THE DOUBLE OF A SIMPLICIAL COMPLEX

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Abstract. We introduce the notion of doubling and r-tupling for simplicial complexes, a notion reminiscent to that of matching complexes in graph theory. We prove a connectivity result for such complexes and relate r-tupling to stabilizing r times faster in homological stability.

1. Introduction

Given a simplicial complex X, one can define a new simplicial complex, its *double* $D(X)$, whose vertices are the edges of X, and where a collection of $p+1$ edges form a p-simplex in $D(X)$ if they are disjoint, and together form a $2p+1$-simplex of X. One can likewise define the r-tupling $D^r(X)$ of X for any natural number, where the vertices of $D^r(X)$ are now the $(r-1)$-simplices of X. (See Definition 2.1.)

Recall that a simplicial complex is called weakly Cohen-Macaulay (wCM for short) of dimension n if it is $(n-1)$–connected, and the link of any p–simplex is $(n-p-2)$–connected. The goal of this note is to prove the following result:

Theorem A. *Suppose X is wCM of dimension n. Then its r-tupling $D^r(X)$ is wCM of dimension $\left\lfloor \frac{n-r+1}{r+1} \right\rfloor$.*

The doubling of a simplicial complex is closely related to the notion of a *matching complex* in graph theory. Recall that the matching complex of a graph Γ is the simplicial complex whose vertices are the edges of Γ, and higher simplices the disjoint collections; a top simplex in that simplicial complex is a *complete matching* of the vertices in the graph. When $X = \Delta^n$ is the n-simplex, the double $D(\Delta^n)$ identifies with the matching complex of the complete graph on $n+1$ vertices. More generally, the r-tupling $D^r(\Delta^n)$ identifies with what is known as the matching complex of the complete r-hypergraph on $n+1$-vertices, see e.g. [1].

The connectivity properties of matching complexes of the complete graph and r-hypergraphs have been studied by several authors, and the above result can be seen as a generalization of [2, Cor 4.2] (when $r = 2$) and [1, Thm 1.2] (for all r's) that give the case $X = \Delta^n$ in our theorem via the identification just explained. Our proof in fact relies on these earlier results. When $r = 2$, the result is known to be sharp for Δ^n, see [14, Thm 1.3, Thm 1.6], building on [4].

Relationship to homological stability. Our study of r-tupling of simplicial complexes arose in homological stability, in questions related to *destabilization complexes*. Recall that a sequence of groups

$$G_1 \to \cdots \to G_n \to G_{n+1} \to \dots$$

satisfies homological stability if the map $H_i(G_n) \to H_i(G_{n+1})$ is an isomorphism when n is large enough. Stability holds for example for $G_n = GL_n(R)$ for many rings R, or $G_n = \Sigma_n$ the symmetric group on n letters, or $G_n = \mathrm{Aut}(F_n)$ the automorphism group of the free group of rank n, see e.g.; [13] or [17] for further examples, references and an introduction to the subject.

Homological stability results for families of groups are typically proved using the action of the groups on certain simplicial spaces, or simplicial complexes. As shown in [13], when

Date: June 12, 2025.

the groups $\{G_n\}$ together form a braided monoidal groupoid, as is most often the case in examples, there is a family of semi-simplicial sets W_n, called the destabilization complexes, whose connectivity properties govern homological stability in the sense that, if W_n is $\left(\frac{n-a}{k}\right)$–connected for all n and some $k \geq 2$, then the map $H_i(G_n) \to H_i(G_{n+1})$ is an isomorphism for all $i \leq \frac{n-a+1}{k}$. See Theorems A and 3.1 in [13].

Conjecture C in the same paper states that a form of converse of this result should hold, in the sense that if homological stability holds in such a fashion, then, at least homologically, the destabilization complexes W_n should be highly connected. As we will explain now, Theorem A can be interpreted as proving this conjecture in the special case of "fast stabilization".

We will use the following observation: if a sequence $G_1 \to G_2 \to G_3 \to \dots$ satisfies homological stability, so does the sequence $G_2 \to G_4 \to G_6 \to \dots$, or more generally $G_r \to G_{2r} \to G_{3r} \to \dots$ for any $r \geq 1$. The above conjecture, if true, would imply the following: whenever the destabilization complex W_n associated to the standard "+1" stabilization is highly connected, so should the destabilization complex W_n^r associated to the "$+r$" stabilization for any r. A consequence of Theorem A is that, under mild assumptions, this is indeed the case. See Proposition 4.6 for a precise statement. To be able to apply our main result, we will use that, in good conditions, the connectivity of the semi-simplicial sets W_n and W_n^r is determined by that of their underlying simplicial complexes S_n and S_n^r, see Definition 4.2 and [13, Thm 2.10].

Note that, as detailed in Remark 4.7, the connectivity bounds for W_n^r we obtain this way do not give optimal stability results in general. For symmetric groups, taking $G_n = \Sigma_n$, the simplicial complex S_n identifies with the n-simplex Δ^n. In that particular case, we know that the connectivity bounds we obtain for W_n^r are optimal, but it also happens in this case that the resulting stability slope remains optimal for any r. See the remark for more details.

The complexes W_n^r and S_n^r come with an action of the group G_{rn}. Hence their first non-trivial homology defines a family of representations of the groups. In the particular case of the symmetric groups, where $S_n = \Delta^n$, Bouc identifies the first non-trivial homology group of the double $D(S_n)$, as symmetric groups representations, as certain direct sums of Specht modules, see [4, Sec 3.2].

A variant of the r-tupling construction has arisen in current work by Belmont, Quigley, and Vogeli, who study an equivariant version of homological stability in the formalism of [13]. In their case, for a group G, the G-equivariant analogue $W_n(G)$ of the destabilization complex W_n has p-simplices given by ordered $(k_0 + ... + k_p)$-tuples of distinct vertices in the non-equivariant W_n, where $k_i = |G/H_i|$ for some subgroup $H_i \leq G$. They show, using a similar argument to the one given here, that high connectivity of the complexes W_n implies high connectivity of the complexes $W_n(G)$.

Using the E_k–cells formalism of [8], homological stability can be proved using the splitting complex instead of the destabilization complex. Connectivity properties of the destabilization complex and splitting complex are essentially equivalent by [12, Prop 7.1]. It should thus be possible to formulate and prove a version of our main result aimed at splitting complexes.

Organization of the paper: We define the doubling and r-tupling of a simplicial complex in Section 2, and relate that notion to that of matching complexes. The proof of Theorem A is given in Section 3. Finally Section 4 details the relationship to homological stability.

Acknowledgments. This project started at the Women in Topology workshop held at MSRI in 2017. We thank the organizers of the program for setting this up and MSRI for hosting us for the week. We also thank the Newton Institute in Cambridge for support and hospitality both during the program Homotopy Harnessing Higher Structures in 2018 and Equivariant Homotopy Theory in Context in 2025, where work on this paper was undertaken (EPSRC grants EP/K032208/1, EP/R014604/1, and EP/Z000580/1). The last author was also supported by the Danish National Research Foundation through the Copenhagen Centre for Geometry and Topology (DNRF151).

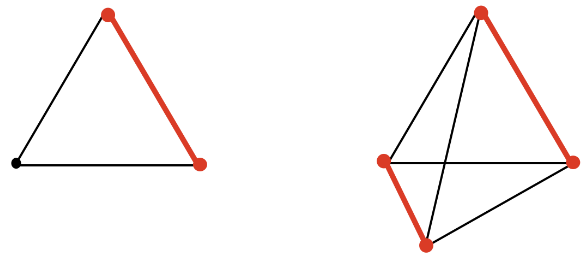

FIGURE 1. Vertex of $D(\Delta^2)$ and edge of $D(\Delta^3)$

2. DOUBLING, r-TUPLING, AND MATCHING COMPLEXES

Recall that a simplicial complex $X = (X_0, \mathcal{P})$ is the data of a set of vertices X_0 together with a collection $\mathcal{P}$ of subsets of X_0 including all the singletons and closed under taking subsets. A p-simplex of X is then a subset $\tau = \{x_0, \ldots, x_p\} \in \mathcal{P}$ of cardinality $p + 1$. We denote by X_p the collection of p-simplices of X.

Definition 2.1. The *r-tupling* of a simplicial complex $X = (X_0, \mathcal{P})$ is the simplicial complex $D^r(X) = (X_{r-1}, \mathcal{P}^r)$ with vertices the $(r-1)$–simplices of X and p-simplices the collections $\{\tau_0, \ldots, \tau_p\}$ with the property that $\bigcup_{i=0}^{p} \tau_i \in X_{(p+1)r-1}$. When $r = 2$, we call $D^2(X) = D(X)$ the *double* of X.

For example, the double $D(\Delta^2)$ has three vertices and no edges, while $D(\Delta^3)$ has four vertices and two (disjoint) edges (see Figure 1).

The *matching complex* of a graph Γ is the simplicial complex $\mathcal{M}(\Gamma)$ with vertices the edges of Γ, and p–simplices the collections $\{e_0. \ldots, e_p\}$ of pairwise disjoint edges. (See eg., [5, Def 3.4].)

Matching complexes were first introduced by Bouc [4] to study posets of subgroups. The poset of simplices of the double $D(\Delta^n)$ identifies with the poset of elementary abelian 2-subgroups of the symmetric group Σ_{n+1} that are generated by transpositions. Bouc began the study of the connectivity properties of $D(\Delta^n)$ as well as the Σ_{n+1}-representation defined by its first non-trivial homology group.

Let K_{n+1} be the complete graph on n vertices. Note that there is an isomorphism $D(\Delta^n) \cong \mathcal{M}(K_{n+1})$. Indeed, 1-skeleton of Δ^n is the complete graph K_{n+1}, giving an isomorphism on the set of vertices. And higher simplices are defined in the same way in both cases, since any subset of vertices defines a simplex in Δ^n.

In [1], Athanasiadis considers the more general *r–hypergraph matching complexes* $\mathcal{M}_n(r)$, where $\mathcal{M}_n(2) = \mathcal{M}(K_n)$ is the matching complex of the complete graph, and $\mathcal{M}_n(r)$ is defined more generally as the simplicial complex whose vertices are the subsets of $[n] = \{1, \ldots, n\}$ of cardinality r, and simplices the collections of pairwise disjoint such. We have the more general identification $D^r(\Delta^n) \cong \mathcal{M}_{n+1}(r)$.

Recall that simplicial complex X is called *Cohen-Macaulay* (abbreviated *CM*) *of dimension* n if it is of dimension n, and the link of any p-simplex in X is $(n-p-2)$–connected. We say that X is *weakly* Cohen-Macaulay (or *wCM*) of dimension n if only the connectivity condition for the links is satisfied. This is equivalent to requiring that the n-skeleton of X is CM of dimension n since any wCM complex of dimension n has actual dimension at least n because the links of $(n-1)$–simplices in such a complex are required to be non-empty.

It was shown in [2] that $\mathcal{M}_n(r)$ is wCM of dimension $\nu(r) = \left\lfloor \frac{n-2}{2r-1} \right\rfloor$, while in [1], it is shown that its $\mu_n(r)$–skeleton is shellable for

$$\mu_n(r) = \left\lfloor \frac{n-r}{r+1} \right\rfloor,$$

where *shellable* is a condition known to be stronger than CM, see e.g. [3, App.]. When $r = 2$, the two bounds agree, but for $r > 2$, the second bound is better.

In particular, it follows that

Theorem 2.2. [1] *$D^r(\Delta^n)$ is wCM of dimension* $\left\lfloor \dfrac{n+1-r}{r+1} \right\rfloor$.

Proof. We have that $D^r(\Delta^n)$ is isomorphic to $\mathcal{M}_{n+1}(r)$, which is wCM of dimension $\lfloor \frac{n+1-r}{r+1} \rfloor$, since its skeleton of that dimension is shellable by [1, Thm 1.2], and shellable implies (homotopy) CM, see e.g. [3, App.], and hence wCM of dimension equal to the dimension of the skeleton. □

Note that the non-trivial homology groups of the complexes $\mathcal{M}_n(r)$ have been studied as symmetric groups representations, see e.g.; [4], [15] and [16], also for the relationship to other complexes associated to groups.

3. Deducing the connectivity of $D^r(X)$ from that of $D^r(\Delta^n)$

Our proof of Theorem A is inspired by the proof of Theorem 3.6 in [9], and we will use as input the known connectivity of $D^r(\Delta^n)$.

For a simplicial complex X, let sX denote its first barycentric subdivision. As in [9], we denote by X_m the subposet of sX consisting of all the simplices of X with at least m vertices (i.e. of dimension at least $m-1$). From [9, Lem 3.8], we know that X_m is at least $(n-m)$–connected if X is wCM of dimension n. The idea of the proof is to use $X_{(k+2)r}$ as an intermediate space to show that the r-tupling of X is k–connected, because locally, in $X_{(k+2)r}$, we always have a simplex of X with at least $r(k+2)$ vertices, which corresponds to a simplex of $D^r(X)$ of dimension at least $k+1$. And $(k+1)$–simplices is exactly what is needed to fill in a k-sphere with a disc.

We will consider X_m as a simplicial complex with vertices the simplices of X of dimension at least $m-1$, and simplices the chains of such. And we start with an enhancement of the connectivity result of [9] just mentioned, that promotes "connectivity" to "wCM".

Lemma 3.1. *Suppose X is wCM of dimension n. Then X_m is wCM of dimension $n-m+1$.*

Proof. We know from [9, Lem 3.8] that X_m is $(n-m)$–connected, so we are left to show that the link of a p-simplex in X_m is at least $(n-m-p-1)$–connected for any $p \geq 0$.

A p–simplex of X_m is a simplex $\sigma_0 < \cdots < \sigma_p$ of the first barycentric subdivision of X with σ_0 at least an $(m-1)$–simplex of X. Suppose that σ_p is a q-simplex, where $q \geq p+m-1$. The link of such a simplex is the join $L_d * L_m * L_u$ with

- the "down-link" L_d: the subposet of X_m of simplices τ that are faces of σ_0,
- the "middle-link" L_m: the subposet of simplices τ with $\sigma_i < \tau < \sigma_{i+1}$,
- the "up-link" L_u: the subposet of simplices τ that contain σ_p as a face.

Write m_i for the number of vertices of σ_i. Then $m \leq m_0 < \cdots < m_p$.

The down-link L_d identifies with $(\partial\Delta^{m_0-1})_m$, which is (m_0-m-2)–connected as $\partial\Delta^{m_0-1}$ is CM of dimension m_0-2.

The middle-link $L_m = L_1 * \cdots * L_p$ splits as the join of the sublinks L_i of simplices τ with $\sigma_{i-1} < \tau < \sigma_i$. Now we can write $\sigma_i = \sigma_{i-1} * \Delta^{m_i-m_{i-1}-1}$ and L_i identifies with $\partial\Delta^{m_i-m_{i-1}-1}$, as we can add any strict face of this extra simplex to σ_{i-1}. So L_i is $(m_i - m_{i-1}-3)$–connected.

Finally the up-link L_u identifies with the first barycentric subdivision of the link of σ_p in X, since larger simplices are automatically large enough, and any larger simplex is obtained by joining a simplex from the link. It is hence $(n-m_p-1)$–connected by our assumption on X. Hence the whole link has connectivity at least

$$(m_0-m)+\sum_{i=1}^{p}(m_i-m_{i-1}-1)+(n-m_p+1)-2 = n-m-p-1$$

as needed. □

The wCM property will allow us to use [7, Thm 2.4], which tells us that when we use the connectivity of X_m to extend a map $f : S^k \to X_m$ from sphere S^k to a disc D^{k+1}, we can do this in such a way that the map is simplexwise injective on the new simplices.

For σ a p-simplex of $D^r(X)$, we will denote $\delta(\sigma)$ the underlying $((p+1)r-1)$-simplex of X.

Lemma 3.2. *For τ a p-simplex of $D^r(X)$,*

$$\mathrm{Link}_{D^r(X)}\,\tau \cong D^r(\mathrm{Link}_X\,\delta\tau).$$

Proof. By definition, a simplex ν is in $\mathrm{Link}_{D^r(X)}\,\tau$ if and only if $\nu * \tau$ is a simplex of $D^r(X)$, which happens, again by definition, if and only if $\delta(\nu) * \delta(\tau)$ is a simplex of X. This can be reformulated as saying that $\delta(\nu)$ is in $\mathrm{Link}_X\,\delta\tau$, which happens if and only if $\nu \in D^r(\mathrm{Link}_X\,\delta\tau)$. □

Proof of Theorem A. We will show that

$$D^r(X) \text{ is } \frac{n-2r}{r+1} = (\frac{n-r+1}{r+1} - 1)\text{-connected for any wCM complex } X \text{ of dimension } n.$$

This connectivity result gives the desired wCM property since, by Lemma 3.2, if τ is a p-simplex of $D^r(X)$, then $\mathrm{Link}_{D^r(X)}\,\tau \cong D^r(\mathrm{Link}_X\,\delta\tau)$. Now if X is wCM of dimension n, then $\mathrm{Link}_X\,\delta\tau$ is wCM of dimension $n - r(p+1) - 1$. Hence we can apply the connectivity result to $\mathrm{Link}_X\,\delta\tau$, which gives that the link is at least $\frac{n-r(p+1)-1-2r}{r+1}$-connected. Finally we check that

$$\frac{n-r(p+1)-1-2r}{r+1} = \frac{n-rp-3r-1}{r+1} \geq \frac{n-rp-3r-1-p}{r+1} = \frac{n-r+1}{r+1} - p - 2$$

giving the required connectivity condition on links.

So fix a simplicial complex X that is wCM of dimension n and let $k \leq \frac{n-2r}{r+1}$. Then $(r+1)k \leq n-2r$, or equivalently, $k \leq n - r(k+2)$.

Let $f : S^k \to D^r(X)$ be a map, which we assume simplical for some triangulation K of S^k. The plan of the proof is as follows:

- step 1 Use f to construct a map $\hat{f} : K_1 \to X_{(k+2)r}$, for K_1 a subdivision of K. By [HW, Lem 3.8], we know that $\hat{f}$ extends to a map $g : D^{k+1} \to X_{(k+2)r}$, simplicial with respect to a triangulation L_1 of the disc restricting to K_1 on the boundary. We will choose g to be simplexwise injective.
- step 2 Use g to construct a map $h : L_2 \to D^r(X)$, for L_2 a subdivision of L_1, with the property that $h|_{S^k} \simeq f$.

We start with step 1. We will build $\hat{f}$ and K_1 inductively on the skeleta of the first barycentric subdivision sK of K, with the following property:

- $(*)$ For each vertex v of K_1 in the interior of a simplex $\sigma_0 < \sigma_1 < \cdots < \sigma_p$ of sK, we have that $\delta(f(\sigma_0)) \leq \hat{f}(v)$.

We start with the vertices of sK. Such a vertex corresponds to a simplex σ of K. Because X is wCM of dimension $n \geq k + r(k+2)$, we can, if the dimension of $\delta(f(\sigma))$ is not large enough, pick a $((k+2)r-1)$–simplex $\hat{f}(\sigma)$ containing $\delta(f(\sigma))$ as a face, as any simplex in a wCM complex of dimension n is the face of a simplex of dimension n. The injectivity condition is automatically satisfied on vertices.

Now suppose that we have defined $\hat{f}$ on the $(m-1)$–skeleton of sK simplexwise injective and satisfying $(*)$. Let $\underline{\sigma} = (\sigma_0 < \cdots < \sigma_m)$ be an m–simplex of sK. By induction, we have thus defined $\hat{f}$ on a subdivision of the boundary $\partial\underline{\sigma}$, so that the image of $\hat{f}$ on any vertex always contains at least $\delta f(\sigma_0)$ as a face. Now σ_0 is a p–simplex of K for some $p \leq k$, with $f(\sigma_0)$ a p'–simplex of $D^r(X)$ for some $p' \leq p$, so that $\delta f(\sigma_0)$ is a simplex of X with $r(p'+1)$

vertices for some $p' \leq p \leq k$. Note that $r(p'+1) \leq r(k+1) < r(k+2)$. We can thus consider the restriction of $\hat{f}$ to $\partial\underline{\sigma}$ as a map

$$\hat{f}|_{\partial\underline{\sigma}} = \delta f(\sigma_0) * \hat{f}_1 : \partial\underline{\sigma} \simeq S^{m-1} \longrightarrow X_{r(k+2)}$$

with $\hat{f}_1 : \partial\underline{\sigma} \to \mathrm{Link}_X(\delta f(s_0))_{r(k+2)-r(p'+1)}$. This latter complex is wCM of dimension $n - r(k+2) + r(p'+1) + 1$ by Lemma 3.1, so we can extend $\hat{f}_1$ to a map $\hat{f}_2 : D^m \to \mathrm{Link}(\delta f(\sigma_0))_{r(k+2)-r(p'+1)}$ in a simplexwise injective way as long as $m-1 \leq n - r(k+2) + r(p'+1)$, which holds since $m - 1 \leq k - 1 \leq n - r(k+2) - 1$ by assumption. Now define $\hat{f} = \delta f(\sigma_0) * \hat{f}_2$ in the interior of the simplex. This satisfies condition $(*)$ by construction, is still simplexwise injective, and hence gives the induction step, once this is done for all the m simplices of sK.

By assumption $k \leq n - r(k+2)$, so that [9, Lem 3.8] gives a null-homotopy $g : D^{k+1} \to X_{(k+2)r}$ of $\hat{f}$, simplicial with respect to a triangulation L_1, agreeing with K_1 in the boundary of the disc. By Lemma 3.1 and using [7, Thm 2.4], we can assume that g is simplexwise injective.

For step 2, we want to use g to construct a map $\hat{g} : D^{k+1} \to D^r(X)$, with D^{k+1} now triangulated by a subdivision L_2 of L_1. We build $\hat{g}$ inductively on the skeleta of L_1, satisfying that

$(**)$ For each vertex v of L_2 in the interior of a simplex τ of L_1 with $g(\tau) = \alpha_0 < \cdots < \alpha_p$ in $X_{(k+2)r}$, we have that $\hat{g}(v)$ is an $(r-1)$-face of α_p. And if τ is in the boundary, lying in the interior of the simplex $\sigma_0 < \cdots < \sigma_j$ of sK, we require more specifically that $\hat{g}(v) = f(w)$ for w a vertex of σ_0.

Note that the condition on the boundary is compatible with the condition on all simplices by condition $(*)$: if v is in the interior of a simplex τ of L_1 and in the interior of the simplex $\sigma_0 < \cdots < \sigma_j$ of sK, then the vertices of τ are necessarily interior to faces of $\sigma_0 < \cdots < \sigma_j$, and hence by $(*)$, each α_i contains at least $\delta f(\sigma_0)$ as a face, so faces of $\delta f(\sigma_0)$ are also faces of $g(\alpha_p)$.

To start the induction, let v be a vertex of L_1. If $v \in K_1$, we set $\hat{g}(v) = f(w)$ with w chosen as in $(**)$, and if not, we pick $\hat{g}(v)$ to be some $(r-1)$-face of the simplex $g(v)$, which is possible as $(k+2)r \geq r$. This satisfies $(**)$ by construction.

Note that this choice of $\hat{g}$ on the vertices of $K_1 \subset L_1$ extends linearly to a map

$$\hat{g}|_{K_1} : K_1 \longrightarrow D^r(X)$$

since any simplex $\tau = \{v_0, \ldots, v_j\}$ of K_1 lies inside a simplex $\sigma_0 < \cdots < \sigma_j$ of sK, and $\hat{g}(v_i) = f(a_{s_i})$ for a_{s_i} some vertex of σ_j. For the same reason, the map $\hat{g}|_{|K_1|}$ is homotopic to f, by linear interpolation.

Now suppose that we have defined $\hat{g}$ on the $(m-1)$–skeleton of L_1 satisfying $(**)$ and consider τ an m–simplex of L_1, not in the boundary, with $g(\tau) = \alpha_0 < \cdots < \alpha_m$ in X_{2k+4}, where we use that g is simplexwise injective. By induction, we have defined $\hat{g}$ on its boundary so that the value of $\hat{g}$ on every vertex of $\partial\tau$ is an $(r-1)$-face of α_m. Hence we have

$$\hat{g}|_{\partial\tau} : \partial\tau \simeq S^{m-1} \to D^r(\alpha_m).$$

Now α_m is a simplex of X of dimension at least $(k+2)r + m - 1$ since α_0 has at least $(k+2)r$ vertices.

We have that $D^r(\alpha_m)$ is $\left(\frac{(k+2)r+m-r}{r+1} - 1\right)$–connected by Theorem 2.2. Now $m \leq k+1$, so

$$\frac{(k+2)r + m - r}{r+1} - 1 \geq \frac{mr + r + m - r}{r+1} - 1 = m - 1,$$

as needed. So we can extend $\hat{g}$ in the interior of τ, satisfying $(**)$ by construction. This finishes the proof. □

4. Doubling and destabilization complexes

The r-tupling of a simplicial complex arises in the context of homological stability, when "stabilizing fast" in the way described in the introduction. We make this precise here. We start by recalling the destabilization complexes of [13]. To do so, we will adopt for this section the categorical language of that paper, where groups are considered as automorphism groups in appropriate categories, and stabilization maps are induced by a monoidal structure in the category.

Let $(\mathcal{C}, \oplus, 0)$ be a monoidal category, and A, X objects in $\mathcal{C}$. We will consider groups of the form $G_n = \mathrm{Aut}(A \oplus X^{\oplus n})$ with stabilization maps of the form

$$\sigma_1 : G_n = \mathrm{Aut}(A \oplus X^{\oplus n}) \xrightarrow{\oplus X} G_{n+1} = \mathrm{Aut}(A \oplus X^{\oplus n+1}) \tag{4.1}$$

adding the identity on the added X-summand. For example symmetric groups Σ_n can be interpreted as automorphisms in the category of finite sets and injections, with disjoint union as monoidal structure. We refer to [13, Sec 5] for further examples including general linear groups, unitary groups, automorphisms of free groups or mapping class groups. See also [13, sec 1.1] for how to construct an appropriate category $\mathcal{C}$ from the groups $\{G_n\}_{n\geq 1}$, a "sum" $G_n \times G_m \to G_{n+m}$ and a braiding $b_{n,m} : G_{n+m} \to G_{m+n}$.

When 0 is initial in $\mathcal{C}$, we can associate to the pair of objects (A, X) a semi-simplical set, and underlying simplicial complex:

Definition 4.2. [13, Defn 2.1 and 2.8] Let $(\mathcal{C}, \oplus, 0)$ be a monoidal category with 0 initial and (A, X) a pair of objects in $\mathcal{C}$.

(1) Define $W_n(A, X)_\bullet$ as the semi-simplicial set with p-simplices
$$W_n(A, X)_p = \mathrm{Hom}_{\mathcal{C}}\left(X^{\oplus p+1}, A \oplus X^{\oplus n}\right),$$
and face maps d_i precomposing with $X^{\oplus i} \oplus \iota_X \oplus X^{\oplus p-i}$ for $\iota_X : 0 \to X$ the unique map.

(2) Define $S_n(A, X)$ to be the simplicial complex with set of vertices $W_n(A, X)_0$ and such that $\{f_0, \dots, f_p\}$ forms a p-simplex if there exists a p-simplex $f \in W_n(A, X)_p$ with set of vertices $\{f_0, \dots, f_p\}$.

A p-simplex of a semi-simplicial set has $p + 1$ vertices, obtained from applying repeated face maps, but, unlike in a simplicial complex, the vertices of a simplex might not be distinct and might not determine the simplex.

For a simplex $f : X^{\oplus p+1} \to A \oplus X^{\oplus n}$ in $W_n(A, X)$, the vertices of f are its restrictions
$$f_i = f \circ (\iota_{X^{\oplus i-1}} \oplus X \oplus \iota_{X^{\oplus n-i}}) : X \to A \oplus X^{\oplus n}$$
to each of the X-summands of the source. We will here stay away from pathological examples and restrict ourselves to categories that are *locally standard at* (A, X) in the sense of [13]. This is precisely the categories with the property that simplices of $W_n(A, X)$ have all their vertices distinct, and are determined by the ordered collection of their vertices, see [13, Prop 2.6].

For a simplicial complex X, denote X^{ord} the semi-simplicial set with a p-simplex for every p-simplex of X and every choice of ordering of its vertices. In practice, the following two situations arise in examples:

(A) $W_n(A, X) = S_n(A, X)^{ord}$ (when the groupoid $\mathcal{G}$ is symmetric monoidal)
(B) $W_n(A, X) = S_n(A, X)$ (when the groupoid $\mathcal{G}$ is braided but not symmetric).

Examples 4.3. Let $(\mathcal{C}, \oplus, 0) = (FI, \sqcup, \emptyset)$ be the category of finite sets and injections, with disjoint union as monoidal structure, $A = \emptyset$ and $X = \{*\}$. Then

(1) $W_n(A, X)$ is the semi-simplicial set of *injective words* in $[n] = \{1, \dots, n\}$, with p-simplices
$$W_n(A, X)_p = \mathrm{Hom}_{FI}([p + 1], [n]) = \{(a_0, \dots, a_p) \in [n]^{p+1} \mid a_i \neq a_j \text{ when } i \neq j\}.$$

(2) $S_n(A,X)$ is isomorphic to the $(n-1)$-simplex Δ^{n-1}.

Here we are in situation (A), with $W_n(A,X) = S_n(A,X)^{ord}$.

The semi-simplicial set $W_n(A,X)$ is called the *destabilization complex*, and Theorem A in [13] says that, under good conditions, its connectivity ensures homological stability for the maps (4.1). Moreover, the connectivity of the simplicial complexes $S_n(A,X)$ often determines that of the semi-simplicial sets $W_n(A,X)$, see [13, Thm 2.10].

Given (A,X) in $(\mathcal{C},\oplus,0)$ as above, we can also consider the stabilization maps

$$\sigma_r : \mathrm{Aut}(A\oplus X^{\oplus n}) \xrightarrow{\oplus X^{\oplus r}} \mathrm{Aut}(A\oplus X^{\oplus n+r}) \tag{4.4}$$

that go r times faster. Homological stability for this fast stabilization is ruled by the semi-simplicial sets $W_n(A,X^{\oplus r})$. Because homological stability for the maps stabilizing one X at a time implies homological stability for the maps stabilizing r X's at once, it is to be expected that the connectivity of the semi-simplicial sets $W_n(A,X)$ should imply a connectivity result for the semi-simplicial sets $W_n(A,X^{\oplus r})$. We will see that this is the case, under good conditions, by first using that the connectivity properties of $W_n(A,X)$ and $W_n(A,X^{\oplus r})$ are tightly related to those of $S_n(A,X)$ and $S_n(A,X^{\oplus r})$, and then relating the latter simplicial complexes to the r-tupling construction on the former. Understanding what happened to these complexes under fast stabilization was our motivation for proving Theorem A.

Recall from [9, Def 3.2] that a simplicial complex Y is a complete join over a simplicial complex X if there is a surjective map of simplicial complexes $\pi : Y \to X$ such that $\langle y_0,\dots,y_p\rangle$ is a p-simplex of Y if and only if $\langle \pi(y_0),\dots,\pi(y_p)\rangle$ is a p-simplex of X. This last condition can be reformulated as saying that if $\sigma = \langle x_0,\dots,x_p\rangle$ is a p-simplex of X, then $\pi^{-1}(\sigma)$ is the join $\pi^{-1}(x_0) * \cdots * \pi^{-1}(x_p)$, which explains the name *complete join complex.*

Proposition 4.5. *Let $(\mathcal{C},\oplus,0)$ be a monoidal category with 0 initial, locally standard at (A,X). Then $S_n(A,X^{\oplus r})$ is a complete join complex over $D^r(S_{nr}(A,X))$.*

Proof. By definition, a vertex of $S_n(A,X^{\oplus r})$ is a morphism $X^{\oplus r} \to A\oplus (X^{\oplus r})^{\oplus n}$ in $\mathcal{C}$, which can be interpreted as an $(r-1)$-simplex of $W_{nr}(A,X)$. On the other hand, a vertex of $D^r(S_{nr}(A,X))$ is an $(r-1)$-simplex of $S_{nr}(A,X)$, that is a collection $\{f_1,\dots,f_r\}$ of morphisms $f_i : X \to A\oplus X^{\oplus n}$ such that there exists $f \in W_{nr}(A,X)$ in $\mathcal{C}$ with vertices f_i, i.e. such that $f : X^{\oplus r} \to A\oplus (X^{\oplus r})^{\oplus n}$ restricts to f_i on the ith factor. In particular, we get a surjective map on vertex sets

$$\pi : S_n(A,X^{\oplus r})_0 \cong W_{nr}(A,X)_{r-1} \longrightarrow S_{nr}(A,X)_{r-1} \cong D^r(S_{nr}(A,X))_0.$$

taking a vertex of $S_n(A,X^{\oplus r})$, thought of as a simplex σ of $W_{nr}(A,X)$, to its underlying simplex in $S_{nr}(A,X)$.

Now a collection of vertices $\{g_0,\dots,g_p\}$ in $S_n(A,X^{\oplus r})$ defines a simplex if and only if there is a simplex $g : (X^{\oplus r})^{\oplus p+1} \to A\oplus (X^{\oplus r})^{\oplus n}$ in $W_n(A,X^{\oplus r})$ appropriately restricting to $g_0,\dots,g_p$. Interpreting g as an $(r(p+1)-1)$-simplex of $W_{rn}(A,X)$, it has vertices the union of the vertices of the g_i's. Hence $\{g_0,\dots,g_p\}$ is a p-simplex of $S_n(A,X^{\oplus r})$ if and only if the union of the vertices of the g_i's form a simplex in $S_{nr}(A,X)$, which happens if and only if $\{\pi(g_0),\dots,\pi(g_p)\}$ is a p-simplex of $D^r(S_{nr}(A,X))$, proving the claim. □

To be able to use the above result to deduce a connectivity result for $S_n(A,X^{\oplus r})$ and $W_n(A,X^{\oplus r})$ from the connectivity result for $S_n(A,X)$, we need to know that the simplicial complexes $S_n(A,X)$ are wCM. This is a property that holds in all standard stability examples, and follows for example if the category $\mathcal{C}$ is locally homogeneous and locally standard at (A,X) satisfying condition (A) of [13, Sec 2.1], see [13, Thm 2.10,Cor 2.13]. These many conditions are for example automatically satisfied under the local standardness assumption if $\mathcal{C}$ is build from a groupoid (in the sense of [13, sec 1.1] that is symmetric monoidal and satisfies cancellation, see Theorem 1.10 and Proposition 2.9 in that paper.

Proposition 4.6. *Let $(\mathcal{C}, \oplus, 0)$ be a monoidal category with 0 initial, locally standard at (A,X). Suppose that $S_n(A,X)$ is wCM of dimension $\frac{n+k-a}{k}$ for all n. Then $S_n(A,X^{\oplus r})$ is wCM of dimension $\frac{nr+k(2-r)-a}{k(r+1)}$ for any $n,r\geq 1$.*

If moreover, $W_n(A,X)$ satisfies condition (A) or (B) as above (see also [13, Sec 2.1]*), we have that if $W_n(A,X)$ is $\frac{n-a}{k}$-connected for all n, implies that $W_n(A,X^{\oplus r})$ is $\frac{nr+k(1-2r)-a}{k(r+1)}$-connected for any $n,r\geq 1$*

Note that the above assumptions are satisfied for basically all known examples, and in particular all the examples treated in [13].

Proof. For the first part of the proposition, the assumption gives that $D^r(S_{nr}(A,X))$ is wCM of dimension

$$\frac{nr+k-a-kr+k}{k(r+1)} = \frac{nr+k(2-r)-a}{k(r+1)}$$

by Theorem A. Combining Proposition 4.5 with [9, Prop 3.5] then gives that $S_n(A,X^{\oplus r})$ is wCM of the same dimension, proving the claim.

For the second part, in case (A) we have $W_n(A,X)=S_n(A,X)^{ord}$ is the ordered complex of $S_n(A,X)$, with simplices all the possible orderings of the vertices of simplices of $S_n(A,X)$. Picking an ordering of the vertices of $S_n(A,X)$ defines a splitting of the forgetful map $W_n(A,X)\to S_n(A,X)$ so that $W_n(A,X)$ is $\frac{n-a}{k}$-connected implies that the same holds for $S_n(A,X)$. Since we assumed that $S_n(A,X)$ is wCM, the converse holds by [13, Prop 2.14]. Hence the result in that case follows from the first part. And in case (B), $W_n(A,X)\cong S_n(A,X)$ so that the result follows directly from the first part in that case. □

Remark 4.7 (Stability bounds and sharpness of the connectivity of the destabilization complex)**.** Note that the above result is not optimal from the point of view of homological stability. Indeed, under the goodness assumptions of [13], if the original destabilization complexes $W_n(A,X)$ are $\frac{n-a}{k}$–connected for some a, i.e. slope $\frac{1}{k}$-connected, then Theorem A in that paper implies that the map

$$\sigma_r : H_i(\mathrm{Aut}(A\oplus X^{\oplus nr}))\to H_i(\mathrm{Aut}(A\oplus X^{\oplus nr+r}))$$

is an isomorphism for $i\leq \min(\frac{1}{k},\frac{1}{2})nr+b$ for some constant b. Using the above result instead, we get that the complexes $W_n(A,X^{\oplus r})$, in good enough situations, are slope $\frac{r}{k(r+1)}$–connected. Applying Theorem A of [13] to these complexes, we can deduce stability for the same maps σ_r. This however now gives an isomorphism range of the form $i\leq \min(\frac{r}{k(r+1)},\frac{1}{2})n+b'$, which is a strictly worse slope when $\frac{1}{k}\leq\frac{1}{2}$.

In the case of symmetric groups, we actually know that the above connectivity results are sharp. Indeed, let $\mathcal{C}$ be the category of finite sets and injections, with disjoint union as monoidal structure. Then $W_n(\emptyset,[1])$ is the complex of injective words (see Example 4.3), which is known to be wCM of dimension $n-1$, a connectivity that cannot be improved, see e.g. [6, p 613]. (But note that it is slope $1>\frac{1}{2}$ connected!) The connectivity of $W_n(\emptyset,[2])$ coming from the above result is also known to be sharp. Indeed, the connectivity of $D(S_{2n}(\emptyset,[1]))$ given in Theorem 2.2 is known to be sharp, see [14, Thm 1.3,1.6]. This implies that the same holds for $S_n(\emptyset,[2])$, because the projection to $D(S_{2n}(\emptyset,[1]))$ exhibiting it as a join complex admits a splitting. Finally, there is likewise a splitting of the forgetful map $W_n(\emptyset,[2])\to S_n(\emptyset,[2])$, choosing an ordering of the vertices. Hence the connectivity slope $\frac{r}{r+1}=\frac{2}{3}$ for $W_n(\emptyset,[2])$ cannot be improved. As $\frac{2}{3}>\frac{1}{2}$, the stability result in this case would still give a stability bound of slope $\frac{1}{2}$, even if we use the double, and that is in fact known to be best possible homological stability slope for symmetric groups. More generally, if one knew that the connectivity of $W_n(\emptyset,[r])$ given in Theorem 2.2 was likewise best possible, that would still give the correct stability slope for any r because $\min(\frac{r}{r+1},\frac{1}{2})=\frac{1}{2}$ for any r. Of course this works precisely because the original destabilization complex $W_n(A,X)$ was slope 1 connected.

Remark 4.8 (Monoid cells point of view)**.** Following [10, Sec 7], to construct the destablization complex $W_n(A,X)$ it is enough to have a monoidal map from the monoid of braid groups $\mathcal{B}=\coprod_{n\geq 0} B_n$ to $\mathcal{G}=\coprod_{n\geq 0} G_n$, if the latter is a monoidal groupoid. This endows $B\mathcal{G}$ with the structure of E_1-module over the E_2-algebra on $B\mathcal{B}$. In [11, Rem 3.10], the authors consider the effect on the destabilization complex of reversing the braiding in $\mathcal{G}$. One can interpret the reversed complex as coming from precomposing the map $\mathcal{B}\to\mathcal{G}$ by the automorphism of $\mathcal{B}$ flipping the braid. The resulting complex is homeomorphic to the original one, via a homeomorphism that reverses the order of the face maps in the semi-simplicial sets. In this language, the construction presented here comes instead from pre-composing the map $\mathcal{B}\to\mathcal{G}$ with a self-map of the braided category that is multiplication by r on the objects and takes r-fold copies of the strings in each braid. This was pointed out to us by Oscar Randal-Williams.

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