18.022: Multivariable calculus - solution to pset 2, ver 1 - fall 2006

1. 1.4.3

   $(1)(2)(3)+(3)(7)(-1)+(5)(0)(0)-(1)(7)(0)-(3)(0)(3)-(5)(2)(-1)=-5$

2. 1.4.6

   $$\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} (-2)(1)-(1)(1) \\ (1)(1)-(3)(1) \\ (3)(1)-(1)(-2) \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 5 \end{pmatrix}$$

3. 1.4.10

   Area of parallelogram $= 2 \times$ Area of triangles $= 2 \times \frac{1}{2} \times \| \overrightarrow{AB} \times \overrightarrow{AC} \|$

   $=\| [(3-1)i+(2-1)j] \times [(1-1)i+(3-1)j] \| = \| 4k \| = 4$

4. 1.4.12

   $$\overrightarrow{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -1 \end{pmatrix}$$

   $$\hat{u} = \frac{\overrightarrow{u}}{\| \overrightarrow{u} \|} = \begin{pmatrix} \sqrt{3}/9 \\ -5\sqrt{3}/9 \\ -\sqrt{3}/9 \end{pmatrix}$$

5. 1.4.18

   volume $= |(\overrightarrow{a} \times \overrightarrow{b}) \cdot \overrightarrow{c}|$

   $$= \left| \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \right| = 3$$

6. 1.4.20

   $$LHS = (\overrightarrow{a} \times \overrightarrow{b}) \cdot \overrightarrow{c} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_2b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

   $= a_2b_3c_1 - a_3b_2c_1 + a_3b_1c_2 - a_1b_3c_2 + a_1b_2c_3 - a_2b_1c_3$

   $$RHS = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

7. 1.4.25

   (a) $\overrightarrow{a} \times \overrightarrow{b}$

   (b) $\frac{2\overrightarrow{a} \times \overrightarrow{b}}{\| \overrightarrow{a} \times \overrightarrow{b} \|}$

(c) It follows from p.21 that $\left(\frac{\overrightarrow{a}\cdot\overrightarrow{b}}{\overrightarrow{a}\cdot\overrightarrow{a}}\overrightarrow{a}\right)$

(d) $\frac{\|\overrightarrow{b}\|\overrightarrow{a}}{\|\overrightarrow{a}\|}$

(e) $\overrightarrow{a}\times(\overrightarrow{b}\times\overrightarrow{c})$

(f) $(\overrightarrow{a}\times\overrightarrow{b})\times\overrightarrow{c}$

8. 1.4.40

(a) $\overrightarrow{\omega}=12k;\ \overrightarrow{r}=2i-j+3k;\ \overrightarrow{v}=\overrightarrow{\omega}\times\overrightarrow{r}=(12,24,0)$

(b) angle swept $=12-4\pi\approx-0.566$;
$(2-\cos\left(-0.566+\arctan(-1/2)\right),-1-\sin\left(-0.566+\arctan(-1/2)\right),3)$
$\approx(1.48,-1.85,3)$

9. 1.5.2

perpendicular to $i-2k\Rightarrow x-2z=C$

contains $(9,5,-1)\Rightarrow 9-2(-1)=C$

$x-2z=11$

10. 1.5.3

$Ax+By+Cz=1$

passes through $(3,-1,2)\Rightarrow 3A-B+2C=1$

passes through $(2,0,5)\Rightarrow 2A+5C=1$

passes through $(1,-2,4)\Rightarrow A-2B+4C=1$

Solving these simultaneous equations $\Rightarrow A=5/25, B=-4/25, C=3/25$

$5x-4y+3z=25$

11. 1.5.5

parallel to $5x-4y+z=1\Rightarrow 5x-4y+z=C$

passes through $(2,-1,-2)\Rightarrow 5(2)-4(-1)+(-2)=C$

$C=12$

$5x-4y+z=12$

12. 1.5.6

$t=7\Rightarrow(13,25,0); t=1\Rightarrow(1,7,6)$

$Ax+By+Cz=1$

passes through $(13,25,0)\Rightarrow 13A+25B=1$

passes through $(1,7,6)\Rightarrow A+7B+6C=1$

passes through $(2,5,0)\Rightarrow 2A+5B+=1$

Solving these equations, we have $A=-4/3, B=11/15, C=-7/15$

$-20x+11y-7z=15$

13. 1.5.7

    perpendicular to the line $\Rightarrow 3x - 2y - z = C$

    contains $(1, -1, 2) \Rightarrow 3(1) - 2(-1) - (2) = C$

    $3x - 2y - z = 3$

14. 1.5.9

    $x + 2y - 3z = 5; 15x + 15y - 3z = 3$

    Solving these equations, we have $14x + 13y = -1$

    $x = 0 \Rightarrow (0, -2/13, -25/13); y = 0 \Rightarrow (-1/7, 0, -12/7)$

    $x = -t/7, y = -2/13 + 2t/13, z = -23/13 + 5t/91$

15. 1.5.11

    $8x - 6y + 9Az = 6; -6Ax - 6y - 12z = -18$

    Equating the ratios between coefficients $\Rightarrow \frac{-6A}{-12} = \frac{8}{9A}$

    $A = \pm 4/3$

    Substituting back, we eliminated $A = 4/3$. Therefore, $A = -4/3$ is the only solution.

16. 1.5.22

    $t = 0 \Rightarrow B = (5, 3, 8); P_0 = (-11, 10, 20); \overrightarrow{BP_0} = (-16, 7, 12); \overrightarrow{a} = (-1, 0, 7)$

    $\overrightarrow{a} \times \overrightarrow{BP_0} \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix} \times \begin{pmatrix} -16 \\ 7 \\ 12 \end{pmatrix} = \begin{pmatrix} -49 \\ -100 \\ -7 \end{pmatrix}$

    It follows from p.44 that

    $D = \frac{\|\overrightarrow{a} \times \overrightarrow{BP_0}\|}{\|a\|} = \sqrt{249}$

17. 1.5.23

    Following p.45, we have

    $\overrightarrow{n} = \begin{pmatrix} 8 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -8 \\ 24 \end{pmatrix}$

    $\overrightarrow{OB_1}(t = 0) = (-1, 3, 5); \overrightarrow{OB_2}(t = 0) = (0, 3, 4); \overrightarrow{B_1B_2} = (1, 0, -1)$

    $\left(\frac{\overrightarrow{n} \cdot \overrightarrow{B_1B_2}}{\overrightarrow{n} \cdot \overrightarrow{n}}\right) \overrightarrow{n} = \frac{25\sqrt{641}}{641}$

18. 1.6.9

    Let $\overrightarrow{A} = \overrightarrow{a} - \overrightarrow{c}$ and $\overrightarrow{B} = \overrightarrow{c} - \overrightarrow{b}$. Apply the triangle inequality:

    $\| \overrightarrow{a} - \overrightarrow{b} \| = \| \overrightarrow{A} + \overrightarrow{B} \| \leq \| \overrightarrow{A} \| + \| \overrightarrow{B} \| = \| \overrightarrow{a} - \overrightarrow{c} \| + \| \overrightarrow{c} - \overrightarrow{b} \|$

19. 1.6.11

$$(\overrightarrow{a} + \overrightarrow{b}) \cdot (\overrightarrow{a} + \overrightarrow{b}) = (\overrightarrow{a} - \overrightarrow{b}) \cdot (\overrightarrow{a} - \overrightarrow{b})$$

$$\overrightarrow{a} \cdot \overrightarrow{a} + 2\overrightarrow{a} \cdot \overrightarrow{b} + \overrightarrow{b} \cdot \overrightarrow{b} = \overrightarrow{a} \cdot \overrightarrow{a} - 2\overrightarrow{a} \cdot \overrightarrow{b} + \overrightarrow{b} \cdot \overrightarrow{b}$$

$$\overrightarrow{a} \cdot \overrightarrow{b} = 0$$

20. 1.6.21

$$\begin{vmatrix} 7 & 0 & -1 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & -3 & 0 & 2 \\ 0 & 5 & 1 & -2 \end{vmatrix} = 7\begin{vmatrix} 0 & 1 & 3 \\ -3 & 0 & 2 \\ 5 & 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 2 & 0 & 3 \\ 1 & -3 & 2 \\ 0 & 5 & -2 \end{vmatrix}$$

$$= 7(10 - 9 - 6) - 1(12 + 15 - 20) = -42$$