18.022 Problem set 9

1. 4.3.3

$$\nabla f = \langle 5, 2\rangle$$

$$\nabla g = \langle 10x, 4y\rangle$$

so $10x = \lambda 5$ and $4y = \lambda 2$. Solving this gives $x = y$. Now requireing that $g(x, y) = 14$ gives critical points $(x, y) = \pm(\sqrt{2}, \sqrt{2})$.

2. 4.3.8

$$\nabla f = \langle 1, 1, 1\rangle$$

$$\nabla g_1 = \langle -2x, 2y, 0\rangle$$

$$\nabla g_2 = \langle 1, 0, 2\rangle$$

we want to solve $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$. Looking at the second component gives $\lambda_1 = \frac{1}{2y}$. Looking at the third component gives $\lambda_2 = \frac{1}{2}$. This leaves us with the following equation from the first component:

$$1 = -\frac{x}{y} + \frac{1}{2}$$

so $y = -2x$ and z is unconstrained. We must now put this into our two constraint equations $g_1 = 1$ and $g_2 = 1$. The first gives the equation

$$4x^2 - x^2 = 1$$

so $x = \pm\frac{1}{\sqrt{3}}$. The second equation says that $z = \frac{1-x}{2}$, so we have the following 2 critical points:

$$(x, y, z) = \left(\pm\frac{1}{\sqrt{3}}, \mp\frac{2}{\sqrt{3}}, \frac{1 \mp \frac{1}{\sqrt{3}}}{2}\right)$$

3. 4.3.17

We wish to maximise the function $f(x, y, z) = xyz$ subject to the constraint that $g(x, y, z) = x + y + z = 18$ and $x > 0$, $y > 0$, $z > 0$.

$$\nabla f = \langle yz, xz, xy\rangle$$

$$\nabla g = \langle 1, 1, 1\rangle$$

solving $\nabla f = \lambda \nabla g$ gives that $xy = yz = zx$ and therefore $x = y = z$ if these numbers are non zero. Our critical point will then be $(6, 6, 6)$, giving $f = 6^3$. The value of f on the boundary of our region is 0 so this must be the maximum value of f.

4. 4.3.18 f is a continuous function, and therefore on any compact set, it will some maximum and minimum.

$$\nabla f = \langle 1, 1, -1 \rangle$$

$$\nabla g = \langle 2x, 2y, 2z \rangle$$

Solving $\nabla f = \lambda \nabla g$ gives that $(x, y, z) = (x, x, -x)$. Satisfying the constraint equation $g(x, y, z) = 81$ gives critical points at $\pm 3\sqrt{3}(1, 1, -1)$. The values of f at these points must be the maximum and minimum values. They are $f = \pm 9\sqrt{3}$.

5. 4.3.19

$$\nabla f = \langle 2x + y, 2y + x \rangle$$

The only critical point of our unconstrained f is at 0, where it has the value 0. We must now look for critical points of f constrained to the boundary, where $g(x, y) = x^2 + y^2 = 4$

$$\nabla g = \langle 2x, 2y \rangle$$

so we need to solve

$$2x + y = \lambda(2x)$$
$$x + 2y = \lambda(2y)$$

If $x = 0$, $y = 0$, which doesn't obey the constraint equation, so we can divide the first equation by x and the second by y to obtain

$$\frac{y}{x} = \frac{x}{y}$$

so we have critical points $(\pm\sqrt{2}, \pm\sqrt{2})$, and $(\pm\sqrt{2}, \mp\sqrt{2})$. At $\pm(\sqrt{2}, \sqrt{2})$, $f = 6$. At $\pm(\sqrt{2}, -\sqrt{2})$, $f = 2$. So the maximum and minimum values of f are 6 and 0.

6. 4.3.31

   (a)

$$\nabla f = \langle 1, 1 \rangle$$
$$\nabla g = \langle y, x \rangle$$

   Solving $\nabla f = \lambda \nabla g$ gives $x = y$. To satisfy $xy = 6$, we have $(x, y) = \pm\sqrt{6}(1, 1)$.

(b)

$$f(x, \frac{6}{x}) = x + \frac{6}{x}$$

We can see that this has no maximum by sending $x \to \infty$. It has no minimum because $\lim_{x\to-\infty} f(x, \frac{6}{x}) = -\infty$.

7. 5.2.1 (b) The region D is symmetric with respect to reflection in the y axis. Given some partition of D, we can reflect it to obtain another partition. The boxes of this new partition will have the same area, but the function x^3 will have the opposite sign. Therefore, the Riemann sum using the reflected integral partition will be (-1) times the original Riemann summ. So $\int\int_D x^3 dA = -\int\int_D x^3 dA$ therefore,

$$\int\int_D x^3 dA = 0$$

8. 5.2.10 The depth will be less than

$$4 \times 12 + 10 \times 11 + 10 \times 10 + 6 \times 9 + 5 \times 8 + 5 \times 7 + 5 \times 6 + 5 \times 5 = 442$$

and greater than

$$2 \times 10 + 9 \times 9 + 12 \times 8 + 7 \times 7 + 5 \times 6 + 5 \times 5 + 5 \times 4 + 5 \times 3 = 336$$

so to the nearest $100ft^3$, the pool has a volume of $400ft^3$.

9. 5.2.11

$$\int_{x=0}^{2}\int_{y=0}^{2-x}(1-xy)dydx = \int_0^2 (2 - x - \frac{1}{2}x(2-x)^2)dx = \frac{4}{3}$$

10. 5.2.16

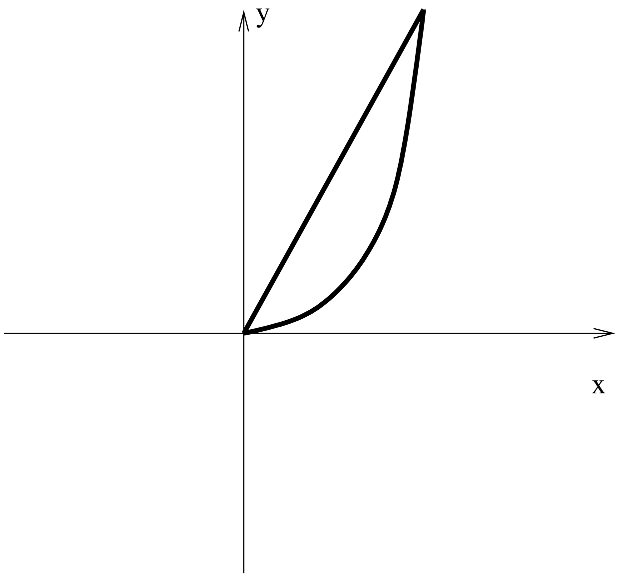
$$\begin{aligned}\int\int_D (x^2+y^2)dA &= \int_{x=0}^{1}\int_{y=x}^{3x}(x^2+y^2)dydx + \int_{x=1}^{x=\sqrt{3}}\int_{y=x}^{\frac{3}{x}}(x^2+y^2)dydx \\ &= \int_0^1 x^3(3-1+9-\frac{1}{3})dx + \int_1^{\sqrt{3}}(3x - x^3 + 9x^{-3} - \frac{1}{3}x^3)dx \\ &= \frac{8}{3} + \frac{9}{2} - \frac{3}{2} - \frac{1}{2} + \frac{9}{2} - 3 + \frac{1}{3} = 7\end{aligned}$$

11. 5.3.1

(a)

$$\int_0^2 \int_{x^2}^{2x} (2x+1)dydx = \int_0^2 (2x-x^2)(2x+1)dx = 4+8-8 = 4$$

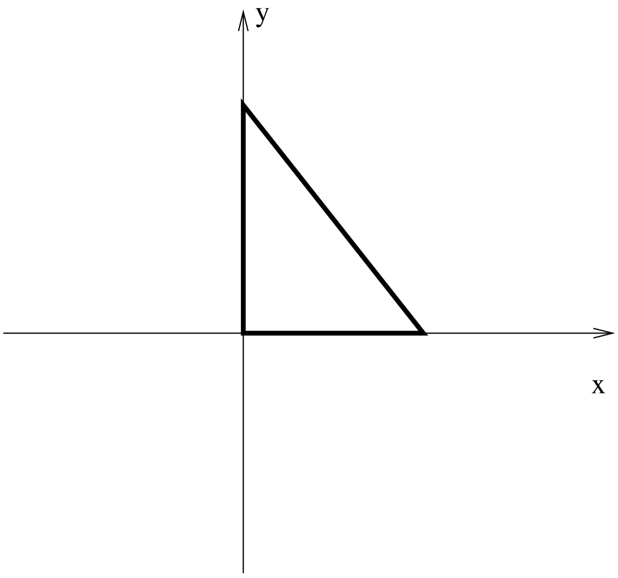
(b) Region of integration



(c)

$$\int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (2x+1)dxdy = \int_0^4 (y-\frac{y^2}{4}+\sqrt{y}-\frac{y}{2})dy = 4+\frac{16}{3}-\frac{16}{3} = 4$$

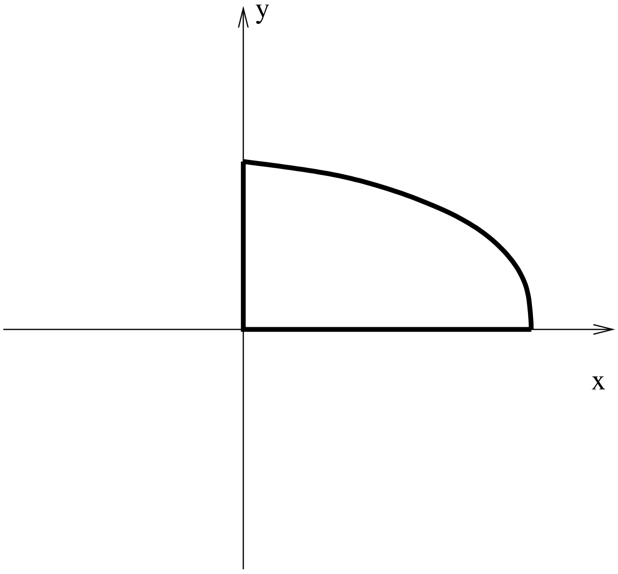
12. 5.3.3



$$\int_0^2 \int_0^{4-2x} ydydx = \int_0^1 2(2-x)^2 dx = \frac{16}{3}$$

$$\int_0^4 \int_0^{2-\frac{y}{2}} ydxdy = \int_0^4 (2y - \frac{y^2}{2})dy = 16 - \frac{32}{3} = \frac{16}{3}$$

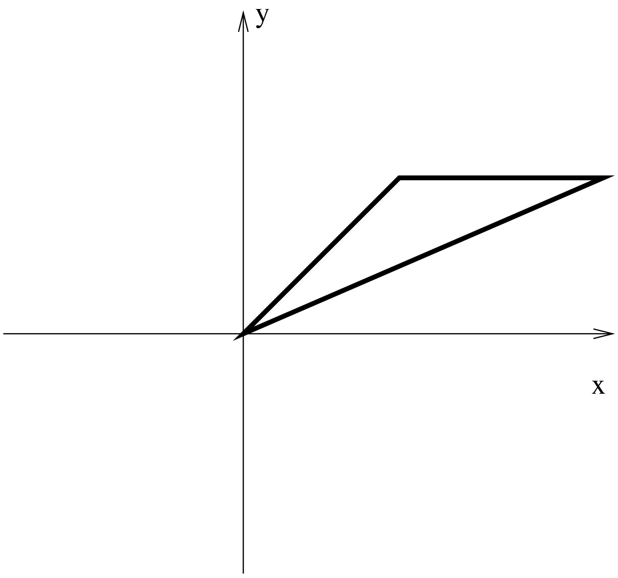
13. 5.3.4



$$\int_0^2 \int_0^{4-y^2} xdxdy = \int_0^2 \frac{1}{2}(4-y^2)^2 dy = 16 - \frac{32}{3} + \frac{16}{5} = \frac{128}{15}$$

$$\int_0^4 \int_1^{\sqrt{4-x}} xdxdy = \int_0^4 x\sqrt{4-x}dx = \int_0^4 (4-u)u^{\frac{1}{2}}du = \frac{64}{3} - \frac{64}{5} = \frac{128}{15}$$

14. 5.3.7



$$\int_0^1 \int_y^{2y} e^x dx dy = \int_0^1 e^{2y} - e^y dy = \frac{e^2}{2} - e + \frac{1}{2}$$

$$\begin{aligned} \int_0^1 \int_{\frac{x}{2}}^{x} e^x dy dx + \int_1^2 \int_{\frac{x}{2}}^{1} e^x dy dx &= \int_0^1 \frac{x}{2} e^x dx + \int_1^2 (1 - \frac{x}{2}) e^x dx \\ &= \frac{e}{2} - \frac{e}{2} + \frac{1}{2} - e^2 + \frac{e}{2} + \frac{3}{2} e^2 - \frac{3}{2} e \\ &= \frac{e^2}{2} - e + \frac{1}{2} \end{aligned}$$

15. 5.3.12

$$\int_0^1 \int_y^{2-y} \sin x dx dy = \int_0^1 (-\cos(2-y) + \cos y) dy = 2 \sin 1 - \sin 2$$