

## 1 Algebraic $K$ -theory and the $p$ -adic $L$ -function

We let  $p$  be an odd prime number and define  $F_p$  to be the  $p$ -completion of the homotopy fiber of the cyclotomic trace map

$$\mathrm{tr}: K(\mathbb{Z}) \rightarrow \mathrm{TC}(\mathbb{Z}; p).$$

Then we have a long-exact sequence of  $p$ -adic homotopy groups

$$\cdots \rightarrow \pi_q(F_p) \rightarrow K_q(\mathbb{Z}; \mathbb{Z}_p) \xrightarrow{\mathrm{tr}_q} \mathrm{TC}_q(\mathbb{Z}; p, \mathbb{Z}_p) \rightarrow \pi_{q-1}(F_p) \rightarrow \cdots$$

We will argue that the following (1)–(3) hold:

1. The value  $L_p(\mathbb{Q}, \omega^{-2k}, 1 + 2k)$  is non-zero if and only if  $\pi_{4k+1}(F_p)$  is zero.
2. If  $L_p(\mathbb{Q}, \omega^{-2k}, 1 + 2k) \neq 0$ , then  $\pi_{4k}(F_p)$  and  $\pi_{4k-1}(F_p)$  are finite and

$$|L_p(\mathbb{Q}, \omega^{-2k}, 1 + 2k)|_p = \#\pi_{4k-1}(F_p) / \#\pi_{4k}(F_p).$$

3. There is an exact sequence

$$0 \rightarrow \mathbb{Z}_p \rightarrow \pi_{4k-2}(F_p) \rightarrow K_{4k-2}(\mathbb{Z}, \mathbb{Z}_p) \rightarrow 0.$$

Here and below, we write  $|a|_p = p^{-v_p(a)}$  for the  $p$ -adic absolute value of  $a$ .

We briefly recall the  $p$ -adic  $L$ -function. The Dirichlet series associated with the Dirichlet character  $\chi$  is defined by

$$L(\mathbb{Q}, \chi, s) = \sum_{n=1}^{\infty} \chi(n)n^{-s} = \prod_p (1 - \chi(p)p^{-s})^{-1}.$$

The series converges for  $\mathrm{Re}(s) > 1$  and admits a unique extension to the meromorphic function  $L(\mathbb{Q}, \chi, s)$  called the Dirichlet  $L$ -function. The values of this function at negative integers are rational numbers. We remark that  $L(\mathbb{Q}, 1, s) = \zeta(s)$  is the Riemann zeta function. Let  $\omega$  be the Teichmüller character. The Kubota-Leopoldt  $p$ -adic  $L$ -function associated with  $\chi$  is the unique continuous function

$$L_p(\mathbb{Q}, \chi, -): \mathbb{Z}_p \setminus \{1\} \rightarrow \mathbb{Q}_p$$

with the property that

$$L_p(\mathbb{Q}, \chi, 1 - i) = (1 - (\chi\omega^{-i})(p)p^{i-1})L(\mathbb{Q}, \chi\omega^{-i}, 1 - i)$$

for all integers  $i \geq 1$ . In this sense, the  $p$ -adic  $L$ -function interpolates the values of the Dirichlet  $L$ -function with the Euler factor corresponding to the prime  $p$  removed. In particular,  $L_p(\mathbb{Q}, \omega^i, 1 - i) = (1 - p^{i-1})\zeta(1 - i)$  for  $i \geq 1$ .

We wish to understand the kernel and cokernel of the cyclotomic trace map

$$\mathrm{tr}_q: K_q(\mathbb{Z}, \mathbb{Z}_p) \rightarrow \mathrm{TC}_q(\mathbb{Z}; p, \mathbb{Z}_p).$$

Let  $f: \mathbb{Z} \rightarrow \mathbb{Z}_p$  be the canonical ring homomorphism. Then, in the diagram

$$\begin{array}{ccc} K_q(\mathbb{Z}, \mathbb{Z}_p) & \xrightarrow{f_q} & K_q(\mathbb{Z}_p, \mathbb{Z}_p) \\ \downarrow \text{tr}_q & & \downarrow \text{tr}_q \\ \text{TC}_q(\mathbb{Z}; p, \mathbb{Z}_p) & \xrightarrow{f_q} & \text{TC}_q(\mathbb{Z}_p; p, \mathbb{Z}_p), \end{array}$$

the right-hand vertical map  $\text{tr}_q$  and lower horizontal map  $f_q$  are isomorphisms. Hence, the kernel and cokernel of the left-hand vertical map  $\text{tr}_q$  are isomorphic to the kernel and cokernel, respectively, of the upper horizontal map  $f_q$ . The latter, in turn, are isomorphic to the kernel and cokernel of the map

$$K_q(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p) \xrightarrow{f_q} K_q(\mathbb{Q}_p, \mathbb{Z}_p).$$

Indeed, this follows immediately from the localization sequence in  $K$ -theory. To understand this map, we consider the induced map of the spectral sequences from motivic cohomology to algebraic  $K$ -theory,

$$\begin{aligned} E_{s,t}^2 &= H^{t-s}(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(t)) \Rightarrow K_{s+t}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p) \\ E_{s,t}^2 &= H^{t-s}(\text{Spec } \mathbb{Q}_p, \mathbb{Z}_p(t)) \Rightarrow K_{s+t}(\mathbb{Q}_p, \mathbb{Z}_p). \end{aligned}$$

The motivic cohomology groups in the  $E^2$ -terms are isomorphic to the corresponding étale cohomology groups, if  $s \leq t$ , and are zero, otherwise. Indeed, this is the statement of the affirmed Beilinson-Lichtenbaum conjectures. We will not distinguish notationally between motivic cohomology groups and the associated étale cohomology groups.

Let  $i \neq 0, 1$  be an integer which may be positive or negative. Then the étale cohomology groups  $H^q(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i))$  and  $H^q(\text{Spec } \mathbb{Q}_p, \mathbb{Z}_p(i))$  are finitely generated  $\mathbb{Z}_p$ -modules which are non-zero for  $q = 1, 2$  only [4, §1, Satz 5; §2, Satz 2; §3, Satz 4; §4, Lemmas 2 and 5]. Hence, for  $i > 1$  and  $j = 1, 2$ , we have a commutative diagram

$$\begin{array}{ccc} K_{2i-j}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p) & \xrightarrow{f_{2i-j}} & K_{2i-j}(\mathbb{Q}_p, \mathbb{Z}_p) \\ \uparrow & & \uparrow \\ H^j(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i)) & \xrightarrow{f_{2i-j}} & H^j(\text{Spec } \mathbb{Q}_p, \mathbb{Z}_p(i)) \end{array}$$

where the vertical maps are isomorphisms. The lower horizontal map  $f_{2i-j}$  appears in the Tate-Poitou duality sequence which takes the following form [4, §2, Satz 5].

$$\begin{aligned}
0 &\rightarrow H^2(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \\
&\rightarrow H^1(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i)) \xrightarrow{f_{2i-1}} H^1(\mathrm{Spec} \mathbb{Q}_p, \mathbb{Z}_p(i)) \\
&\rightarrow H^1(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \\
&\rightarrow H^2(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i)) \xrightarrow{f_{2i-2}} H^2(\mathrm{Spec} \mathbb{Q}_p, \mathbb{Z}_p(i)) \\
&\rightarrow H^0(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \rightarrow 0
\end{aligned}$$

Here the asterisk indicates the Pontryagin dual. Hence, the kernel and cokernel of the cyclotomic trace map is governed by the groups  $H^q(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))$  with  $i > 1$ . The following conjecture was made by Lichtenbaum [2, Conjecture 9.1] for  $i < 0$  and by Schneider [4, p. 192] in general.

**Conjecture.** *The group  $H^2(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))$  is zero for every integer  $i \neq 0$ .*

The conjecture was proved for  $i < 0$  by Soulé [5, Théorème 5] but it remains open for  $i > 0$ . The conjecture is equivalent to the statement that for every integer  $i \neq 0$ , the group  $H^2(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(1-i))$  is finite. Moreover, by the discussion above, the conjecture for  $i > 1$  is equivalent to the statement that the map

$$f_{2i-1}: K_{2i-1}(\mathbb{Z}, \mathbb{Z}_p) \rightarrow K_{2i-1}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is injective.

We next recall the precise relationship between the values of the  $p$ -adic  $L$ -function and the étale cohomology groups implied by the affirmed Main Conjecture of Iwazawa theory [3, 6]. It was proved by Bayer and Neukirch [1, Theorem 6.1] that if  $i \neq 1$  is an odd integer and if  $L_p(\mathbb{Q}, \omega^{1-i}, i) \neq 0$ , then the groups

$$H^q(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(1-i))$$

are finite for all  $q \geq 0$ , zero for  $q \neq 1, 2$ , and

$$\begin{aligned}
|L_p(\mathbb{Q}, \omega^{1-i}, i)|_p &= \frac{\#H^1(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(1-i))}{\#H^2(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(1-i))} \\
&= \frac{\#H^0(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))}{\#H^1(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))}.
\end{aligned}$$

We remark that in loc. cit., the equality above is stated for the so-called Iwazawa zeta function  $\zeta_I(\mathbb{Q}, \omega^{1-i}, i)$  which is introduced in op. cit., §5 and which depends on a choice of topological generator  $q \in \mathbb{Z}_p^*$ . However, by the affirmed Main Conjecture of Iwazawa theory, it follows that

$$L_p(\mathbb{Q}, \omega^{1-i}, i) = \zeta_I(\mathbb{Q}, \omega^{1-i}, i) \cdot u(q^{1-i} - 1)$$

where  $u(T) \in \mathbb{Z}_p[[T]]^*$ . We also remark that for  $i < 0$  odd, the  $p$ -adic  $L$ -function interpolates the values of the Riemann zeta function at negative integers. Hence, in this case, Lichtenbaum's conjecture

$$|\zeta(i)|_p = \frac{\#H^0(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))}{\#H^1(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))}$$

ensues; see [1, Theorem 6.2].

Now let  $i > 1$  be odd and assume that  $L_p(\mathbb{Q}, \omega^{1-i}, i)$  is non-zero. Then we conclude from the theorem of Bayer-Neukirch and from the Tate-Poitou sequence that the group  $H^2(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))$  is zero, that

$$f_{2i-1}: K_{2i-1}(\mathbb{Z}, \mathbb{Z}_p) \rightarrow K_{2i-1}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is injective, and that the cokernel is related to the  $p$ -adic  $L$ -function as follows.

$$\begin{aligned} |L_p(\mathbb{Q}, \omega^{1-i}, i)|_p &= \frac{\#H^0(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))}{\#H^1(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))} \\ &= \frac{\#H^2(\mathrm{Spec} \mathbb{Q}_p, \mathbb{Z}_p(i))}{\#\mathrm{coker}(f_{2i-1}) \cdot \#H^2(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i))} \\ &= \frac{\#K_{2i-2}(\mathbb{Z}_p, \mathbb{Z}_p)}{\#\mathrm{coker}(f_{2i-1}) \cdot \#K_{2i-2}(\mathbb{Z}, \mathbb{Z}_p)}. \end{aligned}$$

The group  $K_{2i-2}(\mathbb{Z}, \mathbb{Z}_p)$  is zero for all odd integers  $i > 1$  if and only if the Kummer-Vandiver conjecture holds for the prime number  $p$ . The group  $K_{2i-2}(\mathbb{Z}_p, \mathbb{Z}_p)$  is finite cyclic of order  $p^{v_p(i-1)+1}$  if  $p-1$  divides  $i-1$ , and is zero otherwise. Therefore, if the Kummer-Vandiver conjecture holds for the prime  $p$ , and if  $i > 1$  is odd, we find

$$\#\mathrm{coker}(f_{2i-1}) = \begin{cases} p^{v_p(i-1)+1} |L_p(\mathbb{Q}, \omega^{1-i}, i)|_p^{-1} & \text{if } p-1 \text{ divides } i-1 \\ |L_p(\mathbb{Q}, \omega^{1-i}, i)|_p^{-1} & \text{otherwise,} \end{cases}$$

or equivalently,

$$\mathrm{length}(\mathrm{coker}(f_{2i-1})) = \begin{cases} v_p(L_p(\mathbb{Q}, \omega^{1-i}, i)) + v_p(i-1) + 1 & \text{if } p-1 \text{ divides } i-1 \\ v_p(L_p(\mathbb{Q}, \omega^{1-i}, i)) & \text{otherwise.} \end{cases}$$

Next, we let  $i > 1$  be even. Then it follows from [4, §4, Satz 6] that

$$\mathrm{rk}_{\mathbb{Z}_p} H^1(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i)) = \mathrm{rk}_{\mathbb{Z}_p} H^2(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i)).$$

But the right-hand group is finite by Soulé's theorem [5, Théorème 5], and hence, also the left-hand group is finite. This, we claim, implies that

$$H^2(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i)) = 0.$$

Indeed, by the Tate-Poitou sequence, the Pontryagin dual of this group is a subgroup of the finite group  $H^1(\mathrm{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i))$ , and hence, is finite. But it also a quotient of

the group  $H^2(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p(i))$ , and hence, a divisible group. Therefore, the group is zero as claimed. In addition, it follows that the boundary map

$$H^j(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(i)) \rightarrow H^{j+1}(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i))$$

is an isomorphism for all  $j \geq 0$ . For every integer  $i \neq 0$ , we have

$$\begin{aligned} \#H^0(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(i)) &= \max\{p^v \mid [\mathbb{Q}(\mu_{p^v}) : \mathbb{Q}] \text{ divides } i\} \\ \#H^0(\text{Spec } \mathbb{Q}_p, \mathbb{Q}_p/\mathbb{Z}_p(i)) &= \max\{p^v \mid [\mathbb{Q}_p(\mu_{p^v}) : \mathbb{Q}_p] \text{ divides } i\} \end{aligned}$$

and both orders are equal to  $p^{v_p(i)+1}$ , if  $p-1$  divides  $i$ , and equal to 1, otherwise.

## References

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