

# E<sub>∞</sub>-ALG AND p-ADIC H<sub>X</sub>-THEORY

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GENERAL SETUP:  $\mathcal{C} \xrightarrow{F} \mathcal{D}$

$$\text{Map}^h(X, Y) \xrightarrow[?]{} \text{Map}^h(FX, FY) \quad (\text{eg DWYER-KAN FUNCTION CPX'S})$$

$\pi_{0\text{top}}^h(X, Y) = [X, Y]$

QUESTION: HOW MUCH OF  $Y$  CAN WE RECOVER FROM  $FY$   
+ EXTRA STRUCTURE.

EX OF EXTRA STRUCTURE:  $FY$  HAS NATURAL  $(F, F)$ -ACTION.

EXAMPLE (MANDER) E<sub>∞</sub>-ALG STRUCTURE ON COCHAINS.

$$\text{Map}^h(X, Y) \xrightarrow[?]{} \text{Map}_{E_{\infty} \mathbb{F}_p\text{-ALG}}^h(C^*Y, C^*X) \quad C^* = C^*(-; \mathbb{F}_p)$$

$$X = * \quad \text{Map}^h(*, Y) \xrightarrow[?]{} \text{Map}_{E_{\infty} \mathbb{F}_p\text{-ALG}}^h(C^*Y, \mathbb{F}_p)$$

IS  
Y

IN WHICH CASE WE WOULD RECOVER THE SPACE.

MIGHT HOPE TO RECOVER  $Y_p^\wedge$ .

THM (MANDER) (\*\*)

$$\text{Map}^h(X, Y) \xrightarrow{\sim} \text{Map}_{E_{\infty} \mathbb{F}_p\text{-ALG}}^h(C^*(Y; \overline{\mathbb{F}}_p), C^*(X; \overline{\mathbb{F}}_p))$$

$\uparrow$  ALG. CLOSURE

- MODULE:
- p-COMPLETION
  - NILPOTENCY
  - FINITENESS
  - CONNECTED

CONDITIONS ON SPACES.

MAIN THEOREM (VIA INDEX)

$$\text{hom}_{\text{Ho}(s\text{Sets})} (X, Y) \cong_{\substack{\text{NATURAL} \\ \text{IN } X, Y}} \text{hom}_{\text{Ho}(\mathbb{F}_p\text{-ALG})} (C^*Y, C^*X)$$

/

HOMOTOPY CATEGORIES

$C^* = C^*(-; \mathbb{F}_p)$

LEM: THE TECHNIQUES THAT PROVE (\*) WILL IMPLY (\*\*).

LET  $k = \overline{\mathbb{F}_p}$ .

COCHAIN CPX'S = dg  $k$ -MODULE ( $|d| = 1$ )

$X \in s\text{Sets} = \text{Set}^{\Delta^{op}}$  = SIMPLICIAL SETS

$$X: X_0 \rightrightarrows X_1 \rightrightarrows X_2 \dots$$

$C^*(X) \in$  COCHAIN CPX'S

"NORMALISED COCHAIN ON X"

$$C^*(X) = N \left( \begin{array}{c} \prod X \\ \uparrow \\ (k\text{-Mod})^\Delta \end{array} \right) \quad \prod_{X_0} k \rightleftharpoons \prod_{X_1} k \rightleftharpoons \prod_{X_2} k \dots$$

$(kS = \prod_S k, (kS)^* = \text{Hom}(\prod_S k, k) \cong \prod_S k)$

DOUB-KAN:

$$\begin{array}{ccc} (k\text{-Mod})^{\Delta^{op}} & \xrightleftharpoons[N^{-1}]{N} & (\text{NON-DEG COCHAIN CPX'S}) \\ \uparrow & & \uparrow \\ B \in (k\text{-Mod})^\Delta & \xrightleftharpoons[N^{-1}]{N} & (\text{NON-DEG COCHAIN CPX'S}) \end{array}$$

} EQUIV. OF CATEGORIES

$N$  = NORMALISATION

$N^{-1}$  = "DENORMALISATION"

$$N(B) = \text{hom}_{k\text{-Mod}} (N \Delta[-], B)^\Delta \cong \text{hom}_{(k\text{-Mod})^\Delta} (N \Delta[-], B)$$

↑  
SIMPLICIAL  
NORMALIZATION

$\Delta[n] \in s\text{Sets}$

$\mathcal{E} = \text{OPERAD in COCHAIN CPX'S}$

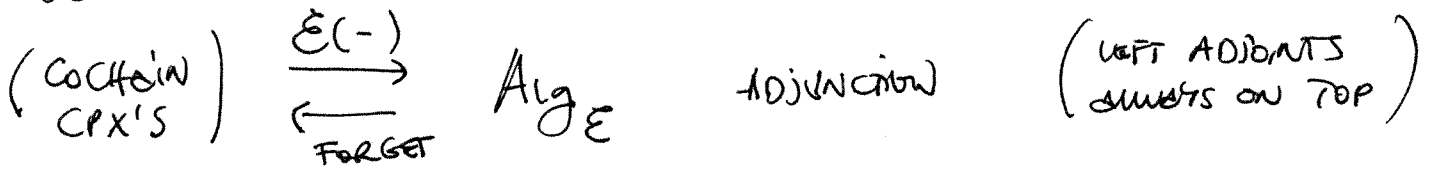
$\bullet \mathcal{E}(n) \simeq k \quad \forall n$

$\bullet \mathcal{E}$  ACTS ON COCHAINS OF  $X \in \text{sSets}$

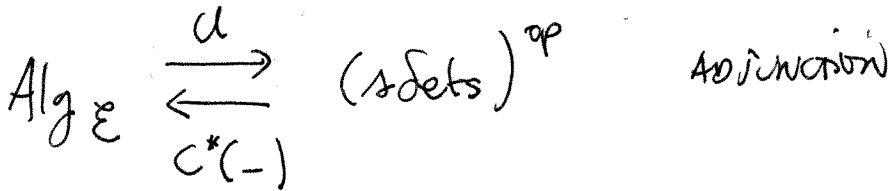
$\bullet$  EACH  $\mathcal{E}(n)$  IS  $\Sigma_n$ -FLAT (i.e.  $\mathcal{E}(n) \otimes_{\Sigma_n} -$  PRESERVES WEAK EQIV.)

THIS EXISTS ...

$\text{Alg}_{\mathcal{E}} = \mathcal{E}\text{-ALG.}$

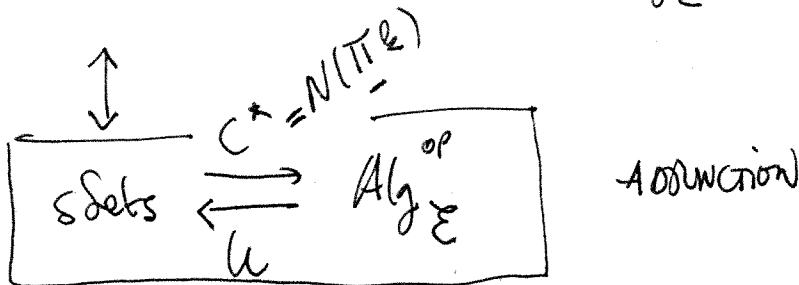


$X \quad \mathcal{E}(X) = \coprod_{t \in X} \mathcal{E}(t) \otimes X^{\otimes t}$ 
FREE ALGEBRA ON X.



$uA := \text{hom}_{\text{Alg}_{\mathcal{E}}}(A, C^* \Delta[-1])$

$\text{hom}_{\text{sSets}^{\text{op}}}(uA, X) \cong \text{hom}_{\text{Alg}_{\mathcal{E}}}(A, C^*X)$



REM:  $C^*$  SENDS MONOMORPHISMS TO LEVELWISE SURJECTIONS  
 $C^*$  PRESERVES W.R.  
 $\hookrightarrow$  IN  $\text{Alg}_{\mathcal{E}}$  W.R. = HOMOMLOGY ISO'S.

TO PROVE THE MAIN THEOREM

GENERAL KEY STEPS:

$$Ho(\mathcal{S}Sets) = (\mathcal{S}Sets + \text{w.e. FORMALITY INVERTED})$$

① USE MODEL CATEGORY TECHNIQUES TO PROVE THAT THE DERIVED ADJUNCTION EXISTS

$$Ho(\mathcal{S}Sets) \begin{array}{c} \xrightarrow{\underline{C}^*} \\ \xleftarrow{\underline{U}} \end{array} Ho(\mathcal{A}lg_{\mathcal{E}}^{op})$$

REM:  $\underline{U}(A) \simeq U(A^c)$

FOR ANY  $A^c \xrightarrow{\sim} A$   
 "COFIBRANT" = BUILT BY ATTACHING CELLS

$$\underline{C}^*(X) \simeq C^*(X^c) \quad \text{FOR ANY } X^c \xrightarrow{\sim} X \text{ IN } \mathcal{S}sets$$

(EVERYTHING IS COFIBRANT IN  $\mathcal{S}sets$ )

UNIT OF ADJUNCTION:  $id \rightarrow \underline{U} \underline{C}^*$

QUESTION: FOR WHAT SPACES  $Y$  IS IT TRUE THAT

$$Y \xrightarrow{\simeq} \underline{U} \underline{C}^* Y$$

CALL SUCH  $Y$ 'S RESOLVABLE.

MAIN: EVERY  $p$ -COMPLETE, FINITE, NILPOTENT, CONNECTED SPACE IS RESOLVABLE. ( $\Rightarrow$  MAIN THM)  $\Rightarrow$

NOTE:  $\text{hom}_{Ho(\mathcal{S}Sets)}(X, \underline{U} A) \cong \text{hom}_{Ho(\mathcal{A}lg_{\mathcal{E}}^{op})}(C^* X, A) \cong \text{hom}_{Ho(\mathcal{A}lg_{\mathcal{E}})}(A, C^* X)$   
 $\cong \text{hom}_{Ho(\mathcal{A}lg_{\mathcal{E}})}(C^* Y, C^* X)$  ( $A = C^* Y$ )  
 $\cong \text{hom}_{Ho(\mathcal{S}Sets)}(X, \underline{U} C^* Y) \cong \text{hom}_{Ho(\mathcal{S}Sets)}(X, Y)$

② SHOW THAT  $K(\mathbb{Z}/p, n)$ ,  $K(\mathbb{C}_p, n)$  ARE RESOLVABLE  $\forall n$ .

③ PROVE THE FOLLOWING:

PROP: IF  $X = \text{li} (X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \dots)$

$$\downarrow \ast$$

$$\text{AND } \text{Coli } H^*(X_n) \xrightarrow{\cong} H^*X$$

THEN  $(X_n \text{ RESOLVABLE } \forall n) \Rightarrow (X \text{ RESOLVABLE})$

④ PROVE THE FOLLOWING:

PROP: IF  $X, Y, Z$  FIBRANT, CONNECTED, OF FINITE  $p$ -TYPE SPACES AND  $Z$  IS SIMPLY CONNECTED

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & Y \\ \downarrow \text{HTM} & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

THEN  $X, Y, Z$  RESOLVABLE

$\Rightarrow X \times_Z Y$  RESOLVABLE

REM: A CONNECTED SPACE IS

- $p$ -COMPLETE
- NILPOTENT
- FINITE  $p$ -TYPE

IFF ITS POSTNIKOV TOWER HAS A PRINCIPAL REFINEMENT IN WHICH EACH FIBER HAS THE HTM TYPE OF  $K(\mathbb{Z}/p, n)$  OR  $K(\mathbb{Z}/p^1, n)$ .

KEY TO ④:

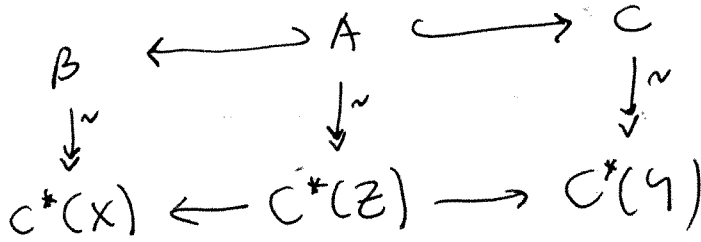
$$A \perp\!\!\!\perp B \simeq A \otimes B$$

$$\simeq A^c \perp\!\!\!\perp B^c \text{ in } \mathcal{E}\text{-Alg}$$

←  $A^c \otimes B^c$  FORGETTING DOWN TO CHAIN CPX'S  
 $\overset{''}{\simeq} A^c \otimes B$   
 $\overset{''}{\simeq} A \otimes B^c$  — MUCH SIMILAR —

PROOF OF ④: SEVERAL KEY STEPS

HAVE  $X \rightarrow Z \leftarrow Y$



EXISTS IN  $\mathcal{E}\text{-Alg}$  BY SMALL OBJ. ARGUMENT  
 $A, B, C$  COFIBRANT  
 $\overset{''}{\simeq}$  BUILT BY ATTACHING COUS

(4a)  $B \underset{A}{\perp\!\!\!\perp} C \simeq B \overset{L}{\otimes} C \quad (\simeq B \otimes C \text{ AS } B, C \text{ COFIBRANT})$

$\overset{''}{\simeq}$  (BUT AS  $B, C$  COFIBRANT) AS CHAIN CPX'S

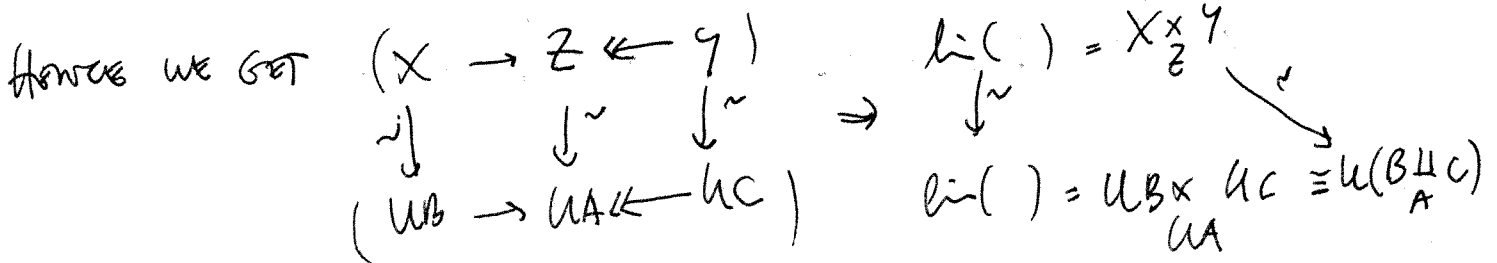
(4b)  $B \underset{A}{\perp\!\!\!\perp} C \xrightarrow[?]{\simeq} C^*(X \overset{x}{\times} \overset{y}{Z}) \simeq C^*(X \overset{h}{\times} \overset{y}{Z})$

$\overset{''}{\simeq}$   $B \underset{A}{\perp\!\!\!\perp} C \simeq C^*(X \underset{C^*(Z)}{\perp\!\!\!\perp} C^*(Y))$

is.  $C^*$  TAKE HIT? PULL-BACKS TO HOMOTOPY PUSH-OUTS

Rem: (4b)  $\Rightarrow$  THE UNIT  $X \overset{x}{\times} \overset{y}{Z} \rightarrow \underline{U} \overset{C^*}{=} (X \overset{x}{\times} \overset{y}{Z})$

IS REPRESENTED BY  $X \overset{x}{\times} \overset{y}{Z} \rightarrow U \underset{A}{\perp\!\!\!\perp} (B \underset{A}{\perp\!\!\!\perp} C)$



We want to prove (4b).

Motivation for proof:  $B \underset{A}{\parallel} C \stackrel{?}{\simeq} C^* \left( X \underset{Z}{\times} Y \right)$

$$B \underset{A}{\parallel} C = \text{colim}_{\Delta^{\text{op}}} (B \parallel C \leftarrow B \parallel A \parallel C)$$

$$\text{colim}_{\Delta^{\text{op}}} \left( B \parallel C \leftarrow B \parallel A \parallel C \right) \xrightarrow{\cong} \text{colim}_{\Delta^{\text{op}}} (B \parallel A \parallel A \parallel C \dots)$$

$\text{Bar}^{\parallel}(B, A, C)$

$$B \underset{A}{\parallel} C \stackrel{?}{\simeq} \text{hocolim}_{\Delta^{\text{op}}} \text{Bar}^{\parallel}(B, A, C)$$

Is?

$$N \text{Bar}^{\parallel}(B, A, C)$$

$$X \underset{Y}{\times} Z = \text{li} (X \times Y \rightarrow X \times Y \times Z)$$

$$\text{li} (X \times Y \rightarrow X \times Y \times Z \rightarrow X \times Y \times Y \times Z \dots)$$

$\text{Cobar}^{\times}(X, Z, Y)$

$$X \underset{Z}{\times} Y \stackrel{?}{\simeq} \text{hocolim}_{\Delta} \text{Cobar}^{\times}(X, Z, Y)$$

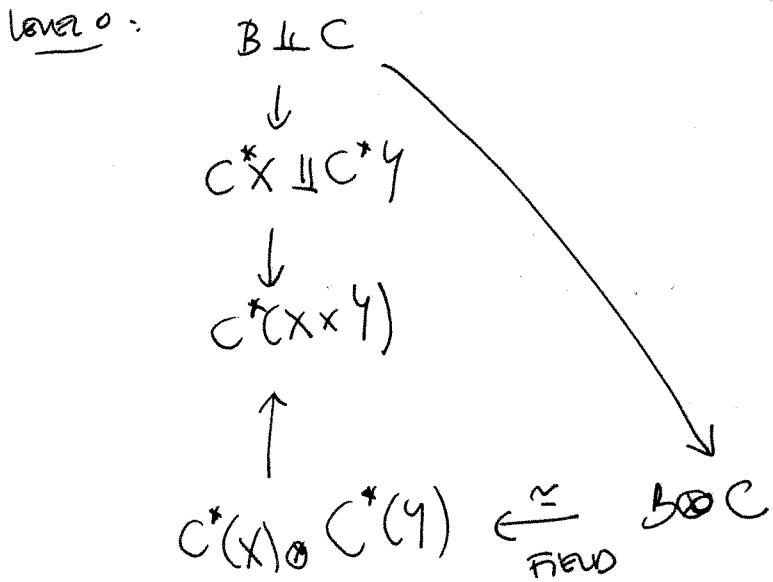
$$C^* \left( \right) \stackrel{?}{\simeq} \text{hocolim}_{\Delta^{\text{op}}} C^* \text{Cobar}^{\times}(X, Z, Y)$$

PROOF SKETCH OF (4b) ASSUMING (4a):

$$B \underset{A}{\parallel} C \longrightarrow C^* (X \times Z \times Y) \xrightarrow[\cong]{\text{E-Moore}} C^* X \underset{C^* Z}{\otimes} C^* Y \xleftarrow[\text{fibre}]{\cong} N \cdot C^* \text{Cobar}^{\times}(X, Z, Y)$$

$$B \underset{A}{\parallel} C \stackrel{?}{\simeq} N \text{Bar}^{\parallel}(B, A, C) \simeq \text{hocolim}_{\Delta^{\text{op}}} \text{Bar}^{\parallel}(B, A, C) \xrightarrow[\cong]{\text{(4a)}} \text{hocolim}_{\Delta^{\text{op}}} C^* \text{Cobar}^{\times}(X, Z, Y)$$

REM:  $\text{Bar}^+(B, A, C) \xrightarrow[\text{LEVELWISE W.E.}]{} \cup \text{Cobar}^+(X, Z, Y)$



WANT TO PROVE (4a)  $B \ll C \stackrel{?}{\simeq} B \otimes C$

THM:  $B, C$  COFIBRANT  $E$ -ALGEBRAS  
 THEN  $B \ll C \stackrel{?}{\simeq} B \otimes C$

STEP 1: PROVE THAT  $B \ll E(Z) \simeq B \otimes E(Z)$

STEP 2: USE STEP 1 IN AN INDUCTION...

PROOF:  $B \leftarrow E(B) \xrightarrow{\simeq} E E B$   
 $E(Z) \leftarrow E(Z) \xrightarrow{\simeq} E(Z)$

REFLEXIVE COEQUIVIZERS IN  
 $\text{Alg}_E$  COMPUTING  
 $B$  AND  $E(Z)$

COIN OF REFLEXIVE COEQUIVIZERS COMPUTED WITH FORGETFUL FUNCTORS

$$B \ll E(Z) \leftarrow E(B) \ll E(Z) \xrightarrow{\simeq} E E(B) \ll E(Z)$$

$$\stackrel{\text{is}}{\simeq} E(B \otimes Z) \simeq \bigoplus_t E(t) \otimes (B \otimes Z)^{\otimes t} \cong \bigoplus_{p+q} E(p+q) \otimes B^{\otimes p} \otimes Z^{\otimes q}$$

$Z$  DOESN'T PLAY A ROLE IN THE MAPS



$$\rightsquigarrow \text{PROP: } B \ll E(Z) \cong \bigoplus_{\Sigma_g} E_B(g) \otimes_{\Sigma_g} Z$$

SOMETHING BUILD BY THE PREVIOUS DIAGRAMS

$$\rightarrow B \ll E(Z) \cong \bigoplus_{\Sigma_g} E_B(g) \otimes_{\Sigma_g} Z^{\otimes g}$$

?  $\downarrow \sim$

$$B \otimes E(Z) = \bigoplus_{\Sigma_g} B \otimes E(g) \otimes_{\Sigma_g} Z^{\otimes g}$$

ENOUGH TO PROVE THAT

AS  $\Sigma_g$ -FLAT ...

