18.022 PSet 4 Solutions

October 2, 2006

Problem 1
(a) Note that $f(x,0) = f(0,y) = 0$, so $f_x(0,0) = f_y(0,0) = 0$. To show that f is differentiable we must show that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - xf_x(0,0) - yf_y(0,0)}{|(x,y) - (0,0)|} = 0.$$

This simplifies to showing that

$$\lim_{(x,y)\to(0,0)} \frac{|xy|}{\sqrt{x^2+y^2}} = 0.$$

Now, $|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2}$. Similarly, $|y| \leq \sqrt{x^2+y^2}$. Hence

$$\left|\frac{|xy|}{\sqrt{x^2+y^2}}\right| \leq \frac{\left(\sqrt{x^2+y^2}\right)^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}.$$

Now for $\epsilon > 0$, let $\delta = \epsilon$. Then for $|(x,y)| = \sqrt{x^2+y^2} < \delta$, $\left|\frac{|xy|}{\sqrt{x^2+y^2}}\right| \leq \sqrt{x^2+y^2} < \epsilon$. Hence f is differentiable at $(0,0)$.

(b) In fact, the partial derivatives do not exist in a neighborhood of $(0,0)$. For any $\epsilon > 0$, $(\epsilon, 0)$ and $(0, \epsilon)$ are within ϵ of $(0,0)$. On the other hand, $f(\epsilon, y) = \epsilon|y|$, which is not differentiable at $y = 0$. Hence $f_y(\epsilon, 0)$ is undefined. Similarly, $f_x(0,\epsilon)$ is undefined.

Problem 2
f is continuous exactly at the points πn where n is an integer. At these points $f(\pi n) = 0$. For $\epsilon > 0$, let $\delta = \epsilon$, then as long as $|x - \pi n| < \delta$,

$$|f(x)-0| = |f(x)| \leq |\sin(x)| = |\sin(n\pi+(x-n\pi))| = |\sin(x-n\pi)| \leq |x-n\pi| < \epsilon.$$

Hence f is continuous at $n\pi$.

For other x, $\sin(x) \neq 0$. Let $\epsilon = |\sin(x)|/2 > 0$.
Case 1 x is rational. Then $f(x) = \sin(x)$. For any $\delta > 0$, there exists a y so that $|y - x| < \delta$ and y is irrational. Then $|f(x) - f(y)| = |\sin(x)| > \epsilon$. So f is not continuous at x.

Case 2 x is irrational. Then $f(x) = 0$. For any $\delta > 0$, there exists a y so that $|x - y| < \delta$, y is rational, and $|\sin(y)| > |\sin(x)|/2$. For such a y, $|f(x) - f(y)| \geq |\sin(x)|/2 = \epsilon$. Hence f is not continuous at x.

Problem 3
Integrating the first equation with respect to x we get

$$f(x, y) = 3xy^2 - x^2 \cos(y) + g(y)$$

for some function g. Integrating the second equation with respect to y, we get that

$$f(x, y) = 3xy^2 - x^2 \cos(y) + 2y + h(x)$$

for some function h. By inspection we note that we can pick $g(y) = 2y$ and $h(x) = 0$ to get

$$f(x, y) = 3xy^2 - x^2 \cos(y) + 2y.$$

We note that this function works by differentiation.

Problem 4
Let $S = \{(x, y) \in \mathbb{R}^2 | 1 \leq x^2 + y^2 < 4\}$. S is neither open nor closed. The boundary of S is $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1 \text{ or } x^2 + y^2 = 4\}$. The part of the boundary where $x^2 + y^2 = 1$ is contained in S, so S is not open. The part of the boundary where $x^2 + y^2 = 4$ is not contained in S, so S is not closed.

Problem 5
Let $S = \{(x, y, z) \in \mathbb{R}^3 | 1 \leq x^2 + y^2 + z^2 \leq 4\}$. S is closed. The boundary of S is $\{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1 \text{ or } x^2 + y^2 + z^2 = 4\}$. Both of these components are contained in S, so S is closed.

Problem 6
We wish to compute

$$\lim_{\rho \to 0} \frac{(\rho \sin \phi \cos \theta)(\rho \sin \phi \sin \theta)(\rho \cos \phi)}{\rho^2} = \lim_{\rho \to 0} \rho(\sin^2 \phi \cos \phi \sin \theta \cos \theta).$$

This is 0, since for $\epsilon > 0$, if $\delta = \epsilon$ and $\rho < \delta$, then

$$|\rho(\sin^2 \phi \cos \phi \sin \theta \cos \theta)| \leq |\rho| < \epsilon.$$

Problem 7
We wish to compute

$$\lim_{\rho \to 0} \frac{(\rho \sin \phi \cos \theta)(\rho \cos \phi)}{\rho^2} = \lim_{\rho \to 0} (\sin \phi \cos \phi \cos \theta).$$

This limit does not exist. This is because no matter how small ρ is, if $(\phi, \theta) = (0, 0)$, then $(\sin \phi \cos \phi \cos \theta) = 0$, but if $(\phi, \theta) = (\pi/2, \pi/4)$, then $(\sin \phi \cos \phi \cos \theta) = 1/2$. Therefore, the values of our function can vary by as much as $1/2$ for arbitrarily small ρ, and hence the limit does not exist.

Problem 8
When $(x, y) \neq (0,0)$,

$$g(x,y) = \frac{x^3 + x^2 + xy^2 + y^2}{x^2 + y^2} = \frac{(x+1)(x^2+y^2)}{x^2+y^2} = x + 1.$$

Therefore, $\lim_{(x,y)\to(0,0)} = 1 \neq g(0,0)$. Hence g is not continuous.

Problem 9
(a) If $|(x,y) - (5,1)| < \delta$, then $\sqrt{(x-5)^2 + (y-1)^2} < \delta$. Therefore, $\delta > \sqrt{(x-5)^2 + (y-1)^2} \geq \sqrt{(x-5)^2} = |x-5|$, and $\delta > \sqrt{(x-5)^2 + (y-1)^2} \geq \sqrt{(y-1)^2} = |y-1|$.
(b) If $|(x,y) - (5,1)| < \delta$, then

$$\begin{aligned} |f(x,y) - 3| &= |2x - 10y + 3 - 3| = |2(x-5) + 10 - 10(y-1) - 10| \\ &= |2(x-5) + 10(y-1)| \leq 2|x-5| + 10|y-1| < 12\delta. \end{aligned}$$

(c) For any $\epsilon > 0$, let $\delta = \epsilon/12$. Then for $|(x,y) - (5,1)| < \delta$, we have by part (b) that

$$|f(x,y) - 3| < 12\delta = \epsilon.$$

Hence $\lim_{(x,y)\to(5,1)} f(x,y) = 3$.

Problem 10
The matrix of partials is

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix} = \begin{bmatrix} yz & xz & xy \\ \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \end{bmatrix}.$$

Substituting in our value of (x, y, z), we get

$$\begin{bmatrix} 0 & -2 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{-2}{\sqrt{5}} \end{bmatrix}.$$

Problem 11
The matrix of partials is

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} & \frac{\partial f_3}{\partial w} \end{bmatrix} = \begin{bmatrix} 3 & -7 & 1 & 0 \\ 5 & 0 & 2 & -8 \\ 0 & 1 & -17 & 3 \end{bmatrix}.$$

And this is independent of the value of (x, y, z, w).

Problem 12
The matrix of partials is

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy & x^2 \\ 1 & 2y \\ -\pi y \sin \pi xy & -\pi x \sin \pi xy \end{bmatrix}.$$

Substituting in our value of (x, y), we get

$$\begin{bmatrix} -4 & 4 \\ 1 & -2 \\ 0 & 0 \end{bmatrix}.$$

Problem 13
The matrix of partials is

$$\begin{bmatrix} \frac{\partial f_1}{\partial s} & \frac{\partial f_1}{\partial t} \\ \frac{\partial f_2}{\partial s} & \frac{\partial f_2}{\partial t} \\ \frac{\partial f_3}{\partial s} & \frac{\partial f_3}{\partial t} \end{bmatrix} = \begin{bmatrix} 2s & 0 \\ t & s \\ 0 & 2t \end{bmatrix}.$$

Substituting in our value of (s, t), we get

$$\begin{bmatrix} -2 & 0 \\ 1 & -1 \\ 0 & 2 \end{bmatrix}.$$

Problem 14
First we note that $(0, 0, 0)$ is not in the domain of f. This is because $f(x, x, x) = \frac{x+x+x}{x^2+x^2+x^2} = \frac{1}{x}$, which does not converge as $x \to 0$. For $(x, y, z) \neq (0, 0, 0)$, the partial derivatives are

$$f_x = \frac{1}{x^2 + y^2 + z^2} - \frac{2x(x + y + z)}{(x^2 + y^2 + z^2)^2},$$
$$f_y = \frac{1}{x^2 + y^2 + z^2} - \frac{2y(x + y + z)}{(x^2 + y^2 + z^2)^2},$$
$$f_z = \frac{1}{x^2 + y^2 + z^2} - \frac{2z(x + y + z)}{(x^2 + y^2 + z^2)^2}.$$

Away from $(0, 0, 0)$, these are all continuous. Therefore, f is differentiable on its domain.

Problem 15
First we note that $(x, 0)$ and $(y, 0)$ are not in the domain of f. This is because $\frac{x}{y}$ is not defined for $y = 0$ and $\frac{y}{x}$ is not defined for $x = 0$. When $x \neq 0$ and $y \neq 0$, the partial derivatives are

$$\frac{\partial f_1}{\partial x} = \frac{y^2}{x^2 + y^4} - \frac{2x^2y^2}{(x^2 + y^4)^2},$$
$$\frac{\partial f_1}{\partial y} = \frac{2xy}{x^2 + y^4} - \frac{4xy^5}{(x^2 + y^4)^2},$$
$$\frac{\partial f_2}{\partial x} = \frac{1}{y} - \frac{y}{x^2},$$
$$\frac{\partial f_2}{\partial y} = \frac{1}{x} - \frac{x}{y^2}.$$

If $x \neq 0$ and $y \neq 0$, these are clearly continuous, and hence f is differentiable on its domain.

Problem 16
Taking partial derivatives, we find that

$$\frac{\partial z}{\partial x} = -4y \sin xy,$$
$$\frac{\partial z}{\partial y} = -4x \sin xy.$$

Evaluating at $(\pi/3, 1)$, we get that

$$\frac{\partial z}{\partial x} = -2\sqrt{3},$$
$$\frac{\partial z}{\partial y} = \frac{-2\pi}{\sqrt{3}}.$$

Hence the tangent plane is defined by

$$z = 2 - 2\sqrt{3}(x - \pi/3) - \frac{2\pi}{\sqrt{3}}(y - 1).$$

This can also be written as

$$\sqrt{3}x + \frac{2\pi\sqrt{3}}{3}y + z = 2 - \frac{4\pi\sqrt{3}}{3}.$$

Problem 17
Taking partial derivatives, we find that

$$\frac{\partial z}{\partial x} = 2x - 6,$$
$$\frac{\partial z}{\partial y} = 3y^2.$$

Therefore, the tangent plane at $(x_0, y_0, x_0^2 - 6x_0 + y_0^3)$ is

$$z = (x_0^2 - 6x_0 + y_0^3) + (2x_0 - 6)(x - x_0) + (3y_0^2)(y - y_0),$$

or

$$(-2x_0 + 6)x + (-3y_0^2)y + z = (x_0^2 - 6x_0 + y_0^3) + (6x_0 - 2x_0^2) - 3y_0^3.$$

This is parallel to $4x - 12y + z = 7$ if and only if $4 = 6 - 2x_0$ and $-12 = -3y_0^2$. Hence we must have $(x_0, y_0) = (1, \pm 2)$. Therefore our two planes are

$$4x - 12y + z = -17$$

and

$$4x - 12y + z = 15.$$