frac{\mathbf{a} \cdot (\|\mathbf{b}\|\,\mathbf{a} + \|\mathbf{a}\|\,\mathbf{b})}{\|\mathbf{a}\|\,\|(\|\mathbf{a}\|\,\mathbf{b} + \|\mathbf{b}\|\,\mathbf{a})\|}\right) = \cos^{-1}\left(\frac{\|\mathbf{a}\|\,\|\mathbf{b}\| + \mathbf{a} \cdot \mathbf{b}}{\|(\|\mathbf{a}\|\,\mathbf{b} + \|\mathbf{b}\|\,\mathbf{a})\|}\right)$$

This expression is symmetric in $\mathbf{a}$ and $\mathbf{b}$, so we get the same expression for the angle between $\mathbf{b}$ and $\|\mathbf{b}\|\,\mathbf{a} + \|\mathbf{a}\|\,\mathbf{b}$, so this vector bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.