E_n – OPERADS

BRUNO VALETTE

0 – PREHISTORY OF OPERADS

GALOIS ALREADY ABSTRACTS THE OPERATIONS AS MATHEMATICAL OBJECTS THEMSELVES.

WHITEHEAD (1898)

LAZARD (1950's)

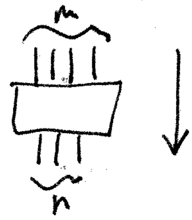
LAWVERE, MACLANE, ADAMS : PROP (PRODUCT AND PERMUTATIONS)

└ WORK WITH HOMOTOPIES (ITERATED BAR CONSTRUCTION, SIMPLICIAL CHAIN CPX)

— ORGANIZE THESE HOMOTOPIES

MACLANE [1965] "THERE ARE SO MANY HOMOTOPIES THAT GENERAL METHODS ARE NEEDED TO TREAT THEM"

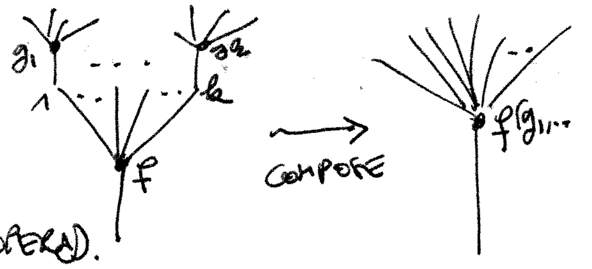
IDEA: $\mathrm{End}(V) := \{\mathrm{Hom}(V^{\otimes n}, V^{\otimes m})\}_{n,m}$



WITH COMPOSITION OF MAPS

PROP MODELED ON THIS EXAMPLE –

$\boxed{m=1}$ $\mathrm{End}(V) := \{\mathrm{Hom}(V^{\otimes n}, V)\}_n$

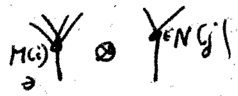


TOYMODEL FOR NOTION OF OPERAD.

LET $(\mathcal{E}, \otimes, e)$ BE A (SYMMETRIC) MONOIDAL CATEGORY.

DEF: $\mathbb{S}$-MODULE / Σ-MODULE $:=$ COLLECTION $\{M(k)\}_{k \geq 0}$ WITH $M(k) \curvearrowleft \Sigma_k$, $M(k) \in \mathcal{E}$ ← "SPACE OF k-ARY OPERATIONS"

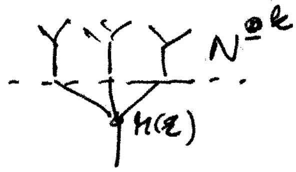
$$M \underline{\otimes} N(k) := \coprod_{i+j=k} \left(M(i) \otimes N(j) \otimes \Sigma_k\right) \Big/ \Sigma_i \times \Sigma_j$$



$\coprod$ ← COPRODUCT IN $\mathcal{E}$; $/\Sigma_i \times \Sigma_j$ ← COEQUALIZER

UNIT $\underline{\otimes}$: $(e, 0, 0, \dots)$ (ARITY $0, 1, 2, \dots$)

$$M \circ N = \left(\coprod_{k \geq 0} M(k) \otimes N^{\underline{\otimes} k}\right) \Big/ \Sigma_k$$



$M \circ N$ ← COMPOSITE PRODUCT

MONOIDAL PRODUCT WITH UNIT $(0, e, 0, \dots)$ (ARITY $0, 1, 2, \dots$)

(LINEAR ON THE LEFT BUT NOT ON THE RIGHT — IF $\mathcal{E}$ ABELIAN

$(M \oplus M') \circ N \cong M \circ N \oplus M' \circ N$.)

(ALGEBRAS, THOUGHT OF AS OPERADS IN ARITY 1, ARE LINEAR ON THE LEFT AND RIGHT. PROPS ARE NOT LINEAR AT ALL IN GENERAL.)

DEF: A (NON-SYMMETRIC) OPERAD IS A MONOID IN ($\mathbb{S}$-MODULES, $\circ$) [NON-SYMMETRIC: IF WE REMOVE Σ_k-ACTIONS]

$$\begin{cases} P \circ P \xrightarrow{\gamma} P & \text{ASSOCIATIVE} \\ I\!d \xrightarrow{\eta} P & \text{UNIT} \end{cases}$$

EX: $\mathcal{E} =$ Top, POSETS, dg-MODULES

1 – Recognition Principle

Goal: Study iterated loop spaces $\qquad I = [0,1] \subset \mathbb{R}$

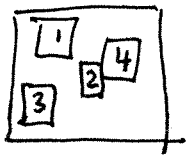
$$\Omega^n Y = \mathrm{Map}\big((I^n, \partial I^n); (Y, *)\big)$$

Def: n-cube operad $\mathcal{C}_n$:

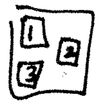
$\mathcal{C}_n(k) = \{$ affine embeddings of k n-cubes I^n inside I… / such that the interior of the images are disj…

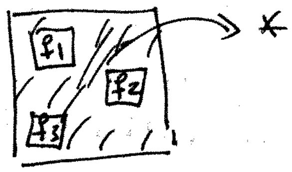
$x \mapsto ax + b$

(or $x \mapsto \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} x + b$)



Composition product: shrink and insert.

$\Omega^n Y$ is a $\mathcal{C}_n$-algebra:  $\in \mathcal{C}_2(3)$, $f_1, f_2, f_3 : I^2 \to$

$\rightsquigarrow$ 

Prop: [Boardman-Vogt / May]

X (homotopy type of CW-cpx) connected (or $\pi_0(X)$ group…)

If X is a $\mathcal{C}_n$-algebra $\Rightarrow$ it is homotopically equiva… to $\Omega^n Y$ fo… some Y

Proof: Bar construction $B(L; P; R)$ $\qquad$ P operad, L right P-mo…, R left P-mo…

Simplicial with $B_n(L; P; R) := L \circ P^{\circ n} \circ R / \sim$ ($\sim$ basepoint identifications)

$$B(\mathcal{C}_n; \mathcal{C}_n; X) \xrightarrow{\sim} B(\Omega^n S^n; \mathcal{C}_n; X) \xrightarrow{\sim} \Omega^n \underbrace{B(S^n; \mathcal{C}_n; X)}_{\text{"}Y}$$

$B(\mathcal{C}_n; \mathcal{C}_n; X) \xrightarrow{\sim} X$

[Ref: Markl-Shnider-Stasheff, Ch 2 — 15 pages easy going…
May, Iterated loop spaces
Boardman-Vogt

$\underline{n=1}$: SUGAWARA, STASHEFF [00]
ASSOCIAHEDRA : $K \xrightarrow{\sim} \mathcal{C}_1$

$$\mathcal{C}_1 \to \mathcal{C}_2 \to \dots \to \mathcal{C}_n \to \dots \to \operatorname{colim}_n \mathcal{C}_n = \mathcal{C}_\infty$$

$\underline{n=\infty}$: [BV] Ω^∞-SPACES USING LINEAR ISOMETRIES

2- E_n-OPERADS

ONLY $\mathcal{C}_n$ IS TOO RIGID.

<u>DEF</u>: AN E_n-OPERAD P IS AN OPERAD
- WEAKLY HOMOTOPICALLY EQUIV. TO $\mathcal{C}_n$: $P \xleftarrow{\sim} \dots \xrightarrow{\sim} \mathcal{C}_n$
- $P(k)$ IS Σ_k-COFIBRANT ("Σ_k-PROJECTIVE") (*) (NEED W.E. TO BE PRESERVED WHEN TAKING QUOTIENTS BY Σ_k — TOO COMPLICATED TO WORK WITH ANYTHING ELSE THAN FREE)

<u>PROPOSITION</u>: X (HTPY TYPE OF CW-CPX, CONNECTED OR $\pi_0 X$ = GROUP)
IF X IS A P-ALGEBRA, P E_n-OPERAD $\Rightarrow$ $X \simeq \Omega^n Y$ FOR SOME Y.

<u>PROOF</u>: $B(\mathcal{C}_n; \mathcal{C}_n; X) \xrightarrow[(*)]{\sim} B(P; P; X) \xrightarrow{\sim} X$
(*) NEEDED HERE

<u>CHARACTERIZATION</u>: $n=1 \Leftrightarrow$ P NON-SYMMETRIC, $P(k)$ CONTRACTIBLE
$P \xrightarrow{\sim} Ass$ (OPERAD OF ASSOC. ALG.)

$n=\infty \Leftrightarrow$ $P(k)$ Σ_k-COF., $P(k)$ CONTRACTIBLE
$P \xrightarrow{\sim} Com$ (OPERAD OF COMM. ALG.)

$n=2 \Leftrightarrow$ [FIEDOROWICZ] "BRAIDED OPERADS"

GENERAL n $\Leftrightarrow$???

BERGER-CELLULATION METHOD

LET A BE A POSET ($\leftrightarrow$ CATEGORY), $BA := |N(A)|$)

DEF: A SPACE X HAS AN A-CELLULATION IF

$\exists\, C : A \longrightarrow Top$ s.t.
$\alpha \longmapsto C_\alpha$

(i) $\operatorname{Colim}_{\alpha \in A} C_\alpha \cong X$

(ii) $\forall \alpha \in A$, $\operatorname{colim}_{\beta < \alpha} C_\beta \hookrightarrow X$ (PART OF THE STRUCTURE) IS A CLOSED COFIBRATION

(iii) $\forall \alpha$, C_α IS CONTRACTIBLE

PROP: X ADMITS AN A-CELLULATION $\Rightarrow$ $X \simeq BA$

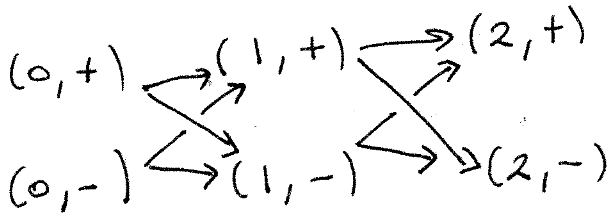
PROOF: $X \xleftarrow[(i)]{\sim} \operatorname{hocolim}_\alpha C_\alpha \xrightarrow{\sim} \operatorname{hocolim}_\alpha * = BA$.

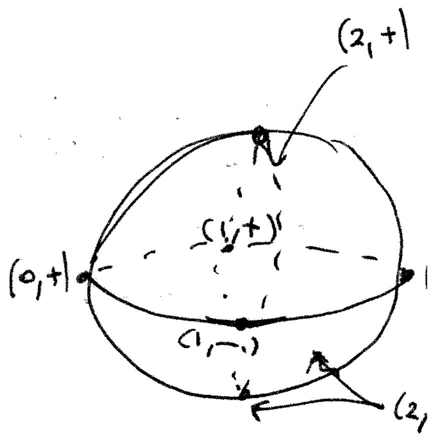
EX: S^{n-1} HAS A $K_n^{(2)}$-CELLULATION

$K_n^{(2)} = \{ (\mu, \sigma) \in \{0,1,\dots,n-1\} \times \Sigma_2 \}$, $\Sigma_2 = \{+,-\}$

$(\mu, \sigma) \leq (\nu, \tau)$ IF $\begin{cases} (\mu,\sigma) = (\nu,\tau) \\ \mu < \nu \end{cases}$

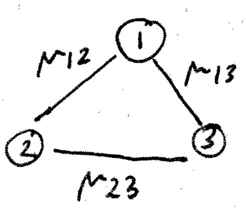
EX: $n = 3$





DEF: $K_n(k) := \{ (\mu, \sigma) \in \{0,1,\dots,n-1\}^{\times \binom{k}{2}} \times \Sigma_k \}$

$\leftrightarrow$ COMPLETE GRAPHS EX: $k = 3$



\+ ORIENTATION ON EDGES GIVEN BY σ
$(i) \longrightarrow (j)$ IF $\sigma(i) > \sigma(j)$
$\longleftarrow$ OTHERWISE

$K(\ell) := \{ (\mu, \sigma) \in \mathbb{N} \quad \times \Sigma_\ell \}$

CLAIM: EACH K_n FORMS AN OPERAD IN POSETS.

PROP: BK_n IS A TOPOLOGICAL E_n-OPERAD.

PROP: LET $\mathcal{P}$ BE A TOPOLOGICAL OPERAD WITH A COMPATIBLE K_n-CELLULATION, i.e. $\mathcal{P}(k)$ HAS A $K_n(\ell)$ CELLULATION s.t. $\forall (c_\alpha, c_{\alpha_1}, \dots, c_{\alpha_\ell}) \in \mathcal{P}(\ell) \times \mathcal{P}(i_1) \times \dots \times \mathcal{P}(i_\ell)$

$$\gamma_{\mathcal{P}}(c_\alpha, c_{\alpha_1}, \dots, c_{\alpha_\ell}) = c_{\gamma_{K_n}(\alpha, \alpha_1, \dots, \alpha_\ell)}$$

THEN $\mathcal{P}$ IS AN E_n-OPERAD.

$\mathcal{C}_n(\ell) \sim \mathrm{Conf}_{\mathbb{R}^n}(\ell)$ —COMPACTIFICATION→ FM-OPERAD

($\mathcal{C}_n(\ell)$: OPERAD ⇝ $\mathrm{Conf}_{\mathbb{R}^n}(\ell)$: "OPERAD UP TO HOMOTOPY")

FM-OPERAD: [TAMARKIN] ~~CELLULAR DECOMPOSITION~~ [GETZLER-JONES]

BATANIN: n-OPERADS (NOT OPERADS...) (ALTERNATIVE TO CELLULAR DECOMPOSITION, TO FIX)

- COMBINATORIAL DEFINITION

PROP: X ACTED ON BY A CONTRACTIBLE n-OPERAD $\Rightarrow X \simeq \Omega^n Y$

3. HOMOTOPY INVARIANCE OF ALG. STRUCTURE

$X \simeq Y$ [STASHEFF]

(X: MONOID ⇝ Y: A_∞-SPACE)

(ALGEBRA DOES NOT MIX WELL WITH HTPY ...

MODELLED BY $\mathrm{Ass} \xleftarrow{\sim} A_\infty$ COFIBRANT OPERAD

(Ass: NOT COF.)

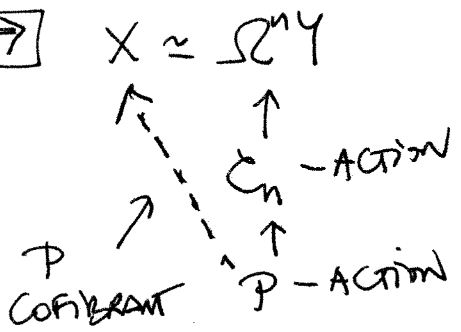
[BOARDMAN-VOGT] NEED TO WORK WITH COFIBRANT OPERADS IN CAT OF OPERADS

PROBLEM: $\mathcal{C}_n$ IS NOT COFIBRANT !

PROPOSITION: LET $\mathcal{P}$ BE A COFIBRANT REPLACEMENT OF $\mathcal{C}_n$.
THEN $X \simeq \Omega^n Y$ IFF X $\mathcal{P}$-ALG.

PF: [⇐] $\mathcal{P} \xrightarrow{\sim} \mathcal{C}_n$
COFIBRANT ⟶ Σ-COFIBRANT [BERGER-MOERDIJK]] ⇒ CAN APPLY PREVIOUS PROP.

[⇒]



$\mathcal{E}$ ⟵ $\mathbb{S}$-MODULES $\overset{F}{\underset{U}{\rightleftarrows}}$ OPERADS

MODEL CAT — $\mathbb{S}$-MODULES: MODEL CAT, COF. GENERATED — TRANSFER — OPERADS: MODEL STRUCTURE

(w.e., fib) ⟵FORGET— (w.e., fib)

COFIBRANT IN dg-MOD = RETRACTS OF QUASI-FREE

COFIBRANT: CAN BE DEFINED BY A LIFTING PROPERTY WITHOUT MODEL STRUCTURE

QUASI-FREE: IN ALGEBRA (C,d) ST. $C = \bigoplus_{i=0}^{\infty} C_i$

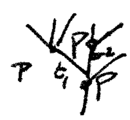
$d : C_i \longrightarrow T\left(\bigoplus_{j=0}^{i-1} C_j\right)$

WHAT YOU THINK
— FIBRATIONS AND W.E. ARE WHAT YOU THINK FROM $\mathbb{S}$-MODULES
NOT TRIVIAL THAT THIS DEFINES A MODEL STRUCTURE ...

ANY OPERAD HAS A COFIBRANT REPLACEMENT.

W-CONSTRUCTION OF [BV]: FUNCTORIAL COFIBRANT REPLACEMENT

$W\mathcal{P} \xrightarrow{\sim} \mathcal{P}$.



4 — 93-94 REVOLUTION.

① Top ⟶ dg. MODULES — WITH MODEL CATEGORY STRUCTURES --

② W-CONSTR. | ΩB (↑ COBAR, ↖ BAR) [GETZLER-JONES, GINZBURG-KAPRANOV]

$\Omega BP \xrightarrow{\sim} P$

└ TREES WITH EDGE LENGTH $t_i = 0$ OR 1.

③ KOSZUL DUALITY THEORY

$\Omega(P^{i}) \xrightarrow{\sim} P$

$P^{i} \subset H_{\cdot}(BP)$ — KOSZUL DUAL CONSTRUCTION

④ INTRODUCTION OF fC_2 : FRAMED LITTLE DISCS [GETZLER]

$\cap$

R_0 = RIEMANN SPHERE OPERAD

$\cap$

R = RIEMANN SURFACE PROP

ALGEBRAS ARE TCFT'S

⑤ CYCLIC OPERADS

[KONTSEVICH IS A GHOST BEHIND EVERYTHING HERE-]

⑥ DELIGNE'S CONJECTURE !

$CH^{*}(A,A)$ $\quad$ $HH^{*}(A,A)$

↑ $\quad$ ↑

MANY MODELS → (⋮) $\quad$ $H_*(e_2) \cong G$

↓

$C_*(e_2)$

(NOTE: BAD QUESTION ~~BEFORE~~ e_2 NOT COFIBRANT...)

HOMOLOGY OPERATION AND HOMOTOPY THEORY

$C^*_{sing}(X) \longleftarrow E_\infty$-ALG [MANDELL]

ENCODES HOMOTOPY TYPE OF X

$$B^n C^\bullet_{sing}(X) \cong C^\bullet_{sing}(\Omega^n X) \cong C^{E_n}(C^\bullet(X))$$

AS AN E_∞-ALG

B PRODUCES AGAIN AN E_∞-ALG

E_n-HOMOLOGY –