18.022 Problem set 11

1. (Exercise 5.5.15)

   Defining $x = r\cos\theta$ and $y = r\sin\theta$, we obtain that $\partial(x,y)/\partial(r,\theta) = r$. This implies that

   $$\int\int_D (x^2+y^2)^{3/2} dA = \int_0^{2\pi}\int_0^3 r^3 \cdot r dr d\theta = 2\pi \cdot \left[\frac{r^5}{5}\right]_0^3 = \frac{486\pi}{5}.$$

2. (Exercise 5.5.17)

   With the same transformation as in the previous exercise, we obtain that

   $$0 \le y \le x \le 3 \Longleftrightarrow 0 \le r\sin\theta \le r\cos\theta \le 3 \Longleftrightarrow \begin{cases} 0 \le r \le 3/\cos\theta \\ 0 \le \theta \le \pi/4. \end{cases}$$

   Hence

   $$\begin{aligned} \int_0^3\int_0^x \frac{dydx}{\sqrt{x^2+y^2}} &= \int_0^{\pi/4}\int_0^{3/\cos\theta} drd\theta = \int_0^{\pi/4} \frac{3d\theta}{\cos\theta} \\ &= \int_0^{\pi/4} \frac{3d\theta}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}} = \int_0^{\pi/4} \frac{3(1+\tan^2\frac{\theta}{2})d\theta}{1-\tan^2\frac{\theta}{2}} \\ &= \begin{bmatrix} t = \tan\frac{\theta}{2} \\ dt = \frac{1}{2}(1+\tan^2\frac{\theta}{2})d\theta \\ 0 \le t \le \tan\frac{\pi}{8} = \sqrt{2}-1 \end{bmatrix} \\ &= \int_0^{\sqrt{2}-1} \frac{6dt}{1-t^2} = \int_0^{\sqrt{2}-1} \left(\frac{3}{1-t} + \frac{3}{1+t}\right) dt \\ &= \left[3\ln\frac{1+t}{1-t}\right]_0^{\sqrt{2}-1} = 3(\sqrt{2}+1). \end{aligned}$$

3. (Exercise 5.5.21)

   We have that $1 \le r \le 1 - \cos\theta$ and hence that

   $$\cos\theta \le 0 \Longleftrightarrow \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}.$$

It follows that the area equals

$$\begin{aligned}
\int_{\pi/2}^{3\pi/2}\int_{1}^{1-\cos\theta} r dr d\theta &= \int_{\pi/2}^{3\pi/2} \frac{1}{2}((1-\cos\theta)^2-1)d\theta \\
&= \int_{\pi/2}^{3\pi/2} \left(\frac{\cos^2\theta}{2} - \cos\theta\right) d\theta \\
&= \int_{\pi/2}^{3\pi/2} \left(\frac{\cos 2\theta + 1}{4} - \cos\theta\right) d\theta \\
&= \left[\frac{\sin^2 2\theta}{8} + \frac{\theta}{4} - \sin\theta\right]_{\pi/2}^{3\pi/2} = 2 + \frac{\pi}{4}.
\end{aligned}$$

4. (Exercise 5.5.23)

   With polar coordinates, we obtain that the region transforms into

$$\begin{cases} \tan^{-1}(\sqrt{3}) \leq \theta \leq \pi \Longleftrightarrow \frac{\pi}{3} \leq \theta \leq \pi \\ 0 \leq r \leq 1. \end{cases}$$

   As a consequence, the integral equals

$$\int_0^1 \int_{\pi/3}^{\pi} r\cos(r^2) d\theta dr = \frac{2\pi}{3}\left[\frac{\sin(r^2)}{2}\right]_0^1 = \frac{\pi \sin 1}{3}.$$

5. (Exercise 5.5.29)

   Switching to cylindrical coordinates, we obtain a region defined by the intersection of $r^2 \leq 1$ and $2r^2 + z^2 \leq 10$. Equivalently,

$$\begin{cases} |z| \leq \sqrt{10-2r^2} \\ 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi. \end{cases}$$

   As a consequence, the volume equals

$$\begin{aligned}
\int_0^{2\pi}\int_0^1\int_{-\sqrt{10-2r^2}}^{\sqrt{10-2r^2}} r dz dr d\theta &= 2\pi \int_0^1 2\sqrt{10-2r^2}\cdot r dr \\
&= 2\pi \left[-\frac{1}{3}(10-2r^2)^{3/2}\right]_0^1 \\
&= \frac{2\pi}{3}\left(10^{3/2} - 8^{3/2}\right) = \frac{4\sqrt{2}\pi}{3}\left(5\sqrt{5} - 8\right).
\end{aligned}$$

6. (Exercise 5.5.31)

Switching to cylindrical coordinates, we obtain a region defined by the inequalities $r^2 + z^2 \leq 25$ and $z \geq 3$. Equivalently,

$$\begin{cases} 0 \leq r \leq \sqrt{25 - z^2} \\ 3 \leq z \leq 5 \\ 0 \leq \theta \leq 2\pi. \end{cases}$$

As a consequence, the integral equals

$$\begin{aligned}
&= \int_0^{2\pi} \int_3^5 \int_0^{\sqrt{25-z^2}} (2 + r^2) r dr dz d\theta = 2\pi \int_3^5 \left[\frac{(2 + r^2)^2}{4} \right]_0^{\sqrt{25-z^2}} dz \\
&= \frac{\pi}{2} \int_3^5 ((27 - z^2)^2 - 4) dz = \frac{\pi}{2} \int_3^5 (z^4 - 54z^2 + 725) dz \\
&= \frac{\pi}{2} \left[\frac{z^5}{5} - 18z^3 + 725z \right]_3^5 dz = \frac{656\pi}{5}.
\end{aligned}$$

7. (Exercise 6.1.1)

(a) Note that $\mathbf{x}'(t) = \langle -3, 4 \rangle$; hence $\|\mathbf{x}'(t)\| = 5$. This implies that

$$\begin{aligned}
\int_{\mathbf{x}} f ds &= \int_0^2 f(\mathbf{x}(t)) \|\mathbf{x}'(t)\| dt \\
&= \int_0^2 25t dt = \left[\frac{25t^2}{2} \right]_0^2 = 50.
\end{aligned}$$

(b) Note that $\mathbf{x}'(t) = \langle -\sin t, \cos t \rangle$; hence $\|\mathbf{x}'(t)\| = 1$. This implies that

$$\begin{aligned}
\int_{\mathbf{x}} f ds &= \int_0^{\pi} f(\mathbf{x}(t)) \|\mathbf{x}'(t)\| dt \\
&= \int_0^{\pi} (\cos t + 2\sin t) dt = [\sin t - 2\cos t]_0^{\pi} = 4.
\end{aligned}$$

8. (Exercise 6.1.3)

Note that $\mathbf{x}'(t) = \langle 1, 1, 3\sqrt{t}/2 \rangle$; hence $\|\mathbf{x}'(t)\| = \sqrt{2 + 9t/4}$. This implies that

$$\begin{aligned}
\int_{\mathbf{x}} f ds &= \int_1^3 f(\mathbf{x}(t)) \|\mathbf{x}'(t)\| dt \\
&= \int_1^3 \sqrt{2 + 9t/4} dt = \left[\frac{8}{27} (2 + 9t/4)^{3/2} \right]_1^3 \\
&= \int_1^3 \frac{35^{3/2} - 17^{3/2}}{27}.
\end{aligned}$$

9. (Exercise 6.1.7)

Note that

$$\begin{aligned}\mathbf{F}(\mathbf{x}(t))\cdot\mathbf{x}'(t) &= \langle 2-\cos t, \sin t\rangle\cdot\langle\cos t,\sin t\rangle\\ &= 2\cos t-\cos^2 t+\sin^2 t = 2\cos t-\cos 2t.\end{aligned}$$

As a consequence,

$$\int_{\mathbf{x}}\mathbf{F}d\mathbf{s} = \int_0^{\pi/2}(2\cos t-\cos 2t)dt = \left[2\sin t-\frac{\sin 2t}{2}\right]_0^{\pi/2} = 2.$$

10. (Exercise 6.1.11)

Note that $\mathbf{x}'(t) = \langle -3\sin 3t, 3\cos 3t\rangle$ We hence obtain that

$$\begin{aligned}\int_{\mathbf{x}} xdy - ydx &= \int_0^{\pi}(\cos(3t)\cdot 3\cos(3t)-\sin(3t)\cdot(-3\sin(3t)))dt\\ &= \int_0^{\pi}3dt = 3\pi.\end{aligned}$$

11. (Exercise 6.1.13)

Note that

$$\mathbf{x}'(t) = e^{2t}\langle 2\cos 3t-3\sin 3t, 2\sin 3t+3\cos 3t\rangle.$$

We hence obtain that the integral equals

$$\begin{aligned}&\int_0^{2\pi}\frac{e^{4t}\cdot(\cos 3t(2\cos 3t-3\sin 3t)+\sin 3t(2\sin 3t+3\cos 3t))}{e^{6t}}dt\\ =& \int_0^{2\pi}2e^{-2t}dt = \int_0^{2\pi}\left[-e^{-2t}\right]_0^{2\pi} = 1-e^{-4\pi}.\end{aligned}$$

12. (Exercise 6.1.19)

Applying Green's formula and using the fact that the given trapezoid is defined by the inequalities $y \le x \le 3$ and $0 \le y \le 1$, we obtain that the integral equals

$$\begin{aligned}&\int_0^1\int_y^3\left(\frac{\partial}{\partial x}(-x-y)-\frac{\partial}{\partial y}(x^2y)\right)dxdy = \int_0^1\int_y^3(-1-x^2)dxdy\\ =& -\int_0^1\left[x+\frac{x^3}{3}\right]_y^3 dy = -\int_0^1\left(12-y-\frac{y^3}{3}\right)dy\\ =& -\left[12y-y^2/2-\frac{y^4}{12}\right]_0^1 = -12+1/2+\frac{1}{12} = -\frac{137}{12}.\end{aligned}$$

13. (Exercise 6.1.21)

Parametrize the line segment as

$$\begin{cases} x = 1 + 4t \\ y = 1 + 2t \\ z = 2 - t; \end{cases}$$

$0 \leq t \leq 1$. Since

$$\begin{aligned}
& yz \cdot \frac{dx}{dt} - xz \cdot \frac{dy}{dt} + xy \cdot \frac{dz}{dt} \\
= \quad & (1 + 2t)(2 - t) \cdot 4 - (1 + 4t)(2 - t) \cdot 2 + (1 + 4t)(1 + 2t) \cdot (-1) \\
= \quad & 8 + 12t - 8t^2 - 4 - 14t + 8t^2 - 1 - 6t - 8t^2 \\
= \quad & 3 - 8t - 8t^2,
\end{aligned}$$

we obtain that the integral equals

$$\int_0^1 \left(3 - 8t - 8t^2\right) dt = \left[3t - 4t^2 - \frac{8t^3}{3}\right]_0^1 = -\frac{11}{3}.$$