18.022 Problem set 3

1. (a) If $g \circ f$ is bijective, f must be injective (in other words one-one). If $f(x) = f(y)$, then $g(f(x)) = g(f(y))$. f may not be surjective.

   (b) g must be surjective (or in other words onto), but may not be injective if f is not surjective.

2. 1.7.39

   (a)

$$\mathbf{e}_r \cdot \mathbf{e}_r = \cos^2\theta + sin^2\theta = 1$$

$$\mathbf{e}_r \cdot \mathbf{e}_\theta = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$$

$$\mathbf{e}_r \cdot \mathbf{e}_z = 0$$

$$\mathbf{e}_\theta \cdot \mathbf{e}_\theta = \sin^2\theta + \cos^2\theta = 1$$

$$\mathbf{e}_\theta \cdot \mathbf{e}_z = 0$$

$$\mathbf{e}_z \cdot \mathbf{e}_z = 1$$

   so these are mutually perpendicular unit vectors.

   (b)

$$\mathbf{e}_\rho \cdot \mathbf{e}_\rho = \sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi = \sin^2\phi + \cos^2\phi = 1$$

$$\begin{aligned}\mathbf{e}_\rho \cdot \mathbf{e}_\phi &= \sin\phi\cos\phi\cos^2\theta + \sin\phi\cos\phi\sin^2\theta - \cos\phi\sin\phi \\ &= \sin\phi\cos\phi - \cos\phi\sin\phi = 0\end{aligned}$$

$$\mathbf{e}_\rho \cdot \mathbf{e}_\theta = -\sin\phi\cos\theta\sin\theta + \sin\phi\sin\theta\cos\theta = 0$$

$$\mathbf{e}_\phi \cdot \mathbf{e}_\phi = \cos^2\phi\cos^2\theta + \cos^2\phi\sin^2\theta + \sin^2\phi = \cos^2\phi + \sin^2\phi = 1$$

$$\mathbf{e}_\phi \cdot \mathbf{e}_\theta = -\cos\phi\cos\theta\sin\theta + \cos\phi\sin\theta\cos\theta = 0$$

$$\mathbf{e}_\theta \cdot \mathbf{e}_\theta = \sin^2\theta + \sin^2\theta = 1$$

   so these are mutually perpendicular unit vectors.

3. 1.7.42

   (a)

$$0 \le \rho \le 3$$

$$0 \le \phi \le \arcsin\left(\frac{1}{3}\right)$$

(b)

$$r \le \frac{1}{\sqrt{8}} z \le (9 - r^2)^{\frac{1}{2}}$$

4. 2.1.1

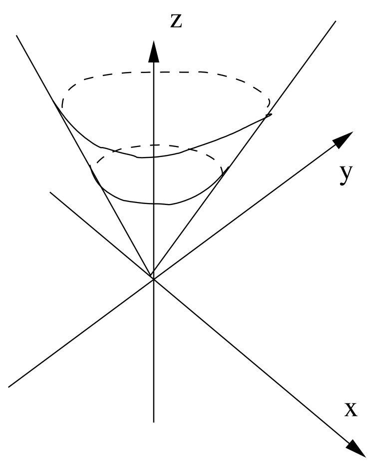
   (a) The domain of f is $\mathbb{R}$ the range of f is $[1, \infty)$.
   (b) f is not injective. $f(1) = f(-1)$
   (c) f is not surjective, as there is no point which maps to 0.

5. 2.1.13

$$f^{-1}(0) = \{0\}$$

$$f^{-1}(1) = \{(x^2 + y^2)^{\frac{1}{2}} = 1\} \text{ a circle of radius 1}$$

$$f^{-1}(2) = \{(x^2 + y^2)^{\frac{1}{2}} = 2\} \text{ a circle of radius 2}$$



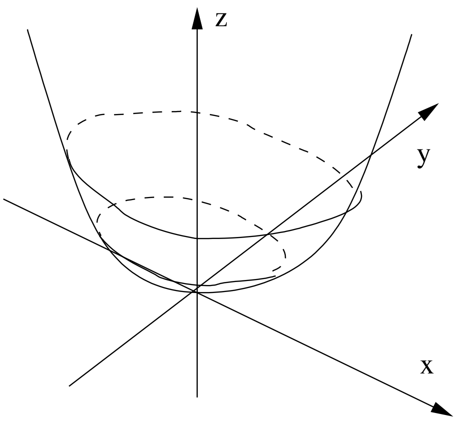
This is a cone.

6. 2.1.14

$$f^{-1}(0) = \{0\}$$

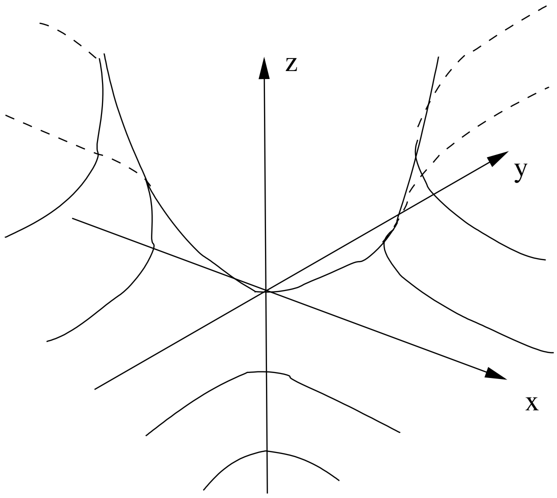
$$f^{-1}(1) = \{4x^2 + 9y^2 = 1\} \text{ an ellipse}$$

$$f^{-1}(4) = \{4x^2 + 9y^2 = 1\} \text{ an ellipse of double the size}$$



7. 2.1.15

$$f^{-1}(0) = \{x = 0\} \cup \{y = 0\}$$
$$f^{-1}(1) = \{yx = 1\}$$
$$f^{-1}(-1) = \{yx = -1\}$$



8. 2.1.27 Two different level curves that represent different values for f cannot intersect if f is continuous because the intersection would represent f taking on two different values at the one point. A level curve representing a single value of f can sometimes look like several curves which intersect.

9. 2.2.1 This set is open.

10. 2.2.2 This set is closed.

11. 2.2.9 This limit does not exist. The function along the line $y = 0$ is 1 but the function along the line $y = -x$ is 0.

12. 2.2.11 This limit does not exist. The limit along $y = 0$ is 2 and the limit along the line $x = 0$ is 1.

13. 2.2.23 Along the line $y = cx$, this limit is given by

$$\lim_{x \to 0} \frac{c^4 x^8}{(x^2 + c^4 x^4)^3}$$

This expression is greater than 0 but less than $c^4 x^2$, so this limit is 0. From this, we might expect the limit to be 0.

Along the path $x = y^2$, we have the limit

$$\lim_{x \to 0} \frac{x^6}{8x^6} = \frac{1}{8}$$

so this limit must not exist!

14. 2.2.28

$$\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^2 + y^2} = \lim_{r \to 0} \frac{r^3 \cos^2\theta \sin\theta}{r^2} = 0$$

15. 2.2.29

$$\lim_{(x,y) \to (0,0)} \frac{x^2}{x^2 + y^2} = \lim_{r \to 0} \frac{r^2 \cos^2\theta}{r^2}$$

This depends on θ so the limit does not exist.

16. 2.2.30

$$\begin{aligned} \lim_{(x,y) \to (0,0)} \frac{x^2 + xy + y^2}{x^2 + y^2} &= \lim_{r \to 0} \frac{r^2(\cos^2\theta + \sin\theta\cos\theta + \sin^2\theta)}{r^2} \\ &= \lim_{r \to 0} (1 + \sin\theta\cos\theta) \end{aligned}$$

This depends on θ, so the limit does not exist.

17. 2.2.32

$$\lim_{(x,y,z) \to (0,0,0)} \frac{x^2 + y^2}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} = \lim_{\rho \to 0} \frac{\rho^2 \sin\phi}{\rho} = 0$$

18. 2.2.39

f restricted to the line $y = 0$ is 1 when $x \neq 0$ but 0 when $x = 0$, so this is not continuous.

19. 2.2.42

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + xy^2 + 2x^2 + 2y^2}{x^2 + y^2} = 2 + \lim_{r\to 0} \frac{r^3(\cos^3\theta + cos\theta \sin^2\theta)}{r^2} = 2$$

so setting $g(0,0) = 2$ will make this function continuous.

20. 2.2.45

    (a) If $|x - 5| < \delta$, then $2\delta > 2|x - 5| = |(2x - 3) - 7|$ as required.
    (b) For any $\epsilon > 0$, if we choose $\delta = \frac{\epsilon}{2}$, then δ will also be positive and if $|x - 5| < \delta$, then $|f - 7| < \epsilon$ so $\lim_{x\to 5} f = 7$.