18.022: Multivariable calculus - solution to pset 8, ver 1 - fall 2006

1. (5 points) 4.1.8

$$\begin{aligned}
f(x,y) &= 1/(x^2+y^2+1) \\
f_x &= -\frac{2x}{(x^2+y^2+1)^2} \\
f_y &= -\frac{2y}{(x^2+y^2+1)^2} \\
f_{xx} &= -\frac{2}{(x^2+y^2+1)^2} + \frac{8x^2}{(x^2+y^2+1)^3} \\
f_{yy} &= -\frac{2}{(x^2+y^2+1)^2} + \frac{8y^2}{(x^2+y^2+1)^3} \\
f_{xy} &= \frac{8xy}{(x^2+y^2+1)^3}
\end{aligned}$$

At $(0,0)$,

$$\begin{aligned}
f &= 1 \\
f_x &= f_y = f_{xy} = 0 \\
f_{xx} &= f_{yy} = -2 \\
p_1 &= 1 \\
p_2 &= 1 - x^2 - y^2
\end{aligned}$$

2. (5 points) 4.1.15

$$\begin{aligned}
f_x &= 3x^2 + 2xy \\
f_y &= x^2 - z^2 \\
f_z &= -2yz + 6z^2 \\
f_{xx} &= 6x + 2y \\
f_{xy} &= 2x \\
f_{xz} &= = f_{yy} = 0 \\
f_{yz} &= -2z \\
f_{zz} &= -2y + 12z
\end{aligned}$$

$$Hf(1,0,1) = \begin{pmatrix} 6 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 12 \end{pmatrix}$$

3. (5 points) 4.1.17

$$p_2 = 1 + \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

4. (5 points) 4.1.18

$$\begin{aligned} f(1,0,1) &= \; 3 \\ Df(1,0,1) &= \; \begin{pmatrix} 3 & 0 & 6 \end{pmatrix} \\ p_2 &= \; 3 + \begin{pmatrix} 3 & 0 & 6 \end{pmatrix} \begin{pmatrix} x-1 \\ y \\ z-1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x-1 & y & z-1 \end{pmatrix} \begin{pmatrix} 6 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 12 \end{pmatrix} \begin{pmatrix} x-1 \\ y \\ z-1 \end{pmatrix} \end{aligned}$$

5. (10 points) 4.1.33

(a)

$$\begin{aligned} f &= \; \cos x \sin y \\ f_x &= \; -\sin x \sin y \\ f_y &= \; \cos x \cos y \\ f_{xx} &= \; -\cos x \sin y \\ f_{xy} &= \; -\sin x \cos y \\ f_{yy} &= \; -\cos x \sin y \end{aligned}$$

At $(0, \pi/2)$,

$$\begin{aligned} f_x &= \; f_y = f_{xy} = 0 \\ f &= \; 1 \\ f_{xx} &= \; f_{yy} = -1 \\ p_2 &= \; 1 - \frac{x^2}{2} - \frac{(y-\pi/2)^2}{2} \end{aligned}$$

(b)

$$\begin{aligned}
R_2(\mathbf{x},\mathbf{a}) = & \frac{1}{6}\sum_{i,j,k=1}^{2} f_{x_ix_jx_k}h_ih_jh_k \\
|R_2(\mathbf{x},\mathbf{a})| \leq & \frac{1}{6}\times 8\times |f_{x_ix_jx_k}||h_i||h_j||h_k| \\
\leq & \frac{8}{6}\times 1\times (0.3)^3 \\
\approx & 0.036
\end{aligned}$$

6. (10 points) 4.1.34

(a)

$$\begin{aligned}
f = & f_x = f_{xx} = e^{x+2y} \\
f_{xy} = & f_y = 2f \\
f_{yy} = & 4f
\end{aligned}$$

At $(0,0)$,

$$\begin{aligned}
f = & f_x = f_{xx} = 1 \\
f_{xy} = & f_y = 2 \\
f_{yy} = & 4 \\
p_2 = & 1 + x + 2y + \frac{x^2}{2} + 2y^2 + 2xy
\end{aligned}$$

(b)

$$\begin{aligned}
R_2(\mathbf{x},\mathbf{a}) = & \frac{1}{6}\sum_{i,j,k=1}^{2} f_{x_ix_jx_k}h_ih_jh_k \\
|R_2| \leq & \frac{1}{6}\times 8\times (1\times 3 + 3\times 4 + 3\times 2 + 1\times 1)\times e^{0.3}\times 0.1^3 \\
\approx & 0.0396
\end{aligned}$$

7. (5 points) 4.2.2(a,c)

(a)

$$\begin{aligned}
g_x = & 2x+2 \\
g_y = & -4y \\
(x,y) = & (-1,0)
\end{aligned}$$

(b) At $(-1, 0)$,

$$\begin{aligned} g_{xx} &= 2 \\ g_{yy} &= -4 \\ g_{xy} &= 0 \\ Hg(-1,0) &= \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix} \end{aligned}$$

Take $u_1 = (1, 0)$ and $u_2 = (0, 1)$. Since $u_1 H u_1^T > 0$ and $u_2 H u_2^T < 0$ and $det(H) \neq 0$, $(-1, 0)$ is a saddle point.

8. (5 points) 4.2.4

$$\begin{aligned} f &= \ln(x^2 + y^2 + 1) \\ f_x &= \frac{2x}{x^2 + y^2 + 1} \\ f_y &= \frac{2y}{x^2 + y^2 + 1} \end{aligned}$$

Therefore, $(0, 0)$ is the only critical point. Put $x = r\cos\theta, y = r\sin\theta$. $f = \ln(r^2 + 1) > \ln(1)$ for any values of r and θ. Thus, $(0, 0)$ is a local minimum point.

9. (5 points) 4.2.5

$$\begin{aligned} f &= x^2 + y^3 - 6xy + 3x + 6y \\ f_x &= 2x - 6y + 3 \\ f_y &= 3y^2 - 6x + 6 \end{aligned}$$

Therefore, $(3/2, 1)$ and $(27/2, 5)$ are the only critical points.

$$\begin{aligned} f_{xx} &= 2 > 0 \\ f_{yy} &= 6y \\ f_{xy} &= -6 \\ f_{xx}f_{yy} - (f_{xy})^2 &= 12y - 36 \end{aligned}$$

$\Rightarrow$ $(3/2, 1)$ is a saddle points since $12(1) - 36 < 0$ and $(27/2, 5)$ is a local minimum points since $f_{xx} > 0$ and $12(5) - 36 > 0$.

10. (5 points) 4.2.12

$$\begin{aligned} f &= e^{-x}(x^2+3y) \\ f_x &= e^{-x}(2x-x^2-3y^2) \\ f_y &= e^{-x}6y \end{aligned}$$

At $(0,0)$ and $(2,0)$, $f_x = f_y = 0$. Therefore they are both critical points.

$$\begin{aligned} f_{xx} &= e^{-x}(x^2-4x+3y^2+2) \\ f_{xy} &= -6ye^{-x} \\ f_{yy} &= 6e^{-x} \end{aligned}$$

At $(0,0)$, $f_{xx} = 2 > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = 12 - 36 < 0 \Rightarrow (0,0)$ is a local maximum point.

At $(2,0)$, $f_{xx} = -2e^{-2} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = e_{-4}(-12) < 0 \Rightarrow (2,0)$ is a saddle point.

11. (10 points) 4.2.22

(a)

$$\begin{aligned} f_x &= 2kx - 2y \\ f_y &= -2x + 2ky \end{aligned}$$

$f_x(0,0) = f_y(0,0) = 0$ for all k. Thus $(0,0)$ is a critical point. $f_{xx}f_{yy} - (f_{xy})^2 = 4k^2 - 4$ and $f_{xx} = 2k$. Therefore when $k > 1$, $f_x x > 0$ $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $(0,0)$ is a local minimum; when $k < -1$, $f_x x < 0$ $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $(0,0)$ is a local maximum.

(b)

$$\begin{aligned} f_x &= 2kx + kz \\ f_y &= -2z - 2y \\ f_z &= kx - 2y + 2kz \end{aligned}$$

$f_x(0,0,0) = f_y(0,0,0) = f_z(0,0,0) = 0$ for all k. Thus $(0,0)$ is a critical point.

$$
\begin{aligned}
d_1 &= 2k \\
d_2 &= -4k \\
d_3 &= -6k^2 - 8k
\end{aligned}
$$

When $k < -4/3$, $d_1 < 0$, $d_2 > 0$, $d_3 < 0$, and hence $(0,0,0)$ is a local maximum point.

12. (10 points) 4.2.23

(a)

$$
\begin{aligned}
f_x &= 2ax \\
f_y &= 2by \\
f_{xx} &= 2a \\
f_{xx}f_{yy} - (f_{xy})^2 &= 4ab
\end{aligned}
$$

$f_x = f_y = 0$ iff $x = y = 0$ Therefore the origin is the only critical point. And $(0,0)$ is a local minimum point if $a > 0$ and $b > 0$; $(0,0)$ is a local maximum point if $a < 0$ and $b < 0$; otherwise $(0,0)$ is a saddle point.

(b)

$$
\begin{aligned}
f_x &= 2ax \\
f_y &= 2by \\
f_z &= 2cz \\
d_1 &= 2a \\
d_2 &= 4ab \\
d_3 &= 8abc
\end{aligned}
$$

$f_x = f_y = f_z = 0$ iff $x = y = z = 0$ Therefore the origin is the only critical point. And $(0,0,0)$ is a local minimum point if $a > 0$, $b > 0$ and $c > 0$; $(0,0,0)$ is a local maximum point if $a < 0$, $b < 0$ and $c < 0$; otherwise $(0,0,0)$ is a saddle point.

(c) For $i = 1, \ldots, n$, $f_{x_i} = 2a_i x_i$ and hence $f_{x_1} = f_{x_2} = \ldots = f_{x_n} = 0$ iff $x_1 = x_2 = \ldots = x_n = 0$.

$d_1 = 2x_1$, $d_2 = 2^2 x_1 x_2, \ldots$, $d_n = 2^n x_1 x_2 \ldots x_n$. The origin is a local minimum if $x_1, x_2, \ldots, x_n$ are all positive; the origin is a local maximum if $x_1, x_2, \ldots, x_n$ are all negative; otherwise, the origin is a saddle point.

13. (10 points) 4.2.30

$$\begin{aligned} f_x &= 2x + y = 0 \\ f_y &= x + 2y - 6 = 0 \end{aligned}$$

$(-2, 4)$ is the only critical point at which $f_{xx} = 2 > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = 3 > 0$. Therefore, $(-2, 4)$ is a local minimum and $f(-2, 4) = -12$. We need to check the values on boundary.

$$\begin{aligned} -9 \leq \; & f(x, 0) &&= x^2 \leq 9 \\ -45/4 \leq \; & f(x, 5) &&= (x + 5/2)^2 - 45/4 \leq 19 \\ 27/4 \leq \; & f(3, y) &&= (y - 3/2)^2 + 27/4 \leq 19 \\ -9 \leq \; & f(-3, y) &&= (y - 9/2)^2 - 45/4 \leq 9 \end{aligned}$$

Therefore, $f(-2, 4) = -12$ is the abs. min. and $f(3, 5) = 19$ is the abs. max.

14. (5 points) 4.2.46(b)

$$\begin{aligned} f_x &= 3ye^x - 3e^{3x} = 0 \\ f_y &= 3e^x - 3y^2 = 0 \end{aligned}$$

$\Leftrightarrow (x, y) = (0, 1)$ and $f(0, 1) = 1$

$$\begin{aligned} f_{xx}(0, 1) &= -6 < 0 \\ f_{xx}f_{yy} - (f_{xy})^2 &= 27 > 0 \end{aligned}$$

$\Rightarrow (0, 1)$ is a local maximum. However, $f(0, y) = -y(y^2 + 3) - 1 \to \infty$ as $y \to -\infty$.