E_∞-ALG AND p-ADIC H_*-THEORY

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GENERAL SET UP: $\mathcal{C} \xrightarrow{F} \mathcal{D}$

$$\mathrm{Map}^h(X,Y) \underset{?}{\xrightarrow{\sim}} \mathrm{Map}^h(FX, FY)$$

(eg DWYER-KAN FUNCTION CPX's $\pi_0 \mathrm{Map}^h(X,Y) = [X,Y]$)

QUESTION: HOW MUCH OF Y CAN WE RECOVER FROM FY + EXTRA STRUCTURE.

EX OF EXTRA STRUCTURE: FY HAS NATURAL (F,F)-ACTION.

EXAMPLE (MANDELL) E_∞-ALG STRUCTURE ON COCHAINS.

$$\mathrm{Map}^h(X,Y) \underset{?}{\xrightarrow{\sim}} \mathrm{Map}^h_{E_\infty\, \mathbb{F}_p\text{-ALG}}(C^*Y, C^*X) \qquad C^* = C^*(-;\mathbb{F}_p)$$

$X = *$ $\quad \mathrm{Map}^h(*,Y) \underset{?}{\xrightarrow{\sim}} \mathrm{Map}^h_{E_\infty\, \mathbb{F}_p\text{-ALG}}(C^*Y, \mathbb{F}_p)$ IN WHICH CASE WE WOULD RECOVER THE SPACE.

IS Y

MIGHT HOPE TO RECOVER $Y^\wedge_p$.

THM (MANDELL) (**)

$$\mathrm{Map}^h(X,Y) \xrightarrow{\sim} \mathrm{Map}^h_{E_\infty\, \mathbb{F}_p\text{-ALG}}(C^*(Y;\overline{\mathbb{F}}_p), C^*(X;\overline{\mathbb{F}}_p))$$

ALG. CLOSURE

MODULO:
- p-COMPLETION
- NILPOTENCY
- FINITENESS
- CONNECTED

CONDITIONS ON SPACES.

MAIN THEOREM (MANDELL)

$$\hom_{Ho(sSets)}(X,Y) \underset{\text{NATURALLY IN } X,Y}{\cong} \hom_{Ho(E_\infty \overline{\mathbb{F}}_p\text{-ALG})}(C^*Y, C^*X)$$

HOMOTOPY CATEGORY

$C^* = C^*(-;\overline{\mathbb{F}}_p)$

REM: THE TECHNIQUES THAT PROVE (*) WILL IMPLY (**).

LET $k = \overline{\mathbb{F}_p}$.

COCHAIN CPX'S = dg k-MODULE ($|d|=1$)

$X \in sSets = Set^{\Delta^{op}}$ = SIMPLICIAL SETS

$X: X_0 \leftleftarrows X_1 \Lleftarrow X_2 \cdots$

$C^*(X) \in$ COCHAIN CPX'S

$\parallel$

NORMALISED COCHAINS ON X

$$C^*(X) = N\Big(\prod_X k\Big) \qquad \prod_{X_0} k \rightrightarrows \prod_{X_1} k \Rrightarrow \prod_{X_2} k \cdots$$

$\prod_X k \in (k\mathrm{Mod})^{\Delta}$

$\left(kS = \coprod_S k, \ (kS)^* = \mathrm{Hom}(\coprod_S k, k) \cong \prod_S k\right)$

DOLD-KAN:

$$(k\text{-Mod})^{\Delta^{op}} \underset{N^{-1}}{\overset{N}{\rightleftarrows}} \begin{pmatrix}\text{NON-DEG}\\ \text{CHAIN CPX'S}\end{pmatrix}$$

$$B \in (k\text{-Mod})^{\Delta} \underset{N^{-1}}{\overset{N}{\rightleftarrows}} \begin{pmatrix}\text{NON-DEG}\\ \text{COCHAIN CPX'S}\end{pmatrix}$$

EQUIV. OF CATEGORIES

$$N(B) = \hom_{k\mathrm{Mod}}(N\,k\underbrace{\Delta[-]}_{\Delta[n] \in sSets}, B)^{\Delta} \cong \hom_{(k\mathrm{Mod})^{\Delta}}(N k\Delta[-], B)$$

SIMPLICIAL NORMALIZATION

N = NORMALISATION

N^{-1} = "DENORMALISATION"

$\mathcal{E}$ = OPERAD ON COCHAIN CPX'S

- $\mathcal{E}(n) \simeq k \quad \forall n$
- $\mathcal{E}$ ACTS ON COCHAINS OF $X \in$ sSets
- EACH $\mathcal{E}(n)$ is Σ_n-FLAT (ie. $\mathcal{E}(n) \otimes_{\Sigma_n} -$ PRESERVES WEAK EQUIV.

THIS EXISTS ...

$Alg_{\mathcal{E}} = \mathcal{E}$-ALG.

$$\begin{pmatrix}\text{COCHAIN}\\ \text{CPX'S}\end{pmatrix} \underset{\text{FORGET}}{\overset{\mathcal{E}(-)}{\rightleftarrows}} Alg_{\mathcal{E}}$$ ADJUNCTION (LEFT ADJOINTS ALWAYS ON TOP)

$$X \qquad \mathcal{E}(X) = \coprod_t \mathcal{E}(t) \otimes_{\Sigma_t} X^{\otimes t}$$ FREE ALGEBRA ON X.

$$Alg_{\mathcal{E}} \underset{C^*(-)}{\overset{\mathcal{U}}{\rightleftarrows}} (sSets)^{op}$$ ADJUNCTION

$$\mathcal{U}A := \hom_{Alg_{\mathcal{E}}}(A, C^*\Delta[-])$$

$$\hom_{sSets^{op}}(\mathcal{U}A, X) \cong \hom_{Alg_{\mathcal{E}}}(A, C^*X)$$

$\updownarrow$

$$\boxed{sSets \underset{\mathcal{U}}{\overset{C^* = N(\Pi^k)}{\rightleftarrows}} Alg_{\mathcal{E}}^{op}}$$ ADJUNCTION

__REM__: C^* SENDS MONOMORPHISMS TO LEVELWISE SURJECTIONS

C^* PRESERVES W.E.

IN $Alg_{\mathcal{E}}$ W.E. = HOMOLOGY ISO'S.

TO PROVE THE MAIN THEOREM, MANDELL USES SEVERAL KEY STEPS:

$Ho(sSets) = (sSets + \text{w.e. FORMALLY INVERTED})$

① USE MODEL CATEGORY TECHNIQUES TO PROVE THAT THE DERIVED ADJUNCTION EXISTS

$$Ho(sSets) \underset{\underline{\underline{u}}}{\overset{\underline{\underline{C}}^*}{\rightleftarrows}} Ho(Alg_{\mathcal{E}}^{op})$$

<u>REM</u>: $\underline{\underline{u}}(A) \simeq u(A^c)$ FOR ANY $A^c \xrightarrow{\sim} A$
"COFIBRANT" = BUILD BY ATTACHING CELLS

$\underline{\underline{C}}^*(X) \simeq C^*(X^c)$ FOR ANY $X^c \xrightarrow{\sim} X$ IN Ssets

(EVERYTHING IS COFIBRANT IN Ssets)

UNIT OF <u>ADJUNCTION</u>: $id \longrightarrow \underline{\underline{u}}\,\underline{\underline{C}}^*$

QUESTION: FOR WHAT SPACES Y IS IT TRUE THAT

$$Y \xrightarrow{\simeq} \underline{\underline{u}}\,\underline{\underline{C}}^* Y$$

CALL SUCH Y'S <u>RESOLVABLE</u>.

<u>MANDELL</u>: EVERY p-COMPLETE, FINITE, NILPOTENT, CONNECTED SPACE IS <u>RESOLVABLE</u>. ($\Rightarrow$ MAIN THM)

<u>NOTE</u>: $\hom_{Ho(sSets)}(X, \underline{\underline{u}}A) \cong \hom_{Ho(Alg_{\mathcal{E}}^{op})}(C^*X, A) \cong \hom_{Ho(Alg_{\mathcal{E}})}(A, C^*X)$

$\cong \hom_{Ho(Alg_{\mathcal{E}})}(C^*Y, C^*X)$ $(A = C^*Y)$

is $\hom_{Ho(sSets)}(X, \underline{\underline{u}}\,\underline{\underline{C}}^*Y) \cong \hom_{Ho(sSets)}(X, Y)$

② SHOW THAT $K(\mathbb{Z}/p, n)$, $K(\mathbb{Z}_p, n)$ ARE RESOLVABLE $\forall n$.

③ PROVE THE FOLLOWING:

PROP: IF $X = \lim (X_0 \twoheadleftarrow X_1 \twoheadleftarrow X_2 \twoheadleftarrow \cdots)$

$$\begin{tikzcd} X_0 \arrow[d, two heads] \\ * \end{tikzcd}$$

AND $\operatorname{Colim} H^*(X_n) \xrightarrow[\cong]{} H^* X$

THEN $(X_n \text{ RESOLVABLE } \forall n) \Rightarrow (X \text{ RESOLVABLE})$.

④ PROVE THE FOLLOWING:

PROP: IF X, Y, Z FIBRANT, CONNECTED, OF FINITE p-TYPE SPACES AND Z IS SIMPLY CONNECTED

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HTPY PULL-BACK}"] & Y \arrow[d, two heads] \\ X \arrow[r] & Z \end{tikzcd}$$

THEN X, Y, Z RESOLVABLE $\Rightarrow X \times_Z Y$ RESOLVABLE.

REM: A CONNECTED SPACE IS

- p-COMPLETE
- NILPOTENT
- FINITE p-TYPE

IFF ITS POSTNIKOV TOWER HAS A PRINCIPAL REFINEMENT IN WHICH EACH FIBER HAS THE HTPY TYPE OF $K(\mathbb{Z}/p, n)$ OR $K(\hat{\mathbb{Z}}_p, n)$.

KEY TO (4): $A \overset{h}{\amalg} B \simeq A \overset{L}{\otimes} B$

$\simeq A^c \amalg B^c$ in $\mathcal{E}$-Alg

$A^c \otimes B^c$ FORGETTING DOWN TO CHAIN CPX'S

$\overset{\text{is}}{A^c \otimes B}$

$\overset{\text{is}}{A \otimes B^c}$ — MUCH SMALLER —

PROOF OF (4): SEVERAL KEY STEPS

HAVE $X \to Z \leftarrow Y$

$$
\begin{tikzcd}
B \arrow[d, "\sim"] & A \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\sim"] & C \arrow[d, "\sim"] \\
C^*(X) & C^*(Z) \arrow[l] \arrow[r] & C^*(Y)
\end{tikzcd}
$$

EXISTS IN $\mathcal{E}$-ALG BY SMALL OBJ. ARGUMENT

A, B, C COFIBRANT $=$ BUILT BY ATTACHING CELLS

(4a) $B \overset{h}{\amalg} C \simeq B \overset{L}{\otimes} C$ ($\simeq B \otimes C$ AS B, C COFIBRANT)

(is $B \amalg C$ AS B, C COFIBRANT) AS CHAIN CPX'S

$\Downarrow$

(4b) $B \underset{A}{\amalg} C \xrightarrow[\simeq]{?} C^*(X \underset{Z}{\times} Y) \simeq C^*(X \overset{h}{\underset{Y}{\times}} Z)$

$\overset{\text{is}}{B \overset{h}{\underset{A}{\amalg}} C} \simeq C^*X \overset{h}{\underset{C^*(Z)}{\amalg}} C^*(Y)$ i.e. C^* TAKE HTPY PULL-BACKS TO HOMOTOPY PUSH-OUTS

Rem: (4b) $\Rightarrow$ THE UNIT $X \underset{Z}{\times} Y \to \underline{U}\,\underline{C}^*(X \underset{Z}{\times} Y)$

is REPRESENTED BY $X \underset{Z}{\times} Y \to U(B \underset{A}{\amalg} C)$

HENCE WE GET

$$
\begin{tikzcd}
(X \arrow[r] \arrow[d, "\sim"'] & Z \arrow[d, "\sim"] & Y) \arrow[l] \arrow[d, "\sim"] \\
(UB \arrow[r] & UA & UC) \arrow[l]
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
\lim(\ ) = X \underset{Z}{\times} Y \arrow[d, "\sim"] \arrow[dr, "\sim"] & \\
\lim(\ ) = UB \underset{UA}{\times} UC & \cong U(B \underset{A}{\amalg} C)
\end{tikzcd}
$$

WE WANT TO USE (4a) TO PROVE (4b).

MOTIVATION FOR PROOF: $B \overset{h}{\underset{A}{\amalg}} C \overset{?}{\simeq} C^*(X \overset{h}{\underset{Z}{\times}} Y)$

$$B \underset{A}{\amalg} C = \mathrm{colim}\,(B \amalg C \leftleftarrows B \amalg A \amalg C)$$

$$\cong$$

$$\mathrm{colim}_{\Delta^{op}} \big(\underbrace{B \amalg C \leftleftarrows B \amalg A \amalg C \ \substack{\leftarrow\\\leftarrow\\\leftarrow} \ B \amalg A \amalg A \amalg C \ \cdots}_{Bar^{\amalg}(B,A,C)} \big)$$

$$B \overset{h}{\underset{A}{\amalg}} C \overset{?}{\simeq} \mathrm{hocolim}_{\Delta^{op}} Bar^{\amalg}(B,A,C)$$

$$\cong ?$$

$$N\, Bar^{\amalg}(B,A,C)$$

$$X \underset{Y}{\times} Z = \lim\,(X \times Y \rightrightarrows X \times Y \times Z)$$

$$\cong$$

$$\lim \big(\underbrace{X \times Y \rightrightarrows X \times Y \times Z \ \substack{\to\\\to\\\to} \ X \times Y \times Y \times Z \ \cdots}_{cobar^{\times}(X,Z,Y)} \big)$$

$$X \overset{h}{\underset{Z}{\times}} Y \overset{?}{\simeq} \mathrm{holim}_{\Delta}\, Cobar^{\times}(X,Z,Y)$$

$$C^*(\ \ ) \overset{?}{\simeq} \mathrm{hocolim}_{\Delta^{op}}\, C^*\, Cobar^{\times}(X,Z,Y).$$

PROOF SKETCH OF (4b) ASSUMING (4a):

$$\begin{tikzcd}
B \underset{A}{\amalg} C \arrow[r] & C^*(X \times_Z Y) \arrow[r, "\text{E-Moore}", "\simeq"'] & C^*X \otimes_{C^*Z} C^*Y & N.C^* Coba^{\times}(X,Z,Y) \arrow[l, "\simeq", "\text{f.type}"'] \\
B \overset{h}{\underset{A}{\amalg}} C \arrow[u, "\cong"] \simeq N\,Bar^{\amalg}(B,A,C) \simeq \mathrm{hocolim}_{\Delta^{op}} Bar^{\amalg}(B,A,C) \arrow[rrr, "\simeq"', "(4a)"] & & & \mathrm{hocolim}_{\Delta^{op}} C^* Cobar^{\times}(X,Z, \arrow[u, "\cong"]
\end{tikzcd}$$

REM: $\mathrm{Bar}^{\amalg}(B,A,C) \xrightarrow[\text{LEVELWISE W.E.}]{} C^*\,\mathrm{Cobar}^{\cdot}(X,Z,Y)$

LEVEL 0:

$$\begin{tikzcd}
B \amalg C \arrow[d] \arrow[ddddr, bend left] & \\
C^*X \amalg C^*Y \arrow[d] & \\
C^*(X\times Y) & \\
& \\
C^*(X)\otimes C^*(Y) \arrow[u] & B\otimes C \arrow[l, "\simeq"', "\text{FIELD}"]
\end{tikzcd}$$

WANT TO PROVE (49) $B \amalg^h C \overset{?}{\simeq} B \otimes^L C$

THM: B, C COFIBRANT $\mathcal{E}$-ALGEBRAS THEN $B \amalg C \simeq B \otimes C$

STEP 1: PROVE THAT $B \amalg \mathcal{E}(Z) \simeq B \otimes \mathcal{E}(Z)$.

STEP 2: USE STEP 1 IN AN INDUCTION...

PROOF:

$$\left.\begin{array}{l} B \leftarrow \mathcal{E}(B) \leftleftarrows \mathcal{E}\mathcal{E}B \\ \mathcal{E}(Z) \leftarrow \mathcal{E}(Z) \leftleftarrows \mathcal{E}(Z) \end{array}\right\}$$

REFLEXIVE COEQUALIZERS IN $\mathrm{Alg}_{\mathcal{E}}$ COMPUTING B AND $\mathcal{E}(Z)$

colim OF REFLEXIVE COEQUALIZERS COMMUTES WITH FORGETFUL FUNCTORS

$$B \amalg \mathcal{E}(Z) \leftarrow \mathcal{E}(B) \amalg \mathcal{E}(Z) \leftleftarrows \mathcal{E}\mathcal{E}(B) \amalg \mathcal{E}(Z)$$

$$\mathcal{E}(B) \amalg \mathcal{E}(Z) \cong \mathcal{E}(B \oplus Z) \cong \bigoplus_t \mathcal{E}(t) \otimes_{\Sigma_t} (B \oplus Z)^{\otimes t} \cong \bigoplus_{p,q} \mathcal{E}(p+q) \otimes_{\Sigma_p} B^{\otimes p} \otimes_{\Sigma_q} Z^{\otimes q}$$

Z DOESN'T PLAY A ROLE IN THE MAPS

$\rightsquigarrow$ <u>PROP</u>: $B \amalg \mathcal{E}(Z) \cong \bigoplus_q \underbrace{\mathcal{E}_B(q)}_{\text{SOMETHING BUILD BY THE PREVIOUS DIAGRAM}} \otimes_{\Sigma_q} Z$

$$\to \begin{tikzcd} B \amalg \mathcal{E}(Z) \arrow[d, "?"', "\sim"] \arrow[r, phantom, "\cong"] & \bigoplus_q \boxed{\mathcal{E}_B(q)} \otimes_{\Sigma_q} Z^{\otimes q} \arrow[d, "?"'] \\ B \otimes \mathcal{E}(Z) \arrow[r, phantom, "="] & \bigoplus_q \boxed{B \otimes \mathcal{E}(q)} \otimes_{\Sigma_q} Z^{\otimes q} \end{tikzcd}$$

ENOUGH TO PROVE THAT

AS/IF Σ_q-FLAT ...