MULTIVARIABLE OPERATIONS AND COSIMPLICIAL SPACES

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$$\mathcal{E} \underset{TC}{\overset{TR}{\rightleftarrows}} \mathcal{X}$$

BARRATT-ECCLES OPERAD — SURJECTION OPERAD

+ Σ_r-FREE BASIS

$\mathcal{X}$ RETRACT OF $\mathcal{E}$ $\to$ $\mathcal{X}(r)$ ACYCLIC $\forall r$ $\Rightarrow$ $\mathcal{X}$ E_∞-OPER

<u>Aim</u>: SHOW THAT $\mathcal{X}(r) = C_*(\mathcal{F}(r))$ WITH $\mathcal{F}(r)$ A REGUL
CW-CPX.

INJECTIVE
ATTACHING M

<u>RECALL</u>: $\mathcal{X}_d(r) = \mathbb{Z}\langle \{1,\dots,r+d\} \overset{f}{\twoheadrightarrow} \{1,\dots,r\} \rangle$
$f(i) \neq f(i+1)\ \forall i$

FREE Σ_r-ACTION: EX: $13132 \rightsquigarrow 132 \in \Sigma_r = \Sigma_3$
(FIRST OCCURENCES)

ACTION OF Σ_r ACTS BY MULT. ON THE ASSOCIATED PERM.

$\mathcal{F}(r) = \{$SPACE OF PARTITIONS OF S^1: $S^1 = \bigcup_{i=1}^{r} M_i$

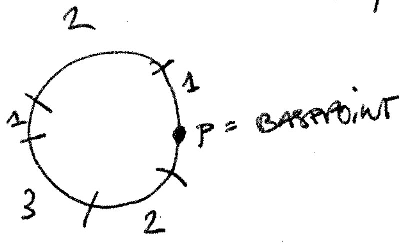
S.t. 1) $M_i \neq \emptyset$ COMPACT 1-MFDS (NOT NEC CONNECTE

2) $\mathring{M}_i \cap \mathring{M}_j = \emptyset$

3) $\mu(M_1) = \dots = \mu(M_r) = \frac{2\pi}{r}$

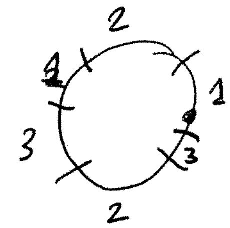
"MEASURE" = LENGTH

<u>EX</u>:



<u>TOPOLOGY</u>: $d(M_1,\dots,M_r; M_1',\dots,M_r') = \sum_{i=1}^{r} \mu(M_i - M_i')$

CLAIM: $\mathcal{J}(r)$ HAS A CELLULAR DECOMPOSITION



$\rightsquigarrow$ 1213231 ASSOCIATED SURJECTION

WHAT ARE THE POINTS OF $\mathcal{J}(r)$ ASSOCIATED TO THIS SURJECTION?

$\underset{1}{(x_1}, \underset{2}{x_2}, \underset{1}{x_3}, \underset{3}{x_4}, \underset{2}{x_5}, \underset{1}{x_6)}$ LENGHTS ARE SUBJECT TO

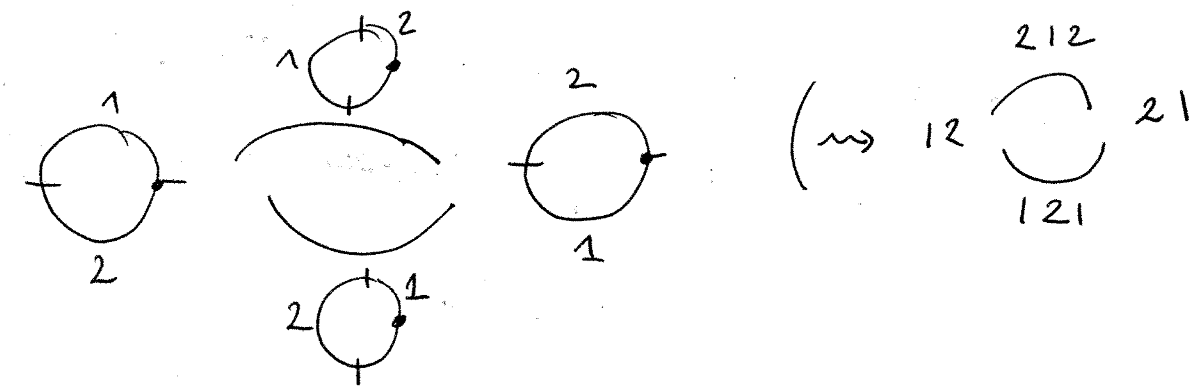
$$\begin{cases} x_1 + x_3 + x_6 = 2\pi/3 & \Delta^2 \\ x_2 + x_5 = 2\pi/3 & \Delta^1 \\ x_4 = \frac{2\pi}{3} & \Delta^0 \end{cases}$$

$\to$ PRODUCT OF SIMPLICES (CELL IN $\mathcal{J}(r)$)

BOUNDARY BY SHRINKING (AS LONG AS THERE ARE SEVERAL COMPONENTS ASSOCIATED TO THIS NUMBER)

EX: $d(12131) = 2131 \pm 1231 \pm 1213$ (IN THE CHAIN CPX!) (HENCE ONLY THE GUYS OF DEGREE -1 COUNT)

EX:



PROBLEM: $\mathcal{J}$ IS NOT A TOPOLOGICAL OPERAD.

$x \in \mathcal{J}(r)$ DEFINE

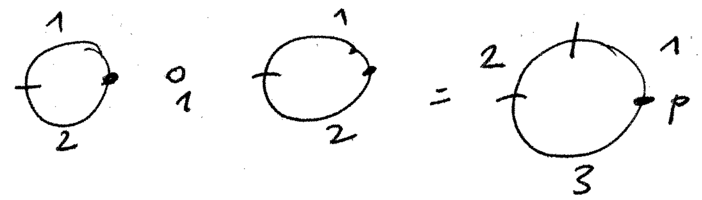
$$\begin{tikzcd} S^1 \arrow[r, "c(x)"] \arrow[dr, "\gamma_j(x)"'] & (S^1)^r \arrow[d, "\pi_j"] \\ & S^1 \end{tikzcd}$$

π_j = PROJ. ON j^{th} FACTOR.

DEGREE 1 MAP COLLAPSES EVERY INTERVAL NOT LABELLED BY j

$c : \mathcal{J} \longrightarrow \mathrm{Coend}(S^1)$ $\qquad \mathrm{Coend}(k) = \{ S^1 \to (S^1)^{\wedge k} \} \, ($

DOES NOT HIT A SUBOPERAD ...



$\mathrm{Mon}(I, \partial I) = \{ f : I \to I$ MONOTONE (WEAKLY INCREASING) $\mid f(0) = 0, f(1) = 1 \}$

$\downarrow$

$\mathrm{Maps}_*(S^1, S^1)$

$\mathcal{J}(r) \times \mathrm{Mon}(I, \partial I) \hookrightarrow \mathrm{Coend}(S^1)(r) = \mathrm{Map}(S^1, (S^1)^{\wedge r})$

$(x, f) \longmapsto c(x) \circ f$

THE IMAGE IS THE SUBOPERAD MS OF

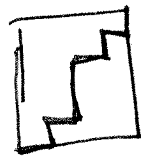
1) "RIGHT-ANGLED" MAPS $g : S^1 \to (S^1)^{r}$ [WEDTERLA

WITH $S^1 = \bigcup_k I_k$ AND $\pi_j \circ g|_{I_k}$ IS CONSTANT $\forall j \neq j_0(k)$

(i.e. MOVES ONLY IN ONE FACTOR OF $(S^1)^{r}$ AT A TIME.)

2) $\pi_j \circ g \in \mathrm{Mon}(I, \partial I)$ FOR EACH j.

THE GRAPH LOOKS LIKE A STAIRCASE



MS IS THE SUBOPERAD OF RIGHT-ANGLED MONOTONE DEGREE 1 MAPS.

$\Rightarrow$ $\mathcal{J} \times \mathrm{Mon}$ IS AN OPERAD

THM : $MS(k) \simeq \mathcal{D}(k) = \mathrm{Tot}(\Delta^\bullet \boxtimes \cdots \boxtimes \Delta^\bullet)$

FROM PASCAL'S TALK

$= \mathrm{Tot}([-]_k(\Delta^\bullet, \ldots, \Delta^\bullet))$

$$\left(\sqcup_k(\Delta^\bullet,\dots,\Delta^\bullet)\right)^s = \underset{\{\underbrace{\underline{k} \xleftarrow{f} T \xrightarrow{h} s}_{\text{obj}}\}}{\operatorname{colim}} \prod_{1\le i\le k} \Delta^{f^{-1}(i)}$$

NOTE: CAN FORGET f's WHICH ARE NOT SURJECTIVE AS WON'T CONTRIBUTE TO THE colim (GET $\emptyset$)

CLAIM 1: $\left(\sqcup_k(\Delta^\bullet,\dots,\Delta^\bullet)\right)^0 \cong \exists(k)$

$$\Big\{\Big(\underline{k} \overset{f}{\twoheadleftarrow} T \to [0],\ 0\le x_t\le 1 \quad t\in T\Big) \,\Big|\, \sum_{f(t)=i} x_t = 1 \ \ \forall 1\le i\le k\Big\}$$

AN SUPPOSE THAT f IS NON-DEG SURJECTION :

$$\begin{tikzcd} \underline{k} & T \arrow[l, "f"'] \arrow[d, "g"] \\ & T' \arrow[ul, "f'"] \end{tikzcd}$$

— CAN "REPLACE" f BY f' - -

CAN ALSO ASSUME $x_i \neq 0 \ \forall i$:

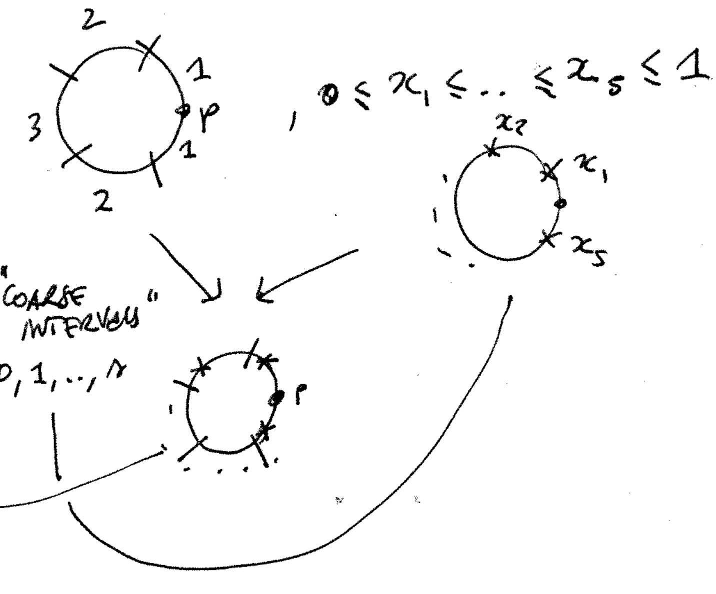
$$\begin{tikzcd} & T_1 \arrow[dl, "f'"'] \arrow[d, hook] \\ \underline{k} & T = T_1 \sqcup \{i\} \arrow[l, "f"] \end{tikzcd}$$

CLAIM 2: $\sqcup_k(\Delta^\bullet,\dots,\Delta^\bullet)^s \cong \sqcup_k(\Delta^\bullet,\dots,\Delta^\bullet)^0 \times \Delta^s \cong \exists(k)\times\Delta^s$

CONSIDER $\underline{k} \overset{f}{\twoheadleftarrow} T \xrightarrow{h} S$ $\quad (f,h)$ s.t. EITHER $f(i)\neq f(i+1)$

OR $h(i)\neq h(i+1)$

ELEMENT IN $\exists(k)\times\Delta^s$: $\quad , 0\le x_1\le \dots \le x_s\le 1$



MANIFOLDS M_i ⟵ "FINE INTERVALS" ⟶ "COARSE INTERVALS"

$\underline{k} \longleftarrow 0,\dots,t \longrightarrow 0,1,\dots,n$

$$\mathcal{D}(\mathcal{E}) \cong \mathrm{Tot}\left(\coprod_{k}(\Delta^*,\dots,\Delta^*)\right)$$
$$= \coprod_{k}(\Delta^*,\dots,\Delta^*)^{\circ} \times \mathrm{Tot}(\Delta^*)$$
$$\cong \mathcal{F}(\mathcal{E}) \times \mathrm{Tot}(\Delta^*) \cong \mathcal{F}(\mathcal{E}) \times \mathrm{Mon}(I, \partial I)$$

FILTRATION: EVERYBODY CAN BE FILTERED HERE ...

RECALL: A SURJECTION HAS COMPLEXITY $\leq n$ IF IT DOES NOT CONTAIN $\underbrace{ijij\cdots}_{n+1}$ $i \neq j$

$\mathcal{X}_n(r) \subseteq \mathcal{X}(r)$ SUBCPX OF SURJ. OF COMPL. $\leq n$

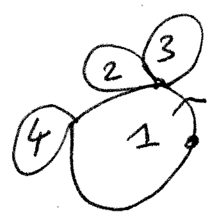
$\rightsquigarrow$ CORRESPONDING FILTRATION OF $\mathcal{F}(r)$ BY $\mathcal{F}_n(r)$

(HAVE $\mathcal{F}_n(2) \cong S^{n-1}$)

$$\coprod_{\mathcal{E}}^{n}(\Delta,\dots,\Delta)^{\circ} \cong \mathcal{F}_n(k)$$

$$MS_n(k) = \mathcal{F}_n(k) \times \mathrm{Mon}(I, \partial I)$$

$MS_2 \longleftrightarrow$ SPINELESS CACTI OPERAD [KAUFFMANN, VORON...
WITH PARAMETRIZATION

$x \in \mathcal{F}_2(r) \rightsquigarrow c(x) : S^1 \to (S^1)^r$
IMAGE OF THE FORM 

CELLULAR OPERADS [BERGER, SEE TALK 1]

IDEA: WRITE AN OPERAD AS UNION OF CONTRACTIBLE PIECES AND LOOK AT THE POSET OF INCLUSIONS AND COMPARE TO THE A POSET ASSOCIATED TO E_n.

DENOTED K_n OPERAD IN POSETS

$|NK_n|$ IS AN E_n-OPERAD

$n=\infty$

$$K(r) = \underset{\substack{\text{TOTAL ORDERS} \\ \text{OF } \{1,\dots,r\}}}{\Sigma_r} \times \mathbb{N}^{\binom{r}{2}}$$

$K_n \subset K$ DETERMINED BY $\{0,\dots,n-1\}^{\binom{r}{2}}$

POSET STRUCTURE: $(\sigma,\mu) < (\tau,\nu)$

if $\forall\ 1 \leq i < j \leq r$ $\quad \mu_{ij} \leq \nu_{ij}$

$\mu_{ij} < \nu_{ij}$ if $\sigma(i) < \sigma(j)$ AND $\tau(i) > \tau(j)$

let $\mathcal{X}(\sigma,\mu) \subset \mathcal{X}$ BE THE SUBCPX GENERATED BY SURJECTIONS SUCH THAT σ IS THE PERMUTATION ASSOCIATED TO THE SURJECTION (ORDER OF FIRST OCCURENCE) AND FOR EACH PAIR $i \neq j$ $\underbrace{iji\cdots j}_{\nexists}$ NOT SUBSEQUENCE OF LENGHT μ_{ij}.

CAN ALSO DEFINE $\mathcal{E}(\sigma,\mu) \underset{TC}{\overset{TR}{\rightleftarrows}} \mathcal{X}(\sigma,\mu)$

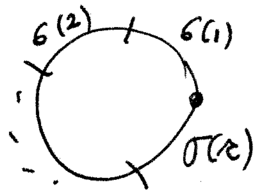
AND $\mathcal{J}(\sigma,\mu)$.

CLAIM: $\mathcal{F}_{(\sigma,\mu)}$ IS A CONTRACTIBLE SUBSPACE.

PROOF SKETCH (?)

~~SUPPOSE $\sigma(1) = 1$ CAN ASSUME THAT BY~~

CONTRACT $\mathcal{F}_{(\sigma,\mu)}$ TO THE POINT



BY FIRST CONTRACTING THE LATER OCCURENCES OF $\sigma(1)$, THAT OF $\sigma(2)$, . . .