THE LATTICE PATH OPERAD AND HOCHSCHILD COCHAINS

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IDEA / DESIRE : PROVE DELIGNE AND CYCLIC DELIGNE CONJ

- 62-63': GERSTENHABER SHOWED THAT $HH^*(A;A)$ CAN BE ENDOWED WITH TWO PRODUCTS: - CUP PRODUCT (COMM.)
  - LIE BRACKET

⇝ "GERSTENHABER ALGEBRA" = ALGEBRA WITH THESE TWO STRUCTURES + RELATIONS

G = ASSOCIATED OPERAD

- 76': [COHEN] $G = H_*(\mathcal{C}_2)$.

- DELIGNE: CAN WE LIFT THE ACTION OF $H_*(\mathcal{C}_2)$ ON $HH^*(A;A)$ TO THE CHAIN LEVEL?

- MORE STRUCTURE ON A: SYMM FROBENIUS ALG

$\Rightarrow HH^*(A;A)$ IS A BV-ALG.

[GETZLER]: $BV = H_*(f\mathcal{C}_2)$ FRAMED LITTLE DISCS.

CYCLIC DELIGNE CONJECTURE: CAN WE LIFT THIS BV-STRUCTURE TO THE CHAIN LEVEL?

FOR THIS, WE INTRODUCE
- $\mathcal{L}$ = LATTICE PATH OPERAD
- $\mathcal{L}^c$ = CYCLIC LATTICE PATH OPERAD

(WON'T ACTUALLY DO THIS).

① CONDENSATION OF A COLOURED OPERAD : COLORED OPERAD $\to$ OPERAD

② LATTICE PATH OPERAD

③ CONDENSATION OF THE LATTICE PATH OPERAD

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① CONDENSATION :

SETTING : CATEGORY $\mathcal{E} = (\mathcal{E}, \otimes_{\mathcal{E}}, I_{\mathcal{E}}, \tau_{\mathcal{E}})$ CLOSED SYMM. MONOIDAL

$I_{\mathcal{E}}$: UNIT, $\tau_{\mathcal{E}}$: SYMM.

EX: $\mathcal{E} = Top$, $Sets^{\Delta^{op}}$, $ch(\mathbb{Z})$

FOR MOST OF THE TALK, WILL ASSUME $\mathcal{E}$ HAS A MODEL STRUCTURE

DEF: $\mathcal{C}$ IS AN $\mathcal{E}$-CATEGORY IF $Hom_{\mathcal{C}}(X,Y)$ ARE OBJECTS OF $\mathcal{E}$ $\forall X, Y \in \mathcal{C}$.

FOR SUCH A $\mathcal{C}$, DEFINE $\mathcal{C}^{\otimes k}$ TO ~~BE~~ THE $\mathcal{E}$-CATEGORY WITH OBJECTS $Obj(\mathcal{C}^{\otimes k}) = (x_i)_{1 \le i \le k}$

MORPHISMS $Hom_{\mathcal{C}^{\otimes k}}((x_i),(y_i)) = \bigotimes_{i=1}^{k} Hom_{\mathcal{C}}(x_i, y_i)$

DEF: A FUNCTOR-OPERAD ξ ON AN $\mathcal{E}$-CATEGORY $\mathcal{C}$ IS A SEQUENCE OF FUNCTORS : $\xi_k : \mathcal{C}^{\otimes k} \to \mathcal{C}$

ENDOWED WITH $\mu_{i_1,\dots,i_k} : \xi_k \circ (\xi_{i_1} \otimes \dots \otimes \xi_{i_k}) \longrightarrow \xi_{i_1 + \dots + i_k}$

NAT. TRANSFORMATION S.T.

- ξ_k "TWISTED SYMMETRIC": FOR $\xi_k^{\sigma}(x_1, \dots, x_k) := \xi_k(x_{\sigma(1)}, \dots, x_{\sigma(k)})$ $\exists$ NAT. TRANSF. $\xi_k \to \xi_k^{\sigma}$ S.T. ...
- $\xi_1 = id$, • ASSOCIATIVITY, • COMPATIBILITY WITH ACTION OF Σ_k.

DEF: $X \in \mathrm{Obj}\,\mathcal{C}$ IS AN ALGEBRA OVER $\mathfrak{Z}$ IF

$\exists$ MAPS $\alpha_\ell : \mathfrak{Z}_\ell(X,\dots,X) \longrightarrow X$ COMPATIBLE

WITH EVERYTHING ...

EX: $\mathfrak{Z}_\ell(X_1,\dots,X_\ell) := X_1 \underset{\boxtimes}{\otimes} \cdots \underset{\boxtimes}{\otimes} X_\ell$ [PASCAL'S TALK YESTERDAY]

SOME TENSOR PRODUCT IN

PROPOSITION(*): $X, Y \in \mathrm{Obj}\,\mathcal{C}$, $\mathfrak{Z}$ A FUNCTOR OPERAD. ASSUME Y IS A $\mathfrak{Z}$-ALGEBRA. THEN $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ IS AN ALGEBRA OVER $\mathrm{COEND}_{\mathfrak{Z}}(X)$ WHERE

$$\mathrm{COEND}_{\mathfrak{Z}}(X)(\ell) := \mathrm{Hom}_{\mathcal{C}}(X; \mathfrak{Z}_\ell(X,\dots,X)) \quad \text{OPERAD IN } \mathcal{E}$$

(THIS IS THE USUAL COENDOMORPHISM OPERAD IF $\mathcal{C}$ HAS A MONOIDAL STRUCTURE AND WE DEFINE $\mathfrak{Z}$ AS IN THE EXAMPLE ABOVE.)

PROOF: • $\mathrm{COEND}_{\mathfrak{Z}}(X)(\ell) \otimes \mathrm{COEND}_{\mathfrak{Z}}(X)(i_1) \otimes \dots \otimes \mathrm{COEND}(X)(i_\ell)$

$$\longrightarrow \mathrm{COEND}_{\mathfrak{Z}}(X)(i_1 + \dots + i_\ell)$$

TAKE $(f, g_1, \dots, g_\ell)$ TO

$$X \xrightarrow{f} \mathfrak{Z}_\ell(X,\dots,X) \xrightarrow{\mathfrak{Z}_\ell(g_1\cdots g_\ell)} \mathfrak{Z}_\ell(\mathfrak{Z}_{i_1}(X\dots), \dots, \mathfrak{Z}_{i_\ell}(X\dots X))$$
$$\downarrow \mu_{i_1,\dots,i_\ell}$$
$$\mathfrak{Z}_{i_1+\dots+i_\ell}(X,\dots,X).$$

• REPLACE $\mathrm{COEND}_{\mathfrak{Z}}(X)(i_j)$ BY $\mathrm{Hom}(X,Y)$ $(i_j = 1\ \forall j)$ ABOVE $\rightsquigarrow$ GET ACTION OF $\mathrm{COEND}_{\mathfrak{Z}}(X)$ ON $\mathrm{Hom}(X,Y)$ USING THE $\mathfrak{Z}$-ALG. STRUCTURE OF Y.

COLOURED OPERAD

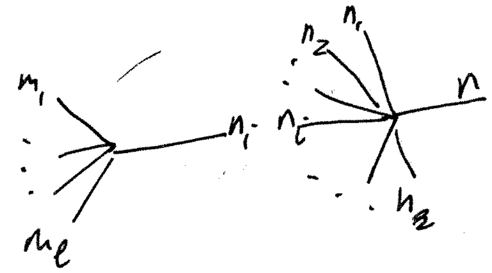
N = SET OF "COLOURS" IN $\mathcal{E}$

DEF: $\mathcal{O}$ IS AN N-COLOURED OPERAD IF

$\mathcal{O} = \{\mathcal{O}(n_1, \ldots, n_\ell ; n)\}_{\substack{n_i, n \in N \\ \ell \in \mathbb{N}}}$ OF OBJECTS OF $\mathcal{E}$ AND COMPOSITION

MAPS $\mathcal{O}(n_1, \ldots, n_\ell ; n) \otimes \mathcal{O}(m_1, \ldots, m_\ell ; n_i) \xrightarrow{\circ_i}$

$\mathcal{O}(n_1, \ldots, n_{i-1}, m_1, \ldots, m_\ell, n_{i+1}, \ldots, n_\ell ; n)$



UNIT FOR EACH COLOUR: $I \to \mathcal{O}(n, n)$.

NOTE: IF $|N| = 1$, $\mathcal{O}$ IS JUST A USUAL OPERAD — NON-SYMM.

DEF: UNDERLYING CATEGORY ASSOCIATED TO $\mathcal{O}$:

CATEGORY $\mathcal{O}_u$ WITH $\mathrm{Obj}(\mathcal{O}_u)$ = SET OF COLOURS

$\mathrm{Mor}_{\mathcal{O}_u}(n, m) = \mathcal{O}(n; m)$ = UNARY OPERATIONS.

WE CAN DEFINE A FUNCTOR

$$\mathcal{O}(-, \ldots, - ; -) : \mathcal{O}_u^{op} \otimes \cdots \otimes \mathcal{O}_u^{op} \otimes \mathcal{O}_u \longrightarrow \mathcal{E}$$

$$(n_1, \ldots, n_\ell ; n) \longmapsto \mathcal{O}(n_1, \ldots, n_\ell ; n).$$

LET $\mathcal{E}^{\mathcal{O}_u}$ BE THE $\mathcal{E}$-CATEGORY OF FUNCTORS $\mathcal{O}_u \to \mathcal{E}$

AND $\mathcal{E}$-NATURAL TRANSFORMATIONS.

(ALGEBRA OVER A COLOURED OPERAD IS ONE OBJECT PER COLOUR AND STRUCTURE MAPS AS USUAL.)

DEFINE $\mathfrak{Z}(\mathcal{O})_k : \mathcal{E}^{\mathcal{O}_u} \otimes \dots \otimes \mathcal{E}^{\mathcal{O}_u} \longrightarrow \mathcal{E}^{\mathcal{O}_u}$ (

BY $\mathfrak{Z}(\mathcal{O})_k(X_1, \dots, X_k)(n) = \mathcal{O}(-, \dots, -; n) \otimes_{\mathcal{O}_u \otimes \dots \otimes \mathcal{O}_u} X_1(-) \otimes \dots \otimes X_k(-$

(COEND)

$$= \coprod_{n_1, \dots, n_k} \mathcal{O}(n_1, \dots, n_k; n) \otimes X_1(n_1) \otimes \dots \otimes X_k(n_k) \Big/ \sim$$

$\mathfrak{Z}(\mathcal{O})$ IS A FUNCTOR OPERAD IN $\mathcal{E}^{\mathcal{O}_u}$.

EXAMPLE: $\mathcal{O}$ OPERAD $\rightarrow$ $\mathcal{O}_u$ HAS ONE OBJECT AND MORPH = (

$\rightarrow$ AN OBJECT OF $\mathcal{E}^{\mathcal{O}_u}$ IS AN $\mathcal{O}(1)$-MODULE.

(SUPPOSE $\mathcal{E} = \mathrm{Mod}_k$).

$$\mathfrak{Z}(\mathcal{O})_k(M_1, \dots, M_k) = \mathcal{O}(k) \otimes_{\mathcal{O}(1) \otimes \dots \otimes \mathcal{O}(1)} M_1 \otimes \dots \otimes M_k$$

PROP: $\{\mathfrak{Z}(\mathcal{O})_k\}$, $k \geq 0$ GIVES A FUNCTOR OPERAD AND THE CATEGORY OF $\mathcal{O}$-ALG IS ISOMORPHIC TO THE CATEGORY OF $\mathfrak{Z}(\mathcal{O})$-ALGEBRAS.

PF: X $\mathcal{O}$-ALG.: $\{X(n)\}_{n \in N}$ + MAPS $\mathcal{O}(n_1, \dots, n_k; n) \otimes X(n_1) \otimes \dots \otimes X($
$\downarrow$
$X(n)$

$\rightsquigarrow$ DEFINED A MAP $\mathfrak{Z}(\mathcal{O})_k(X(n_1), \dots, X(n_k))(n) \rightarrow X(n)$.

CONDENSATION AND TOTALISATION

$$\mathcal{O} \longmapsto \mathrm{Coend}_{\mathcal{Z}(\mathcal{O})}(\delta)$$

COLOURED OPERAD

$$\delta : \mathcal{O}_u \longrightarrow \mathcal{E} \qquad \left(\mathcal{C} = \mathcal{E}^{\mathcal{O}_u}\right)$$

δ - CONDENSATION : COLOURED OPERADS $\longrightarrow$ OPERAD

$$\mathrm{Tot}_\delta : \mathrm{Alg}_{\mathcal{O}} \longrightarrow \mathrm{Alg}_{\mathrm{Coend}_{\mathcal{Z}(\mathcal{O})}(\delta)}$$

$$A \longmapsto \mathrm{Tot}_\delta(A) = \mathrm{Hom}_{\mathcal{E}^{\mathcal{O}_u}}(\delta, A)$$

A can be seen as a functor $\mathcal{O}_u \to \mathcal{E} : n \mapsto A(n)$

$\mathrm{Tot}_\delta(A)$ is a $\mathrm{Coend}_{\mathcal{Z}(\mathcal{O})}(\delta)$ - ALGEBRA BY PROPOSITION $(*)$.

EX: $\mathcal{E} = \mathrm{Top}$, $\quad \delta = \Delta^\bullet : \Delta \longrightarrow \mathrm{Top}$

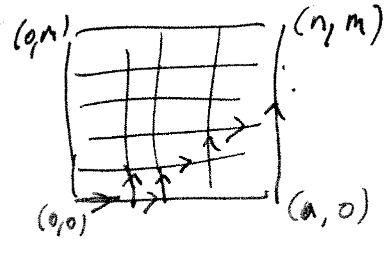
WILL CONSTRUCT $\mathcal{O}$ s.t. $\left|\begin{array}{l} \mathrm{Tot}_\delta(A) = \mathrm{Tot}(A) \\ \mathcal{O}^u = \Delta \end{array}\right.$

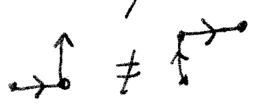
② THE LATTICE PATH OPERAD

(COLOURED OPERAD)

$[n]$ = CATEGORY GENERATED BY $(0 \to 1 \to \dots \to n) = l_n$

$[n] \otimes [m]$ = CAT. FREELY GENERATED BY THE GRID $l_n \times l_m$



$\triangle$: 

$Cat_{*,*}$ = A CATEGORY OF BIPOINTED SMALL CATEGORIES AND FUNCTORS PRESERVING THE DISTINGUISHED POINTS

MONOIDAL CATEGORY : $(A,(a_0,a_1)) \otimes (B,(b_0,b_1))$

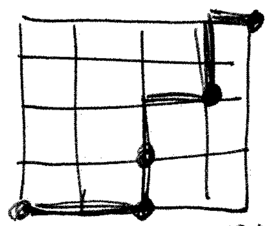
$$= (A \otimes B, (a_0 \otimes b_0, a_1 \otimes b_1))$$

<u>EX</u>: $[n]$ IS BIPOINTED BY $(0,n)$.

<u>DEF</u>: THE <u>LATTICE PATH OPERAD</u> $\mathcal{L}$ IS AN $\mathbb{N}$-COLORED OPERAD IN Sets WITH

$$\mathcal{L}(n_1,\dots,n_\ell\,;n) := \underbrace{\mathrm{Hom}_{Cat_{*,*}}([n+1],[n_1+1]\otimes\cdots\otimes[n_\ell+1])}$$

<u>n-STEPS</u> PATHS IN THE GRID $[n_1+1]\times\dots\times[n_\ell+1]$ FROM $(0,\dots,0)$ TO $(n_1+1,\dots,n_\ell+1)$.



$(n=3)$
$\ell=2$

n POINTS MARKED ON THE PATH (+ END POINTS)

A PATH DEFINES AND IS DEFINED BY A SEQUENCE OF INTEGERS $(a_i)_{1\le i\le n_1+\dots+n_\ell+\ell}$ WHICH CONTAINS n_1+1 TIMES n_1, n_2+1 TIMES n_2, ... SUBDIVIDED IN $n+1$ SUBSTRINGS

<u>EX</u>: 11 | 2 | 21 | 221 .

EXAMPLE OF COMPOSITION: 1 || 1 $\underline{2}$ | 3 | $\underline{2}$ $\begin{matrix}0\\2\end{matrix}$ 3 | 12

$\mathcal{L}(1,\underline{1},0;4)$ $\qquad$ $\mathcal{L}(0,0,0;\mathbf{1})$

(SUBSTITUTION + RELABELLING) = 1 || 14 | 5 | 23 $\in \mathcal{L}(1,0,0,0,0;4)$

$$\mathcal{L}(\underbrace{0,\dots,0}_{k};0) \cong \Sigma_k \qquad \rightsquigarrow \mathcal{L} \longrightarrow \Sigma$$

<u>JOYAL-DUALITY</u> : $\underset{\cup}{\mathrm{Cat}_{*,*}}([n+1],[m+1]) \cong \mathrm{Cat}([m],[n])$

$x(0)=0 \quad x(1) \quad \dots \quad m+1 = x(n+1)$

$\Updownarrow$

$11|1..||..|1$ WITH $m+1$ 1'S AND n BARS.

$$\rightsquigarrow \varphi : \begin{array}{l} 0 \longrightarrow 0 \\ 1 \longrightarrow 0 \\ 2 \longrightarrow 0 \\ 3 \longrightarrow 1 \\ \vdots \\ m+1 \longrightarrow n \end{array}$$

<u>LEMMA</u>: $\mathcal{L}_u(m,n) = \mathrm{Cat}_{*,*}([n+1];[m+1]) \cong \mathrm{Cat}([m],[n])$
$\cong \Delta([m],[n])$

$\rightarrow \mathcal{L}_u = \Delta$

<u>FILTRATION ON $\mathcal{L}$</u> BY COMPLEXITY AS ~~BEFORE~~.

$\rightsquigarrow \mathcal{L}_m$ SUBOPERAD OF COMPLEXITY $\leq m$.

$\mathcal{L}_0 = \mathrm{Morph}(\mathcal{L}_u) = \mathrm{Morph}(\Delta)$ (NO CHANGE OF DIRECTION)
(COLOURED OPERAD CONCENTRATED IN ARITY 1)

$(\mathcal{L}_m)_u = \mathcal{L}_u \quad \forall m$.

THM: FOR $\mathcal{E} = \mathrm{Top} \hookleftarrow \mathrm{Sets}$, $\mathfrak{z}(\mathcal{L}_m) \cong \square^m$

FROM PASCAL'S TALK.

PROP(**): $\mathcal{E}$ COCOMPLETE, CLOSED SYMM. MONOIDAL CAT THEN THE CATEGORY OF $\mathcal{L}_2$-ALG. IN $\mathcal{E}$ IS ISOMORPH. TO THE CATEGORY OF MULTIPLICATIVE NON-Σ OPERAD. IN $\mathcal{E}$.

($\mathcal{L}_1$-ALGEBRAS RELATE TO $\square$-MONOIDS IN PASCAL'S TALK)

THERE IS A MAP $\mathcal{L} \longrightarrow K \; (\to \Sigma)$

COMPLETE GRAPH OPERAD

RESPECTING THE FILTRATION

THM: δ = "STANDARD SYSTEM OF SIMPLICES"

$\delta : \Delta \cong \mathcal{L}_u \longrightarrow \mathcal{E}$ $\quad$ ($\mathcal{E} = \mathrm{Top}, \ldots$

IN A MONOIDAL MODEL CATEGORY $\mathcal{E}$, ASSUME THAT $\mathcal{L}$ IS "δ-REDUCTIBLE" (OK IN THE CASES OF INTEREST ...) THEN δ-CONDENSATION OF $\mathcal{L}_m$, $\mathrm{CoEND}_{\mathfrak{z}(\mathcal{L}_m)}(\delta)$, IS AN E_m-OPERAD IN $\mathcal{E}$. (WHEN THIS IS A WELL-DEFINED NOTION IN $\mathcal{E}$ ...)

APPLICATION: $CH^{n}(A,A) = \mathrm{End}_A(n)$

A ASSOCIATIVE $\Rightarrow \mathrm{End}_A$ IS A MULT. OPERAD

$\Rightarrow \mathrm{End}_A$ IS AN $\mathcal{L}_2$-ALG

PROPOSITION (**)

$\Rightarrow \mathrm{Tot}_{\delta_{\mathbb{Z}}}(\mathrm{End}_A)$ IS A $\mathrm{COEND}_{\mathfrak{Z}(\mathcal{L}_2)}(\delta_{\mathbb{Z}})$-ALG

$\mathrm{Tot}_{\delta_{\mathbb{Z}}}(\mathrm{End}_A) \cong \oplus CH^*(A,A)$

$\mathrm{COEND}_{\mathfrak{Z}(\mathcal{L}_2)}(\delta_{\mathbb{Z}})$: E_2-OPERAD

+ RECOVER YESTERDAY'S THEOREMS WITH $\mathcal{E} = Top$.

+ CYCLIC VERSION —

+ GENERALIZED DELIGNE'S CONJECTURE $E_n \rightsquigarrow E_{n+1}$?