**18.022: Multivariable calculus - solution to pset 13, ver 1 - fall 2006**

1. (25 points)

   (a) Using Frenet-Serret formulas (p.206 of Susan Jane Colley) and the relations $\overrightarrow{T} \times \overrightarrow{N} = \overrightarrow{B}, \overrightarrow{N} \times \overrightarrow{B} = \overrightarrow{T}, \overrightarrow{B} \times \overrightarrow{T} = \overrightarrow{N}$, we have

$$\begin{aligned}
\frac{\partial g}{\partial s} &= \overrightarrow{x}'(s) + r\cos \overrightarrow{N}'(s) + r\sin t \overrightarrow{B}'(s) \\
&= \overrightarrow{T}(s) + r\cos t(-\kappa \overrightarrow{T}(s) + \tau \overrightarrow{B}(s)) - r\sin t \tau \overrightarrow{N}(s) \\
&= (1 - r\kappa \cos t)\overrightarrow{T} + r\tau \cos \overrightarrow{B} - r\tau \sin t \overrightarrow{N} \\
\frac{\partial g}{\partial t} &= -r\sin t \overrightarrow{N}(s) + r\cos t \overrightarrow{B}(s) \\
\frac{\partial g}{\partial s} \times \frac{\partial g}{\partial t} &= r\sin t(r\kappa \cos t - 1)\overrightarrow{B}(s) + r\cos t(r\kappa \cos t - 1)\overrightarrow{N}(s)
\end{aligned}$$

   (b) Since r is less than $1/\kappa$,

$$\|\frac{\partial g}{\partial s} \times \frac{\partial g}{\partial t}\| = r(1 - r\kappa \cos t)$$

$$\begin{aligned}
\int_a^b \int_0^{2\pi} \|\frac{\partial g}{\partial s} \times \frac{\partial g}{\partial t}\| dtds &= \int_a^b 2\pi r ds \\
&= 2\pi r \times \text{ length of } C
\end{aligned}$$

2. (5 points) 6.3.3

$$\nabla \times \overrightarrow{F} = \frac{\partial}{\partial x} e^{xy} - \frac{\partial}{\partial y} e^{x+y} \neq 0$$

   Therefore, $\overrightarrow{F}$ is not conservative.

3. (5 points) 6.3.14

$$\nabla \times \overrightarrow{G} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} 2xy \\ x^2 + 2yz \\ y^2 \end{pmatrix} = \begin{pmatrix} 2y - 2y \\ 0 - 0 \\ 2x - 2x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

   Therefore, $\overrightarrow{G}$ is conservative (therefore, $\overrightarrow{F}$ is not).

$$
\begin{aligned}
f_x &= 2xy \\
f_y &= x^2 + 2yz \\
f_z &= y^2 \\
f &= x^2y + g(y,z) \\
f_y &= x^2 + g_y \\
g_y &= 2yz \\
g &= y^2z + h(z) \\
f_z &= g_z = y^2 + h_z \\
h_z &= 0 \\
h &= const \\
f &= x^2y + y^2z + const
\end{aligned}
$$

4. (5 points) 6.3.15

$$
\begin{aligned}
\nabla \times \overrightarrow{F} = N_x - \frac{\partial}{\partial y}(ye^{2x} + 3x^2e^y) &= 0 \\
N_x &= e^{2x} + 2x^2e^y \\
N &= \frac{1}{2}e^{2x} + x^3e^y + g(y)
\end{aligned}
$$

where g is a function of class $\mathcal{C}^2$.

5. (5 points) 6.3.18

$$
\begin{aligned}
\nabla \times \overrightarrow{F} = (-5) - (-5) &= 0 \\
f_x &= 3x - 5y \\
f_y &= 7y - 5x \\
f &= \frac{3}{2}x^2 - 5xy + g(y) \\
f_y &= -5x + g_y \\
g_y &= 7y \\
g &= \frac{7}{2}y^2 + const \\
f &= \frac{3}{2}x^2 - 5xy + \frac{7}{2}y^2 + const
\end{aligned}
$$

Theorem 3.3

$$f(5,2) - f(1,3) = 3/2 - 18 = -33/2$$

Direct integration

$$\begin{aligned}
(x, y) = &\quad (1 + 4t, 3 - t) \text{ for } 0 \leq t \leq 1 \\
dx = &\quad 4dt \\
dy = &\quad -dt
\end{aligned}$$

$$\int_0^1 [3(1 + 4t) - 5(3 - t)](4dt) \; + \int_0^1 [7(3 - t) - 5(1 + 4t)](-dt) = -33/2$$

6. (5 points) 7.1.3

$$\begin{aligned}
T_s = &\quad \begin{pmatrix} e^s \\ 2t^2 e^{2s} \\ -2e^{-s} \end{pmatrix} \\
T_t = &\quad \begin{pmatrix} 0 \\ 2te^{2s} \\ 1 \end{pmatrix} \\
T_s(0, -2) = &\quad \begin{pmatrix} 1 \\ 8 \\ -2 \end{pmatrix} \\
T_t(0, -2) = &\quad \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} \\
N = &\quad T_s \times T_t = \begin{pmatrix} 0 \\ -1 \\ -4 \end{pmatrix}
\end{aligned}$$

Surface:

$$\begin{aligned}
(0, -1, -4) \cdot (x - 1, y - 4, z) = &\quad 0 \\
y + 4z = &\quad 4
\end{aligned}$$

7. (5 points) 7.1.8

$$\mathbf{X} = (x, y, z) = (2 \sin s \cos t, 3 \sin s \sin t, \cos s)$$

$$(x/2)^2 + (y/3)^2 + z^2 = 1$$

which is an ellipsoid with $a = 2, b = 3, c = 1$ (p.89 of Susan Jane Colley) and from spherical coordinates we can see $\mathbf{X}$ maps $0 \leq s \leq \pi$, $0 \leq t \leq 2\pi$ to the whole ellipsoid.

8. (5 points) 7.1.18

$$\begin{aligned}
X =& \ (a\cos t, a\sin t, s) \text{ for } 0 \leq t \leq 2\pi, 0 \leq s \leq h \\
T_s =& \ (0,0,1) \\
T_t =& \ (-a\sin t, a\cos t, 0) \\
T_s \times T_t =& \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -a\sin t \\ a\cos t \\ 0 \end{pmatrix} = \begin{pmatrix} -a\cos t \\ -a\sin t \\ 0 \end{pmatrix} \\
\|T_s \times T_t\| =& \ a
\end{aligned}$$

Surface area:

$$\int_0^{2\pi} adsdt = 2\pi ah$$

9. (10 points) 7.1.22

Let $z = f(x,y) = 9 - x^2 - y^2$. Then $f_x = -2x$ and $f_y = -2y$. Surface area=$\int\int_D \sqrt{4x^2 + 4y^2 + 1}dxdy$. Let $x = s\cos t$ and $y = s\sin t$. Then

$$\begin{aligned}
& \int_0^3 \int_0^{2\pi} \sqrt{4s^2+1}(sdsdt) \\
=& \ \frac{2\pi}{8} \int_0^3 (4s^2+1)^{1/2} d(4s^2+1) \\
=& \ \frac{\pi}{4} \times \frac{2}{3}[37^{2/3} - 1] \\
=& \ \frac{\pi}{6}[37^{3/2} - 1]
\end{aligned}$$

10. (10 points) 7.2.3

$$\begin{aligned}
X(s,t) =& \ (s, t, 2 - 2s + 2t) \text{ for } 0 \leq s \leq 1, -1 \leq t \leq -1 + s \\
T_s =& \ (1, 0, -2) \\
T_t =& \ (0, 1, 2) \\
N =& \ T_s \times T_t = (2, -2, 1)
\end{aligned}$$

$$\int_X \overrightarrow{F} d\overrightarrow{s} = \int_0^1 \int_{-1}^{-1+s} (s, t, 2-2s+2t) \cdot (2, -2, 1) dt ds = \int_0^1 \int_{-1}^{-1+s} 2 dt ds = 1$$

11. (10 points) 7.2.10

12. (10 points) 7.2.14

    Let $X_1$ be a parametrization of the lateral surface, $X_2$ and $X_3$ be two parametrization of the top and the bottom of the cylinder respectively. Then

$$\begin{aligned}
X_1 &= (3\cos t, 3\sin t, s), 0 \le t \le 2\pi, 0 \le s \le 4 \\
N_1 &= (-3\cos t, -3\sin t, 0) \text{ and } \|N_1\| = 3 \\
X_2 &= (s\cos t, s\sin t, 4), 0 \le t \le 2\pi, 0 \le s \le 3 \\
X_3 &= (s\cos t, s\sin t, 0), 0 \le t \le 2\pi, 0 \le s \le 3 \\
N_2 = N_3 &= (0, 0, s) \text{ and } \|N_2\| = \|N_3\| = s
\end{aligned}$$

7.2.10:

$$\begin{aligned}
\int\int_S z ds &= \int_0^{2\pi} \int_0^4 (s)(3) ds dt + \int_0^{2\pi} \int_0^3 (4)(s) ds dt \\
&= (\frac{3}{2} 4^2)(2\pi) + (\frac{4}{2} 3^2)(2\pi) \\
&= 84\pi
\end{aligned}$$

7.2.14

$$\begin{aligned}
\int\int_S (x\overrightarrow{i} + y\overrightarrow{j}) \cdot d\overrightarrow{s} &= \int_0^{2\pi} \int_0^4 \begin{pmatrix} 3\cos t \\ 3\sin t \\ s \end{pmatrix} \cdot \begin{pmatrix} -3\cos t \\ -3\sin t \\ 0 \end{pmatrix} ds dt \\
&= 72\pi
\end{aligned}$$