18.022 Problem set 12

1. (Exercise 6.2.5)

   (a) The region under consideration is defined by the inequalities $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Green's theorem yields that the integral equals

$$\int_0^1 \int_0^1 \left(\frac{\partial}{\partial x} (x^2) - \frac{\partial}{\partial y} (y^2) \right) dxdy = \int_0^1 \int_0^1 (2x - 2y) dxdy$$
$$= \int_0^1 \int_0^1 2xdxdy - \int_0^1 \int_0^1 2ydydx = 0.$$

   (b) The path C consists of four line segments C_1, C_2, C_3, C_4:

   - C_1 is defined by $\{(t, 0), 0 \leq t \leq 1\}$. We obtain that $dx = dt$ and $dy = 0$; hence
$$\int_{C_1} y^2 dx + x^2 dy = \int_0^1 0 dt = 0.$$
   - C_2 is defined by $\{(1, t), 0 \leq t \leq 1\}$. We obtain that $dx = 0$ and $dy = dt$; hence
$$\int_{C_2} y^2 dx + x^2 dy = \int_0^1 dt = 1.$$
   - C_3 is defined by $\{(1 - t, 1), 0 \leq t \leq 1\}$. We obtain that $dx = -dt$ and $dy = 0$; hence
$$\int_{C_3} y^2 dx + x^2 dy = -\int_0^1 dt = -1.$$
   - C_4 is defined by $\{(0, 1 - t), 0 \leq t \leq 1\}$. We obtain that $dx = 0$ and $dy = -dt$; hence
$$\int_{C_4} y^2 dx + x^2 dy = -\int_0^1 0 dt = 0.$$

   Summing, we obtain that
$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 0 + 1 + (-1) + 0 = 0.$$

2. (Exercise 6.2.6)

The region under consideration is defined by the inequalities $0 \le x \le 2$ and $-2 \le y \le \sqrt{1-(x-1)^2}$. Green's theorem yields that the integral equals

$$\begin{aligned}
&\int_0^2 \int_{-2}^{\sqrt{1-(x-1)^2}} \left(\frac{\partial}{\partial x}\left(2x^2\right) - \frac{\partial}{\partial y}(3xy) \right) dydx \\
=\ & \int_0^2 \int_{-2}^{\sqrt{1-(x-1)^2}} xdydx = \int_0^2 (\sqrt{1-(x-1)^2}+2)xdx \\
=\ & \int_0^2 \left(\sqrt{1-(x-1)^2}\cdot(x-1) + \sqrt{1-(x-1)^2} + 2x \right) dx \\
=\ & \left[-\frac{1}{3}(1-(x-1)^2)^{3/2} + x^2 \right]_0^2 + \int_0^2 \sqrt{1-(x-1)^2}\ dx \\
=\ & 4 + \int_0^2 \sqrt{1-(x-1)^2}\ dx.
\end{aligned}$$

Defining $x-1=\sin t$, we obtain $dx = \cos t\ dt$ and $0 \le x \le 2 \iff -\pi/2 \le t \le \pi/2$. Hence

$$\begin{aligned}
\int_0^2 \sqrt{1-(x-1)^2}\ dx &= \int_{-\pi/2}^{\pi/2} \sqrt{1-\sin^2 t}\cos t\ dt \\
&= \int_{-\pi/2}^{\pi/2} \cos^2 tdt = \int_{-\pi/2}^{\pi/2} \frac{\cos 2t + 1}{2} dt \\
&= \left[\frac{\sin 2t}{4} + \frac{t}{2} \right]_{-\pi/2}^{\pi/2} = \frac{\pi}{2}.
\end{aligned}$$

Summing, we obtain that the integral equals $4+\pi/2$.

To compute the integral directly, we note that the path C consists of three line segments C_1, C_2, C_4 and a half-circle C_3:

- C_1 is defined by $\{(t,-2), 0 \le t \le 2\}$. We obtain that $dx = dt$ and $dy = 0$; hence

$$\int_{C_1} 3xydx + 2x^2dy = \int_0^2 (-6t)dt = -3t^2 = -12.$$

- C_2 is defined by $\{(2,t), -2 \le t \le 0\}$. We obtain that $dx = 0$ and $dy = dt$; hence

$$\int_{C_2} 3xydx + 2x^2dy = \int_{-2}^0 8dt = 16.$$

- C_3 is defined by $\{(\cos t+1, \sin t), 0 \le t \le \pi\}$. We obtain that $dx = -\sin t\, dt$ and $dy = \cos t\, dt$; hence

$$\begin{aligned}
&\int_{C_3} 3xydx + 2x^2dy \\
= &\int_0^\pi (3(\cos t+1)\sin t(-\sin t) + 2(\cos t+1)^2\cos t)dt \\
= &\int_0^\pi (-3\cos t\sin^2 t + 2\cos^3 t - 3\sin^2 t + 4\cos^2 t + 2\cos t)dt \\
= &\int_0^\pi (-5\cos t\sin^2 t + 2\cos t - 3 + 7\cos^2 t + 2\cos t)dt \\
= &\left[\frac{-5\sin^3 t}{3} + 2\sin t - 3t + \frac{7\sin 2t}{4} + \frac{7t}{2} + 2\sin t\right]_0^\pi \\
= &\frac{\pi}{2}.
\end{aligned}$$

- C_4 is defined by $\{(0, -t), 0 \le t \le 2\}$. We obtain that $dx = 0$ and $dy = -dt$; hence

$$\int_{C_4} 3xydx + 2x^2dy = \int_0^2 0(-dt) = 0.$$

Summing, we obtain that

$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = -12 + 16 + \frac{\pi}{2} + 0 = 4 + \frac{\pi}{2}.$$

3. (Exercise 6.2.7)

The region under consideration is defined by the inequalities $0 \le x \le 1$ and $0 \le y \le 1$. Taking into account the orientation of the boundary, Green's theorem yields that the integral equals

$$\begin{aligned}
&-\int_0^1\int_0^1 \left(\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(x^2-y^2)\right) dxdy \\
= &-\int_0^1\int_0^1 (2x+2y)dxdy = -\int_0^1 \left[x^2+2xy\right]_0^1 dy \\
= &-\int_0^1 (1+2y)dy = -\left[y+y^2\right]_0^1 = -2.
\end{aligned}$$

4. (Exercise 6.2.8)

   Let D be the region bounded by the ellipse $x^2 + 4y^2 = 4 \iff (x/2)^2 + y^2 = 1$. The area of D equals $2 \cdot 1 \cdot \pi = 2\pi$. Green's theorem yields that the integral equals

$$\begin{aligned} & \int \int_D \left(\frac{\partial}{\partial x}(x - 4y) - \frac{\partial}{\partial y}(4y - 3x) \right) dxdy \\ = & \int_D (-3) dxdy = -3 \cdot \text{Area}(D) = -6\pi. \end{aligned}$$

5. (Exercise 6.2.9)

   Let C be the boundary of the rectangle oriented counterclockwise. By Green' theorem, we have that the area of the rectangle equals $\oint_C xdy$. The path C consists of four line segments C_1, C_2, C_3, C_4:

   - C_1 is defined by $\{(t, 0), 0 \le t \le a\}$. We obtain that $dx = dt$ and $dy = 0$; hence
   $$\int_{C_1} xdy = 0.$$
   - C_2 is defined by $\{(a, t), 0 \le t \le b\}$. We obtain that $dx = 0$ and $dy = dt$; hence
   $$\int_{C_2} xdy = \int_0^b adt = ab.$$
   - C_3 is defined by $\{(a - t, b), 0 \le t \le a\}$. We obtain that $dx = -dt$ and $dy = 0$; hence
   $$\int_{C_3} xdy = 0.$$
   - C_4 is defined by $\{(0, b - t), 0 \le t \le b\}$. We obtain that $dx = 0$ and $dy = -dt$; hence
   $$\int_{C_4} xdy = -\int_0^a 0dt = 0.$$

   Summing, we obtain that

$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 0 + ab + 0 + 0 = 0.$$

6. (Exercise 6.2.10)

   The region under consideration is bounded by the closed path consisting of the line segment $C_1 = \{(t,0) : 0 \le t \le 2\pi\}$ and the path $C_2 = \{(a(t-\sin t), a(1-\cos t)) : 0 \le t \le 2\pi\}$. On the first path, $dy = 0$. On the second path, $dy = a\sin t\; dt$. Using Green's theorem, we obtain that the area of the region equals

$$\begin{aligned}
\int_{C_1} xdy - \int_{C_2} xdy &= 0 - \int_0^{2\pi} a(t-\sin t)\cdot a\sin t\; dt \\
&= -a^2 \int_0^{2\pi} (t\sin t - \sin^2 t)dt \\
&= -a^2 \int_0^{2\pi} \left(t\sin t - \frac{1-\cos 2t}{2}\right) dt \\
&= -a^2 \left[\sin t - t\cos t - \frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} \\
&= -a^2(-2\pi - \pi) = 3\pi a^2.
\end{aligned}$$

7. (Exercise 6.2.12)

   We have that the area of the region D is given by

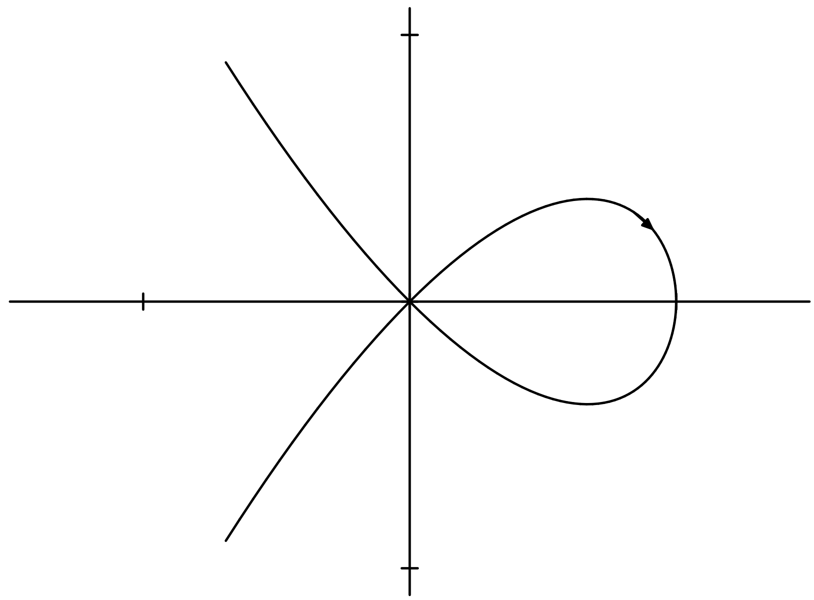
$$\frac{1}{2}\int_{\partial D} xdy - ydx.$$

   Since $dx = -3a\sin t\cos^2 t\; dt$ and $dy = 3a\cos t\sin^2 t\; dt$, we obtain that

$$\begin{aligned}
&\frac{1}{2}\int_{\partial D} xdy - ydx \\
=\;& \frac{1}{2}\int_0^{2\pi} \big(a\cos^3 t\cdot(3a\cos t\sin^2 t) - a\sin^3 t\cdot(-3a\sin t\cos^2 t)\big)\; dt \\
=\;& \frac{3a^2}{2}\int_0^{2\pi} \big(\cos^4 t\sin^2 t + \cos^2 t\sin^4 t\big)\; dt \\
=\;& \frac{3a^2}{2}\int_0^{2\pi} \cos^2 t\sin^2 t\; dt = \frac{3a^2}{2}\int_0^{2\pi} \frac{1+\cos 2t}{2}\cdot\frac{1-\cos 2t}{2} dt \\
=\;& \frac{3a^2}{2}\int_0^{2\pi} \frac{1-\cos^2 2t}{4} dt = \frac{3a^2}{2}\int_0^{2\pi}\left(\frac{1}{4} - \frac{1+\cos 4t}{8}\right) dt \\
=\;& \frac{3a^2}{2}\left[\frac{t}{4} - \frac{t}{8} - \frac{\sin 4t}{32}\right]_0^{2\pi} = \frac{3a^2}{2}\cdot\frac{2\pi}{8} = \frac{3\pi a^2}{8}.
\end{aligned}$$

8. (Exercise 6.2.13)

(a)



(b) The region under consideration is bounded by the path

$$C = \{(1 - t^2, t^3 - t) : -1 \leq t \leq 1\}.$$

Note that this path has a clockwise orientation.

By Green's Theorem, the area is given by

$$\begin{aligned} \oint_C y dx &= \int_{-1}^{1} (t^3 - t) \cdot (-2t) dt = -2 \int_{-1}^{1} (t^4 - t^2) dt \\ &= -2 \left[\frac{t^5}{5} - \frac{t^3}{3} \right]_{-1}^{1} = -2 \left(\frac{2}{5} - \frac{2}{3} \right) = \frac{8}{15}. \end{aligned}$$

9. (Exercise 6.2.19)

Assume that C is oriented counterclockwise; if C is oriented clockwise, then just reorient it and switch sign of the integral. C is the boundary of a region D. Applying Green's theorem, we obtain that the integral equals

$$\begin{aligned} \int \int_D \left(\frac{\partial}{\partial x}(x^3) - \frac{\partial}{\partial y}(3x^2 y) \right) dx dy &= \int \int_D (3x^2 - 3x^2) dx dy \\ &= \int \int_D 0 dx dy = 0. \end{aligned}$$

10. (Exercise 6.2.24)

Let D be the region bounded by C. Applying Green's theorem, we obtain that

$$\oint_C \frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial x} dy = \int\int_D \left(-\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} \right) dxdy$$

$$= -\int\int_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dxdy = -\int\int_D 0 \cdot dxdy = 0.$$