18.022 Problem set 6

1. (a) By the Implicit Function Theorem, it suffices to prove that

$$\begin{vmatrix} \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial w} \\ \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial w} \end{vmatrix} \neq 0$$

at the point $(x, y, z, w) = (3, -1, 1, 0)$. By assumption,

$$\begin{vmatrix} \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial w} \\ \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} = -4,$$

which is indeed nonzero.

(b) Again by the Implicit Function Theorem, it suffices to prove that

$$\begin{vmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial w} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial w} \end{vmatrix} \neq 0$$

at the point $(x, y, z, w) = (3, -1, 1, 0)$. By assumption,

$$\begin{vmatrix} \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial w} \\ \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial w} \end{vmatrix} = \begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix} = -4,$$

which is indeed nonzero.

(c) Write $f(x, y) = (f_1(x, y), f_2(x, y))$. To compute $D\mathbf{f}(3, -1)$, we use the chain rule:

$$\begin{aligned}
0 &= \frac{\partial}{\partial x} F_1(x, y, f_1, f_2) = \frac{\partial F_1}{\partial x} + \frac{\partial f_1}{\partial x}\frac{\partial F_1}{\partial z} + \frac{\partial f_2}{\partial x}\frac{\partial F_1}{\partial w} \\
&= 1 + 2\frac{\partial f_2}{\partial x}, \\
0 &= \frac{\partial}{\partial x} F_2(x, y, f_1, f_2) = \frac{\partial F_2}{\partial x} + \frac{\partial f_1}{\partial x}\frac{\partial F_2}{\partial z} + \frac{\partial f_2}{\partial x}\frac{\partial F_2}{\partial w} \\
&= -1 + 2\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial x}, \\
0 &= \frac{\partial}{\partial y} F_1(x, y, f_1, f_2) = \frac{\partial F_1}{\partial y} + \frac{\partial f_1}{\partial y}\frac{\partial F_1}{\partial z} + \frac{\partial f_2}{\partial y}\frac{\partial F_1}{\partial w} \\
&= -2 + 2\frac{\partial f_2}{\partial y}, \\
0 &= \frac{\partial}{\partial x} F_2(x, y, f_1, f_2) = \frac{\partial F_2}{\partial y} + \frac{\partial f_1}{\partial y}\frac{\partial F_2}{\partial z} + \frac{\partial f_2}{\partial y}\frac{\partial F_2}{\partial w} \\
&= 1 + 2\frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial y}.
\end{aligned}$$

We conclude that

$$0 = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

$$\iff \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} -1 & 2 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & -1 \\ -\frac{1}{2} & 1 \end{pmatrix}.$$

To compute $D\mathbf{g}(3,1)$, the same approach yields the equation

$$0 = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix}$$

$$\iff \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & -2 \\ -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & -1 \\ \frac{1}{4} & -1 \end{pmatrix}.$$

2. (Exercise 3.1.15)

With $\mathbf{x}(t) = te^{-t}\mathbf{i} + e^{3t}\mathbf{j}$, we obtain that

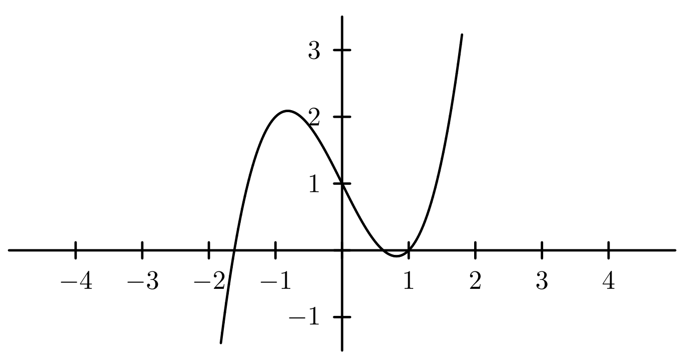
$$\mathbf{v}(t) = (1-t)e^{-t}\mathbf{i} + 3e^{3t}\mathbf{j}.$$

$t = 0$ yields the equation

$$\mathbf{l}(t) = \mathbf{x}(0) + t\mathbf{v}(0) = \mathbf{j} + t(\mathbf{i} + 3\mathbf{j}) = t\mathbf{i} + (1+3t)\mathbf{j}.$$

3. (Exercise 3.1.19)

(a)



(b) We have that $\mathbf{v}(t) = (1, 3t^2 - 2)$; hence the line tangent to $\mathbf{x}(t)$ when $t = 2$ equals

$$\mathbf{l}(t) = \mathbf{x}(2) + (t-2)\mathbf{v}(2) = (2,5) + (t-2)\langle 1, 10\rangle = (t, 10t - 15).$$

(c) We have that $(x, y) = (t, t^3 - 2t + 1)$ is equivalent to

$$y = x^3 - 2x + 1.$$

(d) With $y = x^3 - 2x + 1$, we have that $y' = 3x^2 - 2$. Note that $y(2) = 5$ and $y'(2) = 10$. Hence the tangent line of the curve $y = x^3 - 2x + 1$ when $x = 2$ equals

$$y = 5 + 10(x - 2) = 10x - 15 \Longleftrightarrow (x, y) = (t, 10t - 15).$$

4. (Exercise 3.1.20)

   We want to show that the following parametric equations define a parabola:

$$\begin{cases} x = (v_0 \cos\theta)t, \\ y = (v_0 \sin\theta)t - \frac{1}{2}gt^2. \end{cases}$$

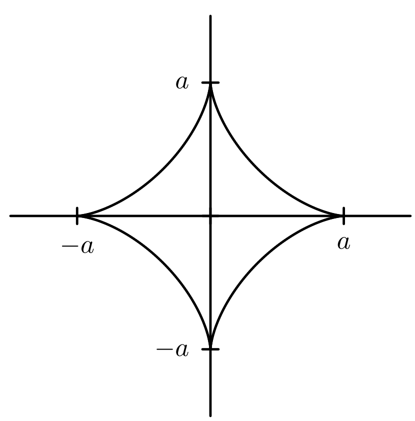
   To see this, note that $t = x/(v_0 \cos\theta)$, which yields that

$$\begin{aligned} y &= (v_0 \sin\theta)\frac{x}{v_0 \cos\theta} - \frac{g}{2} \cdot \frac{x^2}{v_0^2 \cos^2\theta} \\ \Longleftrightarrow y &= \tan\theta \cdot x - \frac{g}{2v_0^2 \cos^2\theta} \cdot x^2. \end{aligned}$$

   This is indeed a parabola.

5. (Exercise 3.2.7)

   The path $\mathbf{x}(t) = (a\cos^3 t, a\sin^3 t)$ has the following shape:



   The length of the path is given by

$$\int_0^{2\pi} \|\mathbf{x}'(t)\| dt.$$

Now,

$$\begin{aligned}\mathbf{x}'(t) &= \langle -3a\cos^2 t\sin t, 3a\sin^2 t\cos t\rangle = 3a\cos t\sin t\cdot\langle -\cos t, \sin t\rangle \\ &= \frac{3a}{2}\sin 2t\cdot\langle -\cos t, \sin t\rangle;\end{aligned}$$

thus

$$\|\mathbf{x}'(t)\| = \frac{3a}{2}|\sin 2t|\sqrt{(-\cos t)^2+\sin^2 t} = \frac{3a}{2}|\sin 2t|.$$

It is clear that $\|\mathbf{x}'(t+\pi/2)\| = \|\mathbf{x}'(t)\|$ for all t; hence

$$\int_0^{\pi/2}\|\mathbf{x}'(t)\|dt = \int_\pi^{3\pi/2}\|\mathbf{x}'(t)\|dt = \int_{\pi/2}^{\pi}\|\mathbf{x}'(t)\|dt = \int_{3\pi/2}^{2\pi}\|\mathbf{x}'(t)\|dt.$$

As a consequence,

$$\int_0^{2\pi}\|\mathbf{x}'(t)\|dt = 4\int_0^{\pi/2}\frac{3a}{2}\sin 2t dt = 4\left[\frac{3a}{2}\frac{(-\cos 2t)}{2}\right]_0^{\pi/2} = 6a.$$

6. (Exercise 3.2.8)

   We want to compute the length of the curve $y = f(x)$, which in parametric form becomes

$$\begin{cases} x = t, \\ y = f(t).\end{cases}$$

   We obtain that

$$\|\mathbf{x}'(t)\| = \sqrt{1+(f'(t))^2}.$$

   As a consequence, the length of the curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$ is equal to

$$\int_a^b \sqrt{1+(f'(x))^2}dx.$$

7. (Exercise 3.2.18)

   (a) *First method.* We have that

$$\kappa = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|^3} = \frac{\|\mathbf{x}'\times\mathbf{x}''\|}{\|\mathbf{x}'\|^3}.$$

   Since $\mathbf{x}$ is parametrized by arclength, we get that

$$\kappa = \|\mathbf{x}'\times\mathbf{x}''\|.$$

Now,

$$\mathbf{x}' \times \mathbf{x}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x' & y' & 0 \\ x'' & y'' & 0 \end{vmatrix} = (x'y'' - x''y')\mathbf{k}.$$

As a consequence, $\kappa = |x'y'' - x''y'|$ as desired.

Second method. Since the curve is parametrized by arclength, we get that

$$\frac{d\mathbf{T}}{ds} = \langle x''(s), y''(s) \rangle;$$

hence

$$\kappa^2 = (x'')^2 + (y'')^2.$$

We want to prove that

$$(x'')^2 + (y'')^2 = (x'y'' - x''y')^2 = (x')^2(y'')^2 + (x'')^2(y')^2 - 2x'x''y'y''.$$

Now, since $(x')^2 + (y')^2 = 1$, we obtain that

$$\begin{aligned} & (x')^2(y'')^2 + (x'')^2(y')^2 - 2x'x''y'y'' \\ = \; & (1 - (y')^2)(y'')^2 + (1 - (x')^2)(x'')^2 - 2x'x''y'y'' \\ = \; & (x'')^2 + (y'')^2 - (y')^2(y'')^2 - (x')^2(x'')^2 - 2x'x''y'y'' \\ = \; & (x'')^2 + (y'')^2 - (x'x'' + y'y'')^2. \end{aligned}$$

Since

$$x'x'' + y'y'' = \frac{1}{2}\frac{d}{dt}((x')^2 + (y')^2) = \frac{1}{2}\frac{d}{dt}(1) = 0,$$

we obtain the desired identity.

(b) We have that

$$\begin{aligned} \mathbf{x}'(s) &= \left\langle -s, \frac{1}{2}\left(-\frac{1}{\sqrt{1-s^2}} - \sqrt{1-s^2} + \frac{s^2}{\sqrt{1-s^2}} \right) \right\rangle \\ &= \left\langle -s, -\sqrt{1-s^2} \right\rangle, \end{aligned}$$

which implies that

$$\|\mathbf{x}'(s)\| = (-s)^2 + (-\sqrt{1-s^2})^2 = s^2 + 1 - s^2 = 1;$$

hence the curve is parametrized by arclength.

Now,

$$\mathbf{x}''(s) = \left\langle -1, \frac{s}{\sqrt{1-s^2}} \right\rangle.$$

Using (a), we obtain that

$$\begin{aligned}\kappa &= |x'y'' - x''y'| = \left|(-s)\cdot\frac{s}{\sqrt{1-s^2}} - (-1)\cdot\left(-\sqrt{1-s^2}\right)\right| \\ &= \left|\frac{-s^2-(1-s^2)}{\sqrt{1-s^2}}\right| = \left|\frac{1}{\sqrt{1-s^2}}\right|.\end{aligned}$$

8. We have that

$$\begin{aligned}\mathbf{r}'(t) &= \langle abe^{bt}\cos t - ae^{bt}\sin t, abe^{bt}\sin t + ae^{bt}\cos t\rangle \\ &= ae^{bt}\langle b\cos t - \sin t, b\sin t + \cos t\rangle.\end{aligned}$$

As a consequence,

$$\begin{aligned}\|\mathbf{r}'(t)\| &= ae^{bt}\sqrt{(b\cos t - \sin t)^2 + (b\sin t + \cos t)^2} \\ &= ae^{bt}\sqrt{b^2+1}.\end{aligned}$$

(a) We have that

$$\lim_{t\to\infty}\|\mathbf{r}'(t)\| = \lim_{t\to\infty} ae^{bt}\sqrt{b^2+1} = a\sqrt{b^2+1}\cdot\lim_{t\to\infty} e^{bt} = 0,$$

because $b<0$.

(b)

$$\begin{aligned}\int_0^s \|\mathbf{r}'(t)\|dt &= \int_0^s a\sqrt{b^2+1}\cdot e^{bt}dt = a\sqrt{b^2+1}\cdot\left[\frac{e^{bt}}{b}\right]_0^s \\ &= \frac{a\sqrt{b^2+1}}{|b|}\cdot(1-e^{bs});\end{aligned}$$

recall that $b<0$. As a consequence,

$$\lim_{s\to\infty}\int_0^s \|\mathbf{r}'(t)\|dt = \frac{a\sqrt{b^2+1}}{|b|}.$$

This limit being finite means that the arc length of the curve $\{\mathbf{r}(t) : 0\le t<\infty\}$ is finite.

9. (a) We have that

$$\mathbf{r}'(t) = \langle\cos t, -\sin t + \frac{1}{2\tan(t/2)\cos^2(t/2)}\rangle = \langle\cos t, -\sin t + \frac{1}{\sin t}\rangle.$$

Since $\sin t > 0$ whenever $t \in (0, \pi)$, this is defined for all t. $\cos t = 0$ if and only if $t = \pi/2$, in which case $-\sin t + \frac{1}{\sin t} = 0$. For all other values on t, $\mathbf{r}'(t)$ is nonzero.

(b) Vector parametric equation for the tangent line at $\mathbf{r}(t_0)$:

$$\begin{aligned}
\mathbf{l}(t) &= \mathbf{r}(t_0) + (t - t_0)\mathbf{r}'(t_0) \Longleftrightarrow \\
\mathbf{l}(t) &= (\sin t_0, \cos t_0 + \ln \tan \frac{t_0}{2}) + (t - t_0) \cdot \langle \cos t_0, -\sin t_0 + \frac{1}{\sin t_0} \rangle.
\end{aligned}$$

(c) The line intersects the y-axis when

$$\sin t_0 + (t - t_0)\cos t_0 = 0 \Longleftrightarrow t = t_0 - \tan t_0.$$

The distance between the points $\mathbf{l}(t_0)$ and $\mathbf{l}(t_0 - \tan t_0)$ is

$$\begin{aligned}
& |t_0 - (t_0 - \tan t_0)| \cdot \|\mathbf{r}'(t_0)\| \\
= \; & |\tan t_0| \cdot \left(\cos^2 t_0 + \left(-\sin t_0 + \frac{1}{\sin t_0} \right)^2 \right)^{1/2} \\
= \; & |\tan t_0| \cdot \left(\cos^2 t_0 + \sin^2 t_0 + \frac{1}{\sin^2 t_0} - 2 \right)^{1/2} \\
= \; & |\tan t_0| \cdot \left(\frac{1 - \sin^2 t_0}{\sin^2 t_0} \right)^{1/2} \\
= \; & |\tan t_0| \cdot \left(\frac{\cos^2 t_0}{\sin^2 t_0} \right)^{1/2} = |\tan t_0| \cdot \frac{1}{|\tan t_0|} = 1.
\end{aligned}$$