18.022 Problem set 14

1. 7.3.11
$$\nabla \times F = \langle x^2 + ye^x \sin yz, 5 + xy, e^x \cos yz - 2xz \rangle$$
We have that if $\partial S_2 = \partial S$, then
$$\int\int_S \nabla \times \mathbf{F} \cdot \mathbf{n} dS = \int_{\partial S} \mathbf{F} \cdot \mathbf{ds} = \int\int_{S_2} \nabla \times \mathbf{F} \cdot \mathbf{n} dS$$
Define S_2 to be the disk $S_2 := \{y = 1, x^2 + z^2 \leq 9\}$. The rightward pointing unit normal to this is $(0, 1, 0)$ (although the meaning of 'rightward' is not entirely clear, we will take it to mean having a positive y component). So we are interested in the integral
$$\int\int_{x^2+z^2\leq 9} (5 + x) dxdz = 45\pi$$

2. 7.3.12 The integral will be the same over any surface with the same boundary, so we can choose $\tilde{S} := \{z = 0, 4 \geq 4x^2 + y^2\}$. The appropriate unit normal to $\tilde{S}$ is $(0, 0, 1)$. We therfore only have to calculate the third component of $\nabla \times \mathbf{F}$.
$$\nabla \times \mathbf{F} = \langle ?, ?, 0 \rangle$$
Therefore
$$\int\int_{\tilde{S}} (\nabla \times \mathbf{F} \cdot \mathbf{n}) dS = 0$$

3. 7.3.13

   (a) $\sin 2t = 2 \cos t \sin t$ is the double angle formula for sin, so $(\cos t, \sin t, \sin 2t)$ is on the surface $z = 2xy$.

   (b) $C = \partial S$, where $S = \{z = 2xy, x^2 + y^2 \leq 1\}$ with normal pointing upwards.
$$\int_C (y^3 + \cos x)dx + (\sin y + z^2)dy + xdz = \int\int_S \langle -2z, -1, -3y^2 \rangle \cdot \mathbf{n} dS$$
$$= \int\int_{x^2+y^2\leq 1} \langle -2z, -1, -3y^2 \rangle \cdot \langle -2y, -2x, 1 \rangle dxdy$$
$$= \int\int_{x^2+y^2\leq 1} (8xy^2 + 2x - 3y^2) dxdy$$
$$= \int_0^{2\pi} \int_0^1 -3r^3 \sin^2 \theta dr d\theta$$
$$= -\frac{3}{4}\pi$$

4. 7.3.16

   Define $\tilde{S} := \{z = 0, 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and orient it so that $\mathbf{n} = (\mathbf{0}, \mathbf{0}, -\mathbf{1})$. If we orient S so that the normal vector points out of the unit cube, then $\tilde{S} \cup S$ is the boundary of the unit cube. Then we can use Gauss' theorem to say the following

$$\begin{aligned}
\int\int_S \mathbf{F}.\mathbf{n} dS &= -\int\int_{\tilde{S}} \mathbf{F}.\mathbf{n} dS + \int_0^1\int_0^1\int_0^1 \nabla\cdot\mathbf{F} dxdydz \\
&= -\int_0^1\int_0^1 -2dxdy + \int_0^1\int_0^1\int_0^1 (2xze^{x^2} + 3 + 7yz^6)dxdydz \\
&= 2 + \int_0^1\int_0^1 (z(e-1) + 3 + 7yz^6)dydz \\
&= 2 + \int_0^1 (z(e-1) + 3 + \frac{7}{2}z^6)dz \\
&= 2 + \frac{e}{2} - \frac{1}{2} + 3 + \frac{1}{2} = 5 + \frac{e}{2}
\end{aligned}$$

5. 7.3.18

   (a)

$$\int\int\int_D \nabla\cdot\mathbf{F} dV = \int\int_{S_7} \mathbf{F}\cdot\mathbf{n} dS - \int\int_{S_5} \mathbf{F}\cdot\mathbf{n} dS = 2a$$

   (b) If $\mathbf{F} = \nabla \times \mathbf{G}$, then Stokes theorem tells us that the integral of $\mathbf{F}$ on any smooth compact surface without boundary is 0 so $a = b = 0$. (Note that $b = 0$ anyway if F is C^1 because the integral on S_r must be small if r is small.)

6. 7.3.19

   (a) On S, $\frac{\partial f}{\partial n} = \frac{\partial f}{\partial \rho} = \frac{2}{\rho}$ where ρ is the radial coordinate in spherical coordiantes. We therefore have

$$\int\int_S \frac{\partial f}{\partial n} dS = \int\int_S \frac{2}{a} dS = \frac{2}{a}(\frac{\pi a^2}{2}) = \pi a$$

   (b)

$$\nabla\cdot(\nabla\mathbf{f}) = \nabla\cdot\langle\frac{2x}{\rho^2}, \frac{2y}{\rho^2}, \frac{2z}{\rho^2}\rangle = \frac{2}{\rho^2}$$

   We can compute our integral in spherical coordinates

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^a \frac{2}{\rho^2}\rho^2 \sin\phi d\rho d\phi d\theta = \pi a$$

(c) Gauss's theorem does not apply to this region because ∇f is not C^1 on this region. We could however apply Gauss's theorem to the region which is given by $\epsilon \leq \rho \leq a$ intersected with the positive octant. The integral on this region is given by $\pi(a-\epsilon)$, the integral of $\frac{\partial f}{\partial n}$ on the extra piece of surface is $\pi\epsilon$, and $\frac{\partial f}{\partial n} = 0$ on the parts of the boundary of this region given by the coordinate planes.

7.

$$dx = x_u du + x_v dv + x_w dw$$
$$dy = y_u du + y_v dv + y_w dw$$
$$dz = z_u du + z_v dv + z_w dw$$

so

$$\begin{aligned} dx \wedge dy &= (x_u du + x_v dv + x_w dw) \wedge (y_u du + y_v dv + y_w dw) \\ &= (x_v y_w - x_w y_u) dv \wedge dw + (x_w y_u - x_u y_w) dw \wedge du + (x_u y_v - x_v y_u) du \wedge dv \end{aligned}$$

$$dy \wedge dz = (y_v z_w - y_w z_v) dv \wedge dw + (y_w z_u - y_u z_w) dw \wedge du + (y_u z_v - y_v z_u) du \wedge dv$$
$$dz \wedge dx = (z_v x_w - z_w x_v) dv \wedge dw + (z_w x_u - z_u x_w) dw \wedge du + (z_u x_v - z_v x_u) du \wedge dv$$

This means that if

$$F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy = G_1 dv \wedge dw + G_2 dw \wedge du + G_3 du \wedge dv$$

then

$$G_1 = (y_v z_w - y_w z_v) F_1 + (z_v x_w - z_w x_v) F_2 + (x_v y_w - x_w y_u) F_3$$
$$G_2 = (y_w z_u - y_u z_w) F_1 + (z_w x_u - z_u x_w) F_2 + (x_w y_u - x_u y_w) F_3$$
$$G_3 = (y_u z_v - y_v z_u) F_1 + (z_u x_v - z_v x_u) F_2 + (x_u y_v - x_v y_u) F_3$$