18.022 Problem set 7

1.

$$\frac{d}{ds}(\mathbf{T}-\tilde{\mathbf{T}}) = \kappa(\mathbf{N}-\tilde{\mathbf{N}})$$

$$\frac{d}{ds}(\mathbf{N}-\tilde{\mathbf{N}}) = -\kappa(\mathbf{T}-\tilde{\mathbf{T}}) + \tau(\mathbf{B}-\tilde{\mathbf{B}})$$

$$\frac{d}{ds}(\mathbf{B}-\tilde{\mathbf{B}}) = -\tau(\mathbf{N}-\tilde{\mathbf{N}})$$

Define the function

$$f(s) := \left\|\mathbf{T}(s)-\tilde{\mathbf{T}}(s)\right\|^2 + \left\|\mathbf{N}(s)-\tilde{\mathbf{N}}(s)\right\|^2 + \left\|\mathbf{B}(s)-\tilde{\mathbf{B}}(s)\right\|^2$$

$$\begin{aligned}\frac{df}{ds} &= 2(\mathbf{T}-\tilde{\mathbf{T}})\cdot\kappa(\mathbf{N}-\tilde{\mathbf{N}}) \\ &\quad + 2(\mathbf{N}-\tilde{\mathbf{N}})\cdot\Big(-\kappa(\mathbf{T}-\tilde{\mathbf{T}}) + \tau(\mathbf{B}-\tilde{\mathbf{B}})\Big) \\ &\quad + 2(\mathbf{B}-\tilde{\mathbf{B}})\cdot\Big(-\tau(\mathbf{N}-\tilde{\mathbf{N}})\Big) \\ &= 0\end{aligned}$$

So $f(s)$ is a constant. As $f(a) = 0$, f is equal to zero everywhere. In particular, this means that $\mathbf{T} = \tilde{\mathbf{T}}$, so $\frac{d\mathbf{r}}{ds} = \frac{d\tilde{\mathbf{r}}}{ds}$, so the fact that $\mathbf{r}$ and $\tilde{\mathbf{r}}$ coincide at one point means that they coincide in the entire interval.

2.

$$\frac{d\mathbf{r}}{ds} = -\frac{a}{c}\sin(\frac{s}{c})\mathbf{i} + \frac{a}{c}\cos(\frac{s}{c})\mathbf{j} + \frac{b}{c}\mathbf{k}$$

(a)

$$\left\|\frac{d\mathbf{r}}{ds}\right\|^2 = \frac{a^2+b^2}{c^2} = 1$$

so this path is parametrized by arclength.

(b)

$$\mathbf{T} = -\frac{a}{c}\sin(\frac{s}{c})\mathbf{i} + \frac{a}{c}\cos(\frac{s}{c})\mathbf{j} + \frac{b}{c}\mathbf{k}$$

$$\frac{d\mathbf{T}}{ds} = \frac{a}{c^2}\Big(-\cos(\frac{s}{c})\mathbf{i} - \sin(\frac{s}{c})\mathbf{j}\Big)$$

so $\kappa = \frac{a}{c^2}$ and

$$\mathbf{N} = -\cos(\frac{s}{c})\mathbf{i} - \sin(\frac{s}{c})\mathbf{j}$$

$$\begin{aligned}\mathbf{B} &= \mathbf{T}\times\mathbf{N}\\ &= \frac{b}{c}\sin(\frac{s}{c})\mathbf{i} - \frac{b}{c}\cos(\frac{s}{c})\mathbf{j} + \frac{a}{c}\mathbf{k}\end{aligned}$$

(c) We already know that $\kappa = \frac{a}{c^2}$.

$$\frac{d\mathbf{B}}{ds} = \frac{b}{c^2}\cos(\frac{s}{c})\mathbf{i} + \frac{b}{c^2}\sin(\frac{s}{c})\mathbf{j} = -\frac{b}{c^2}\mathbf{N}$$

so $\tau = \frac{b}{c^2}$.

3. 3.2.16

$$\frac{d\mathbf{x}}{dt} = \langle 2e^{2t}\sin t + e^{2t}\cos t, 2e^{2t}\cos t - e^{2t}\sin t, 0\rangle$$

$$\frac{ds}{dt} = \left\|\frac{d\mathbf{x}}{dt}\right\| = \sqrt{5}e^{2t}$$

$$\mathbf{T} = \frac{1}{\sqrt{5}}\langle 2\sin t + \cos t, 2\cos t - \sin t, 0\rangle$$

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\frac{1}{\sqrt{5}e^{2t}} = \frac{1}{5e^{2t}}\langle 2\cos t - \sin t, -2\sin t - \cos t, 0\rangle$$

so the curvature $\kappa = \frac{1}{\sqrt{5}e^{2t}}$, and

$$\mathbf{N} = \frac{1}{\sqrt{5}}\langle 2\cos t - \sin t, -2\sin t - \cos t, 0\rangle$$

$$\mathbf{B} = \langle 0, 0, -1\rangle$$

This is constant, so the torsion $\tau = 0$.

4. 3.2.40

$$\mathbf{w} = \tau\mathbf{T} + \kappa\mathbf{B}$$

This along with the fact that $\mathbf{B} = \mathbf{T}\times\mathbf{N}$ is enough to establish the required formulas:

$$\mathbf{T}' = \kappa\mathbf{N} = \mathbf{w}\times\mathbf{T}$$

$$\mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B} = \mathbf{w}\times\mathbf{N}$$

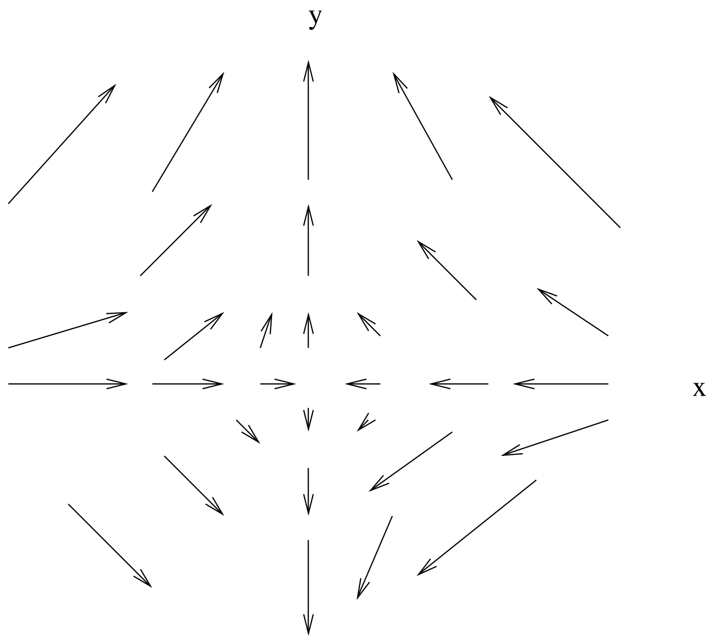
$$\mathbf{B}' = -\tau\mathbf{N} = \mathbf{w}\times\mathbf{B}$$

5. 3.2.41

$$\mathbf{w}' = \tau'\mathbf{T} + \kappa'\mathbf{N} + \tau\mathbf{T}' + \kappa\mathbf{N}' = \tau'\mathbf{T} + \kappa'\mathbf{N} + \mathbf{w} \times \mathbf{w} = \tau'\mathbf{T} + \kappa'\mathbf{N}$$

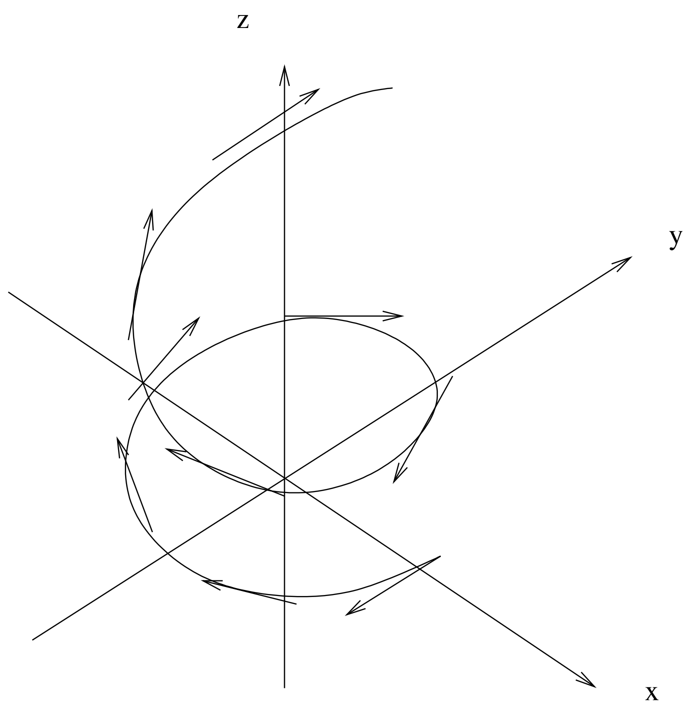
So if $\mathbf{w}$ is constant, then κ and τ are both constant. Question 2 showed that we can choose a helix with any constant (nonzero) κ and τ that we like, and question 1 showed that any other curve with these properties must be congruent. The special case of $\tau = 0$ $\kappa \neq 0$ is a circle. If $\kappa = 0$, this will be a line, however we would need some alternative way of defining the moving frame to make sense of the question in this case.

6. 3.3.3



7. 3.3.19

$$(x, y, z) = (\sin t, \cos t, e^{2t})$$
$$(x', y', z') = (\cos t, -\sin t, 2e^{2t}) = (y, -x, 2z)$$



8. 3.3.20

$$x' = -x \text{ and } x(0) = 2 \Leftrightarrow x = 2e^{-t}$$

$$y' = y \text{ and } y(0) = 1 \Leftrightarrow y = e^{t}$$

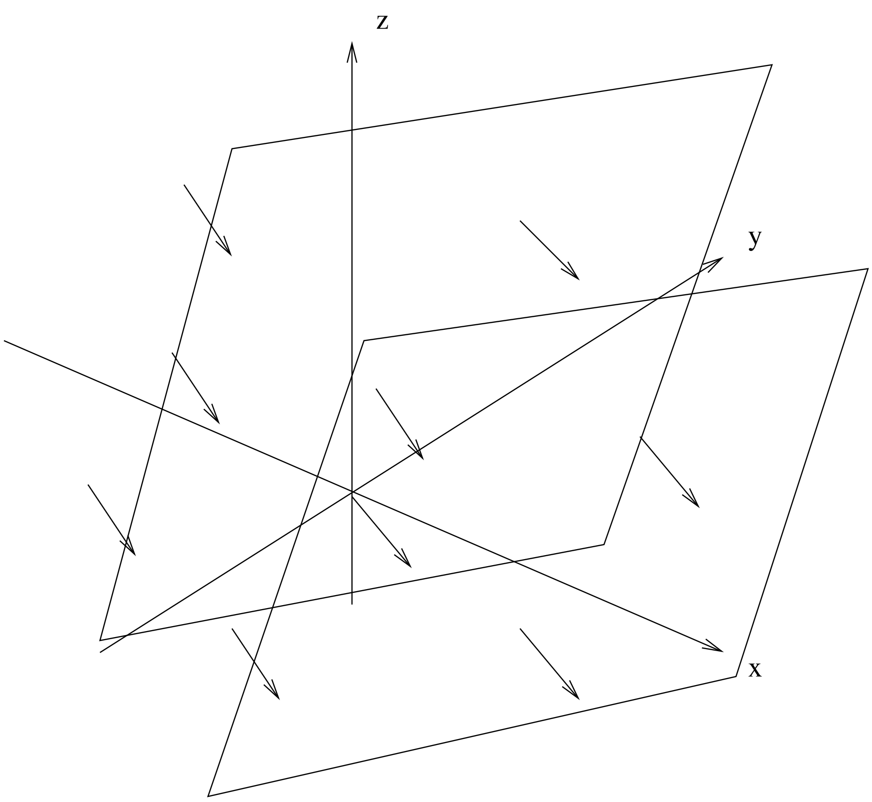
so $(x, y) = (2e^{-t}, e^{t})$.

9. 3.3.23

   (a)

$$F = \nabla(3x - 2y + z)$$

   (b) The equipotential surfaces of F are planes perpendicular to the vector $\langle 3, -2, 1 \rangle$.



10. 3.3.25 If $\frac{dx}{dt} = \nabla f$, then

$$\frac{d}{dt} f(x) = \nabla f \cdot \nabla f \geq 0$$

11. 3.4.1

$$\nabla \cdot \mathbf{F} = 2x + 2y$$

12. 3.4.3

$$\nabla \cdot \mathbf{F} = 3$$

13. 3.4.6

$$\nabla \cdot \mathbf{F} = 1$$

14. 3.4.13

    (a)

$$\nabla \cdot \mathbf{F} > 0$$

    (b)

$$\nabla \cdot \mathbf{F} < 0$$

(c)

$$\nabla \cdot \mathbf{F} > 0 \text{ for } x > 0$$
$$\nabla \cdot \mathbf{F} < 0 \text{ for } x < 0$$

(d) $\nabla \cdot \mathbf{F}$ is hard to determine from the picture.

15. 3.6.43

$$\nabla \times \mathbf{F} = 2e^{-x} \cos z\mathbf{j}$$

So $\mathbf{F}$ can't be the gradient of any function.

16. 3.6.44

$$\nabla \cdot \mathbf{F} = y^2 + 1 + e^x + x^2 e^z \neq 0$$

so $\mathbf{F}$ can't be the curl of any vectorfield of class C^2.