18.022 PSet 10 Solutions

December 4, 2006

Problem 1 (5.4.11, 5 points)
We wish to evaluate

$$\int_0^1 \int_{-2}^2 \int_0^{y^2} 2x - y - z dz dy dx = \int_0^1 \int_{-2}^2 2xy^2 - y^3 - \frac{y^4}{2} dy dx$$
$$= \int_0^1 \frac{32}{3}x - \frac{32}{5} dx$$
$$= \frac{16}{3} - \frac{32}{5} = \frac{-16}{15}.$$

Problem 2 (5.4.12, 5 points)
We wish to evaluate

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{2-x-z} y dy dz dx = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{(2-x-z)^2}{2} dz dx$$
$$= \frac{1}{6} \int_{-1}^1 \left(2 - x + \sqrt{1-x^2}\right)^3 - \left(2 - x - \sqrt{1-x^2}\right)^3 dx$$
$$= \frac{1}{3} \int_{-1}^1 \left(3(2-x)^2 + (1-x^2)\right) \sqrt{1-x^2} dx$$
$$= \frac{1}{3} \int_{-1}^1 \left(13 - 12x + 2x^2\right) \sqrt{1-x^2} dx.$$

Letting $x = \sin(\theta)$, we get that $dx = \cos(\theta) d\theta$. Hence our integral equals,

$$\frac{1}{3} \int_{-\pi/2}^{\pi/2} \left(13 - 12\sin(\theta) + 2\sin^2(\theta)\right) \cos^2(\theta) d\theta = \frac{1}{3}\left(13\frac{\pi}{2} - 12 \cdot 0 + 2\frac{\pi}{8}\right) = \frac{9\pi}{4}.$$

Problem 3 (5.4.15, 5 points)

The region is defined by $x, y, z \geq 0$ and $x + y/2 + z/3 \leq 1$. We wish to evaluate

$$\begin{aligned}\int_0^3 \int_0^{2(1-z/3)} \int_0^{(1-y/2-z/3)} 1 - z^2 dxdydz =& \int_0^3 \int_0^{2(1-z/3)} (1-z^2)(1-y/2-z/3)dydz \\ =& \int_0^3 (1-z^2)\left(2(1-z/3)^2 - (1-z/3)^2\right) dz \\ =& \int_0^3 (1-z^2)(1-z/3)^2 dz \\ =& \int_0^3 1 - \frac{2z}{3} - \frac{8z^2}{9} + \frac{2z^3}{3} - \frac{z^4}{9} \\ =& 3 - 3 - 8 + \frac{27}{2} - \frac{27}{5} \\ =& \frac{1}{10}.\end{aligned}$$

Problem 4 (5.4.19, 5 points)
We want the region where $4x^2 + y^2 \leq z \leq 2 - y^2$. Hence x and y must satisfy $2x^2 + y^2 \leq 1$. Therefore, we need to evaluate

$$\begin{aligned}& \int_{-1}^1 \int_{-\sqrt{(1-y^2)/2}}^{\sqrt{(1-y^2)/2}} \int_{4x^2+y^2}^{2-y^2} 1dzdxdy \\ =& \int_{-1}^1 \int_{-\sqrt{(1-y^2)/2}}^{\sqrt{(1-y^2)/2}} 2 - 4x^2 - 2y^2 dxdy \\ =& \int_{-1}^1 4\sqrt{(1-y^2)/2} - 4y^2\sqrt{(1-y^2)/2} - \frac{4(1-y^2)}{3}\sqrt{(1-y^2)/2}dy \\ =& \frac{4\sqrt{2}}{3} \int_{-1}^1 (1-y^2)^{3/2} dy.\end{aligned}$$

Now, letting $y = \sin(\theta)$, we have that $dy = \cos(\theta)d\theta$ and we get

$$\frac{4\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \cos^4(\theta)d\theta = \frac{4\sqrt{2}}{3}\pi\frac{6}{16} = \frac{\pi\sqrt{2}}{2}.$$

Problem 5 (5.4.22, 5 points)
Our region of integration is $0 \leq y \leq 1$, $0 \leq x \leq 1$, $0 \leq z \leq x^2$. We can perform the iterated integrals in the following orders (with the first letters being the outermost).
x, y, z gives

$$\int_0^1 \int_0^1 \int_0^{x^2} f(x, y, z)dzdydx.$$

x, z, y gives

$$\int_0^1 \int_0^{x^2} \int_0^1 f(x,y,z) dydzdx.$$

y, x, z gives

$$\int_0^1 \int_0^1 \int_0^{x^2} f(x,y,z) dzdxdy.$$

y, z, x gives

$$\int_0^1 \int_0^1 \int_0^{\sqrt{z}} f(x,y,z) dxdzdy.$$

z, x, y gives

$$\int_0^1 \int_0^{\sqrt{z}} \int_0^1 f(x,y,z) dydxdz.$$

z, y, x gives

$$\int_0^1 \int_0^1 \int_0^{\sqrt{z}} f(x,y,z) dxdydz.$$

Problem 6 (5.5.2, 5 points)
(a) T rotates the unit square, sending the corners to $(0,0), (1/\sqrt{2}, 1/\sqrt{2}), (0, 2/\sqrt{2}), (-1/\sqrt{2}, 1/\sqrt{2})$. This rotates the square 45° counter-clockwise about the origin.
(b) T rotates the unit square, sending the corners to $(0,0), (1/\sqrt{2}, 1/\sqrt{2}), (2/\sqrt{2}, 0), (1/\sqrt{2}, -1/\sqrt{2})$. This rotates the square 45° clockwise about the origin.

Problem 7 (5.5.6, 5 points)
T fixes the x-coordinate of a point, and scales the y-coordinate by the value of the x-coordinate. This has the effect of "pinching" the square down to the triangle with vertices $(0,0), (1,1), (1,0)$. T is not one-one since $T(0,0) = (0,0) = T(0,1)$.

Problem 8 (5.5.8, 5 points)
(a) The integral equals

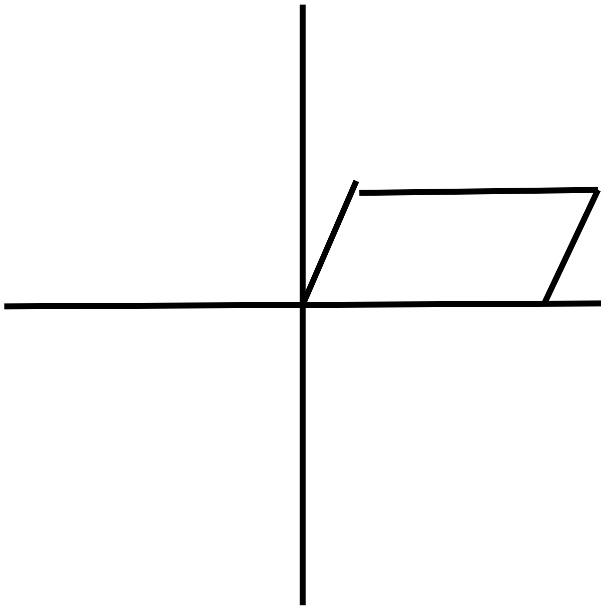
$$\int_0^1 (y/2+2)^2 - (y/2)^2 - 2ydy = \int_0^1 4dy = 4.$$

The region of integration is shown below.
(b) Our original region was defined by $0 \le y \le 1$ and $y/2 \le x \le y/2 + 2$, or $0 \le 2x - y \le 4$. Therefore, our region in the u, v-plane is $0 \le v \le 1$ and $0 \le u \le 4$.
(c) The matrix of partials is

$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}.$$

This has determinant 2, and therefore, we have that our expression equals

$$\frac{1}{2} \int_0^1 \int_0^4 ududv = \frac{1}{2} \int_0^1 8dv = 4.$$



Problem 9 (5.5.9, 5 points)
The region of integration is $0 \leq x \leq 2$ and $x/2 \leq y \leq x/2+1$, or $0 \leq u \leq 2$, $0 \leq v \leq 2$. The matrix of partials is

$$\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}.$$

This has determinant -2, and therefore our integral is

$$\begin{aligned} \frac{1}{2}\int_0^2 \int_0^2 u^5 v e^{v^2} dv du =& \frac{1}{4}\int_0^2 u^5(e^4-1)du \\ =& \frac{8(e^4-1)}{3}. \end{aligned}$$

Problem 10 (5.5.10, 5 points)
We change variables to the coordinates $u = x+y$ and $v = x-2y$. The matrix of partials is

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}.$$

This has determinant -3. Now the region D is enclosed by the lines $v = 0$, $u = v$ and $u = 1$. Hence our integral equals

$$\frac{1}{3}\int_0^1 \int_0^u \sqrt{\frac{u}{v}} dv du = \frac{2}{3}\int_0^1 u du = \frac{1}{3}.$$

Problem 11 (5.5.11, 5 points)
We change variables to the coordinates $u = 2x + y$ and $v = x - y$. The matrix of partials is

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

This has determinant -3. Now the region D is enclosed by the lines $u = 1$, $u = 4$, $v = -1$, $v = 1$. Hence our integral equals

$$\frac{1}{3}\int_1^4 \int_{-1}^1 u^2 e^v dvdu = \frac{1}{3}\int_1^4 u^2(e - e^{-1})du = 7(e - e^{-1}).$$

Problem 12 (5.5.12, 5 points)
We change variables to the coordinates $u = 2x + y - 3$ and $v = 2y - x + 6$. The matrix of partials is

$$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}.$$

This has determinant 2. Now the region D has corners in the (u, v) plane of $(-3, 6), (2, 6), (2, 1), (-3, 1)$. Hence our integral equals

$$\frac{1}{5}\int_{-3}^2 u^2 du \int_1^6 v^{-2} dv = \frac{1}{5}\left(\frac{8+27}{3}\right)\left(\frac{5}{6}\right) = \frac{35}{18}.$$

Problem 13 (On PSet, 10 points)
(a) Note that substituting $z = \sqrt{a}\sin(\theta)$ into

$$\int_{-\sqrt{a}}^{\sqrt{a}} \sqrt{a - z^2} dz$$

yields

$$\sqrt{a}\int -\pi/2^{\pi/2}\sqrt{a}\cos^2(\theta) = a\pi/2.$$

Therefore, the thing we wish to evaluate is

$$\begin{aligned}
&2\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \sqrt{1 - x^2 - y^2 - z^2} dzdydx \\
=&\pi \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 - x^2 - y^2 dydx \\
=&\pi \int_{-1}^1 2(1 - x^2)^{3/2}(1 - 1/3)dx \\
=&\frac{4\pi}{3}\int_0^1 (1 - x^2)^(3/2)dx
\end{aligned}$$

Letting $x = \sin(\theta)$, we get that this equals

$$\frac{4\pi}{3} \int_{-\pi/2}^{\pi/2} \cos^4(\theta) d\theta = \frac{4\pi}{3} \pi \frac{6}{16} = \frac{\pi^2}{2}.$$

(b) Note that the integral equals

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \int_{-\sqrt{1-x^2-y^2-z^2}}^{\sqrt{1-x^2-y^2-z^2}} dwdzdydx.$$

This is the volume of the region $x^2 + y^2 + z^2 + w^2 \leq 1$, or the volume of the unit ball in four dimensions.