18.022: Multivariable calculus - problem set 6 (v.2) - fall 2006

Due by 1:45 PM, Room 2-106, Friday 10/20.

Note that we use 3rd edition of the text for reference and earlier editions may number problems differently. Whilst you may attempt problems in any order, graders will appreciate if you hand in your problems in order. By all means, please try to help the graders.

1. (15 points) Let $\mathbf{F}\colon \mathbb{R}^4 \to \mathbb{R}^2$ be of class C^1 and suppose that

$$D\mathbf{F}(3,-1,1,0) = \begin{pmatrix} 1 & -2 & 0 & 2 \\ -1 & 1 & 2 & 1 \end{pmatrix}.$$

   (a) Show that there exists an open subset $U \subset \mathbb{R}^2$ containing $(3,-1)$ such that, for all $(x,y) \in U$, the system of equations

$$\mathbf{F}(x,y,z,w) = \mathbf{F}(3,-1,1,0)$$

   has a unique solution $(z,w) = (f_1(x,y), f_2(x,y))$.

   (b) Show that there exists an open subset $V \subset \mathbb{R}^2$ containing $(3,1)$ such that, for all $(x,z) \in V$, the system of equations

$$\mathbf{F}(x,y,z,w) = \mathbf{F}(3,-1,1,0)$$

   has a unique solution $(y,w) = (g_1(x,z), g_2(x,z))$.

   (c) Find $D\mathbf{f}(3,-1)$ and $D\mathbf{g}(3,1)$.

2. (10 points) 3.1.15

3. (10 points) 3.1.19

4. (10 points) 3.1.20

5. (10 points) 3.2.7

6. (10 points) 3.2.8

7. (10 points) 3.2.18

8. (10 points) Let $a > 0$ and $b < 0$ be constants, and let $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^2$ be the parametrized differentiable curve defined by

$$\mathbf{r}(t) = (ae^{bt}\cos t, ae^{bt}\sin t).$$

The curve traced out by $\mathbf{r}$ is called the *logarithmic spiral.*

(a) Show that $\mathbf{r}'(t) \to \mathbf{0}$ as $t \to \infty$.

(b) Show that $\lim_{s\to\infty} \int_0^s ||\mathbf{r}'(t)||dt$ is finite. What does this mean?

9. (15 points) Let $\mathbf{r}\colon (0, \pi) \to \mathbb{R}^2$ be the parametrized differentiable curve defined by

$$\mathbf{r}(t) = (\sin t, \cos t + \log\tan\frac{t}{2}),$$

where log is the natural logarithm function. The curve traced out by $\mathbf{r}$ is called the *tractrix.*

(a) Show that $\mathbf{r}'(t)$ is non-zero if $t \neq \pi/2$.

(b) Find a vector parametric equation for the line tangent to tractrix at $\mathbf{r}(t)$.

(c) Show that the length of the segment of the tangent line from $\mathbf{r}(t)$ to the y-axis is constant equal to 1.

Total: 100 points