MICHAEL CHING — CYCLIC OPERADS, BAR & COBAR CONSTR.

GINZBURG – KAPRANOV "KOSZUL DUALITY FOR OPERADS"
GETZLER – JONES "OPERADS ..."
GETZLER – KAPRANOV "CYCLIC OPERADS AND CYCLIC HOMOLOGY"

(1) EXAMPLES OF OPERADS

SYMM. MONOIDAL CATEGORY $\mathcal{C}$

$\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$

$I \in \mathcal{C}$ UNIT

+ NAT. ISO'S $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$

$A \otimes I \cong A$

$A \otimes B \cong B \otimes A$

EX (i) k = COMM. RING $\mathcal{C} = k$-MODULES, $I = k$

(ii) GRADED k-MODULES $V \otimes W \xrightarrow{\cong} W \otimes V$

$v \otimes w \longmapsto (-1)^{|v||w|} w \otimes v$

(iii) DIFF GRADED k-MODULES (dg-k-mod)

(iv) TOP. SPACES, $\otimes = \times$

(v) $\mathcal{C}$ = SYMMETRIC / ORTHOGONAL SPECTRA, $\otimes$ = SMASH PRODUCT

OPERADS Fix $(\mathcal{C}, \otimes)$

AN OPERAD P CONSISTS OF $\{P(n)\}_{n \geq 1}$, $P(n) \in \mathcal{C}$

$\Sigma_n \circlearrowright P(n)$, UNIT MAP $I \to P(1)$

WITH COMPOSITION MAPS SATISFYING ASS., EQUIVARIANCE RELATIONS, UNIT

EX: (a) ASSOCIATIVE ALGEBRAS: "$P(n)$ = COLLECTIONS ON n ELEMENTS IN THE ALGEBRA"

$Ass(n) := k[\Sigma_n]$ WITH COMPOSITIONS

$$k[\Sigma_k] \otimes k[\Sigma_{n_1}] \otimes \cdots \otimes k[\Sigma_{n_k}] \to k[\Sigma_{n_1+\cdots+n_k}]$$

, ...

THERE IS AN UNDERLYING OPERAD OF SETS $Ass(n) = \Sigma_n$ WHOSE ALGEBRAS ARE ASSOCIATIVE MONOIDS

AN ALGEBRA OVER ASS IS AN ASSOCIATIVE k-ALGEBRA.

REMARK: AN OPERAD IS A SEQU. $\{P(n)\}_{n \geq 1}$ IGNORES THE POSSIBILITY OF A UNIT.

(b) Com COMMUTATIVE AND ASSOCIATION OPERAD. $Com(n) = k$

UNDERLYING OPERAD OF SETS: $Com(n) = *$ WITH ALGEBRAS THE COMMUTATIVE MONOIDS

(c) Lie OPERAD: $Lie(n) =$ THE SUBMODULE OF THE FREE LIE ALG. ON n ~~ELEMENTS~~ $x_1 \ldots x_n$ SPANNED BY "MONOMIALS"

$C = dg$-k-MOD

EXAMPLE: $Lie(2) = k$ GENERATED BY ~~[illegible]~~ $[-,-]$ WITH Σ_2 ACTING BY SIGN REPR.

$Lie(3) \cong k \oplus k \cong k\{[[x,y],z], [[y,z],x]\}$ $\circlearrowleft \Sigma_3$ ~~PERMUTE~~ x, y, z

$[[x,y],z]$
$[[y,z],x]$

$[[z,x],y]$ TAKEN CARE OF BY JACOBI $[[y,x],z]$ BY SYMMETRY ...

$Lie(n)$ IS FREE OF RANK $(n-1)!$

A LIE ALGEBRA IS A LIE ALGEBRA $\left(\begin{array}{l} \text{WITH } [x,y] = -[y,x] \\ \text{BUT } 2[x,x] = 0 \end{array}\right)$ UNGRADED

IN GRADED LIE ALGEBRAS $[x,y] = -(-1)^{|x||y|}[y,x]$

TOPOLOGICAL OPERADS $\quad \mathcal{C} = (\text{Top spaces}, \times)$

① LITTLE n-DISCS : $\mathcal{C}_n$

$\mathcal{C}_n(k)$ = SPACE OF COLLECTIONS OF k DISJOINT, LABELLED DISCS INSIDE THE DISC OF RADIUS

~~DATA FOR A POINT IN $\mathcal{C}_n(k)$ = CENTERS + RADII OF THE k DISCS.~~

$\mathcal{C}_n(1) \simeq *$, $\mathcal{C}_n(2) \simeq S^{n-1}$ WITH ANTIPODAL ACTION OF Σ_2

$\mathcal{C}_1 \xrightarrow{\sim} Ass$

Stasheff DEFINED AN OPERAD A_∞ FROM ASSOCIAHEDRA.

BAR CONSTRUCTIONS FOR OPERADS

in k-mod or dg-k-mod

COOPERAD IN $(\mathcal{C}, \otimes)$ IS AN OPERAD IN $(\mathcal{C}^{op}, \otimes)$

i.e. $\{Q(n)\} \circlearrowleft \Sigma_n$ AND CO-COMPOSITIONS $Q(n_1 + \dots + n_k) \to Q(k) \otimes Q(n_1) \otimes \dots \otimes Q(n_k)$

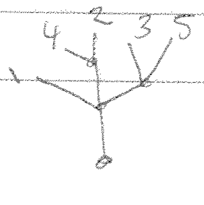
Q dg cooperad $\Rightarrow Q^*$, $Q^*(n) = Hom(Q(n), k)$ dg-operad

P dg operad with $P(n)$ f.d. v.s. $\forall n \Rightarrow P^*(n)$ cooperad

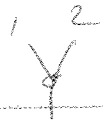
REDUCED BAR CONSTRUCTION

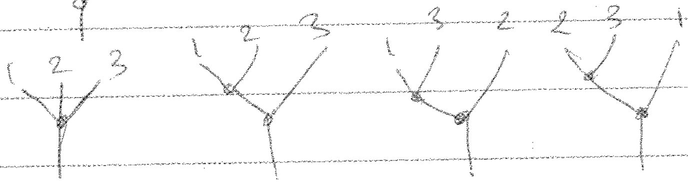
P operad in dg-k-mod. P is REDUCED if $P(1) = k$.

TREES. OUR TREES WILL BE ROOTED, LABELLED, NON-PLANAR, NO 2-VALENT VERTICES



$\leftrightarrow$ SOME FORM OF NON-ORDERED PARTITIONS OF $[n]$

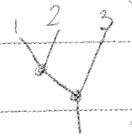
2-LABELLED TREES: 

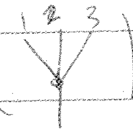
3-LABELLED TREES: 

$\mathcal{P}$ OPERAD IN dg-k-mod

FOR A TREE T, $\mathcal{P}(T) := \bigotimes_{v \in T} \mathcal{P}(\mathrm{in}(v))$

VERTICES ← # OF INCOMING EDGES

e.g. $\mathcal{P}\Big($  $\Big) = \mathcal{P}(2) \otimes \mathcal{P}(2)$

$\mathcal{P}\Big($  $\Big) = \mathcal{P}(3)$

LET e BE AN INTERNAL EDGE IN T. GET $\mathcal{P}(T) \to \mathcal{P}(T/e)$ COLLAPSED

EITHER BY CHOOSING REPRESENTATIVES OF TREES OR

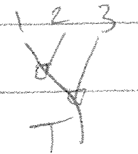
CONSIDER $\mathcal{P}$: (FINITE SETS, BIJ.) $\to$ dg-k-mod

DEFINE $\mathcal{P}(T) = \bigotimes_{v \in T} \mathcal{P}(\mathrm{in}(v))$ SET OF INCOMING EDGES

BAR CONSTRUCTION: $\mathcal{P}$ REDUCED dg-OPERAD

$B(\mathcal{P})(n) = \bigoplus_{\substack{n\text{-LABELLED} \\ \text{TREES } T}} \mathcal{P}(T)$

Σ_n ACTS BY RELABELLING

(12)  T $\quad \mathcal{P}(T) \to \mathcal{P}(T)$

NON-TRIVIAL

GRADING: • INTERNAL GRADING: COMES FROM GRADING ON THE $\mathcal{P}(n)$'s

• TREE GRADING = $|T| + 1$

$|T|$ = # INTERNAL EDGES OF T

DIFFERENTIALS: INTERNAL DIFF + TREE DIFF d_{tree}

$$d_{tree}: B(P)(n) \longrightarrow B(P)(n)$$

$$\uparrow$$

$$P(T) \xrightarrow{\sum d_e} \cancel{\quad}$$

⚠ NEED SIGNS!

$(-1)^{\text{EDGE NUMBER}}$

$$d_e: P(T) \to P(T/e)$$

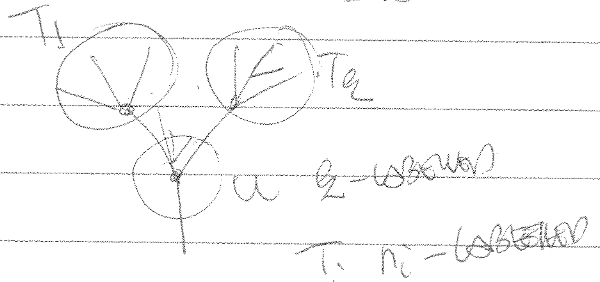
[GINZBURG - KAPRANOV] $\otimes \det$ - TO TAKE CARE OF SIGNS...

$B(P)$ = TOTAL COMPLEX OF THE DOUBLE COMPLEX

$\rightsquigarrow$ Diff graded COOPERAD

$$B(P)(n_1 + \dots + n_k) = \bigoplus_{\substack{T \,:\, n_1+\dots+n_k \\ \text{-LABELLED}}} P(T) \longrightarrow B(P)(k) \otimes B(P)(n_1) \otimes \dots B(P)(n_k)$$

IF



$$P(T) \cong P(u) \otimes P(T_1) \otimes \dots \otimes P(T_k)$$

IF NOT $\longrightarrow 0$

ONLY ONE SUCH DECOMPOSITION IF IT EXISTS BECAUSE OF THE LABELS.