FRAMED DISCS OPERADS AND BATALIN–VILKOVISKY ALGEBRAS

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Abstract

The framed n-discs operad $f\mathcal{D}_n$ is studied as semidirect product of $SO(n)$ and the little n-discs operad. Our equivariant recognition principle says that a grouplike space acted on by $f\mathcal{D}_n$ is equivalent to the n-fold loop space on an $SO(n)$-space. Examples of $f\mathcal{D}_2$-spaces are nerves of ribbon braided monoidal categories. We compute the rational homology of $f\mathcal{D}_n$, which produces higher Batalin–Vilkovisky algebra structures for n even. We study quadratic duality for semidirect product operads and compute the double loop space homology of a manifold as BV-algebra.

1. Introduction

The topology of iterated loop spaces was thoroughly investigated in the seventies. These spaces have a wealth of homology operations parametrized by the famous operads of little discs, denoted by $\mathcal{D}_n$ in the text. The notion of operad was introduced in the first place for this purpose [**2**, **19**]. Such machinery allows, for example, the reconstruction of an iterated delooping if one has full knowledge of the operad action on an iterated loop space. Moreover any connected space acted on by the little discs is weakly homotopy equivalent to an iterated loop space. This fact is the celebrated recognition principle.

Our main objective is to extend this theory by adding the operations *rotating* the discs. The operad generated by the little n-discs $\mathcal{D}_n$ and the rotations in $SO(n)$ is the framed n-discs operad $f\mathcal{D}_n$, first introduced in [**10**]. Our recognition principle for framed n-discs (Theorem 3.1) says that a connected (or grouplike) space acted on by the framed n-discs operad is weakly homotopic to an n-fold loop space on an $SO(n)$-space.

Thus the looping and delooping functors induce a categorical equivalence between $SO(n)$-spaces and spaces acted on by the framed n-discs operad, under the correct connectivity assumptions. The main technique consists in presenting the framed little discs as a semidirect product of the little discs and the special orthogonal group.

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For a group G and a G-operad $\mathcal{A}$, that is, an operad in the monoidal category of G-spaces, one can define an operad $\mathcal{A} \rtimes G$ in the category of spaces, the *semidirect product* of $\mathcal{A}$ and G. The definition is such that the framed n-discs $f\mathcal{D}_n$ is the semidirect product $\mathcal{D}_n \rtimes SO(n)$. Semidirect products of this type have the following property: a space X is an $\mathcal{A} \rtimes G$-algebra if and only if X is an $\mathcal{A}$-algebra in the category of G-spaces. This allows us to study $f\mathcal{D}_n$-algebras by studying $\mathcal{D}_n$-algebras $SO(n)$-equivariantly.

One can actually construct an operad $\mathcal{D}_n \rtimes G$ for any representation $G \to O(n)$ of a topological group. Our recognition principle constructs loop and deloop functors between the category of G-spaces and the category of $\mathcal{D}_n \rtimes G$-spaces, inducing equivalences of 'naive' homotopy categories. If G is trivial, this is May's original recognition principle.

We next investigate the homology of the spaces with an action of the framed discs operad. We consider semidirect products for graded operads: if H is a graded Hopf algebra and P a graded operad acted on by H, one can construct a graded operad $P \rtimes H$ which is such that a $P \rtimes H$-algebra in the category of chain complexes is exactly a P-algebra in the category of differential graded H-modules. The homology functor (with field coefficients) commutes with the semidirect product construction. The homology of $f\mathcal{D}_n$ can thus be studied as the semidirect product $H(\mathcal{D}_n) \rtimes H(SO(n))$. This approach yields a conceptual proof of the fact that the rational homology of an algebra over the framed 2-discs is a Batalin–Vilkovisky algebra [**10**], and computes more generally the rational homology of the framed n-discs operad for any n (Theorem 5.4). For n even this produces Batalin–Vilkovisky structures in higher degrees.

In section 6, we define a notion of quadratic duality for semidirect products. We dualize $P \rtimes H$ by dualizing P in the category of H^{op}-modules. We show that the Batalin–Vilkovisky operad is self-dual up to a shift.

As an application, we explain how to compute the rational homology of a double loop space on an S^1-manifold M as a BV-algebra, starting from the complex of differential forms on M together with a derivation induced by the action (Theorem 6.5). This extends results in [**10**], where M is a double suspension, and [**11**], where only the Gerstenhaber algebra structure is considered.

We end the article by relating the framed 2-discs to the ribbon braid groups, extending work of Fiedorowicz in the case of the little discs and the braid groups. We show that classifying spaces of ribbon braided monoidal categories are double loop spaces on S^1-spaces (Theorem 7.7). This follows from the equivalence between the framed 2-discs and a ribbon braid groups operads. We also give a criterion for an operad to be equivalent to the framed little 2-discs operad (Theorem 7.3).

2. Equivariant operads

We work in the category Top of compactly generated weak Hausdorff topological spaces. Let G be a topological group. The category of left G-spaces, denoted G-Top , is a symmetric monoidal category by the Cartesian product. We can thus consider operads in this category, which we call *G-operads*.

Let $\mathcal{A}$ be a G-operad. So $\mathcal{A}$ consists of a sequence of G-spaces $\mathcal{A}(k)$ for $k \in \mathbb{N}$, with G-equivariant operad structure maps and symmetric group actions. Note that the unit $1 \in \mathcal{A}(1)$ is also preserved by the G-action.

We will denote the action of an element $g \in G$ on an element $a \in \mathcal{A}(k)$ by ga.

The following notion first appeared in [**17**].

DEFINITION 2.1 Let $\mathcal{A}$ be a G-operad. Define $\mathcal{A} \rtimes G$, the *semidirect product* of $\mathcal{A}$ and G, to be the

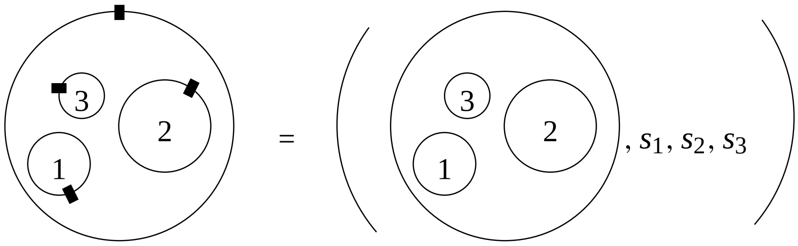


Fig. 1 Element of $f\mathcal{D}_2(3) = \mathcal{D}_2(3) \times (S^1)^3$

following operad in Top, for $k \in \mathbb{N}$:

$$(\mathcal{A} \rtimes G)(k) = \mathcal{A}(k) \times G^k$$

with Σ_k acting diagonally on the right, permuting the components of G^k and acting on $\mathcal{A}(k)$, and the map

$$\gamma : (\mathcal{A} \rtimes G)(k) \times (\mathcal{A} \rtimes G)(n_1) \times \cdots \times (\mathcal{A} \rtimes G)(n_k) \longrightarrow (\mathcal{A} \rtimes G)(n_1 + \cdots + n_k)$$

given by

$$\gamma((a, \mathbf{g}), (b_1, \mathbf{h}^1), \ldots, (b_k, \mathbf{h}^k)) = (\gamma_{\mathcal{A}}(a, g_1 b_1, \ldots, g_k b_k), g_1\mathbf{h}^1, \ldots, g_k\mathbf{h}^k),$$

where $\mathbf{h}^i = (h^i_1, \ldots, h^i_{n_i})$ and $g_i.\mathbf{h}^i = (g_i h^i_1, \ldots, g_i h^i_{n_i})$. The unit in $\mathcal{A} \rtimes G(1)$ is $(1, e)$, formed from the units of $\mathcal{A}$ and G.

The G-equivariance of γ_A is necessary for the associativity of the structure map of the semidirect product operad.

Note that the semidirect product $\mathcal{A} \rtimes G$ can be thought of as the semidirect product of the operad $\mathcal{A}$ and the operad $\mathcal{G}$, where $\mathcal{G}(k) = G^k$ and the operad structures maps of $\mathcal{G}$ are given by the right part of γ above.

EXAMPLE 2.2 The example we have in mind is the framed discs operad $f\mathcal{D}_n$. Let $\mathcal{D}_n$ be the little n-discs operad of Boardman and Vogt. Hence $\mathcal{D}_n(k)$ is the space of embeddings $\coprod_k D^n \to D^n$ of k copies of the unit n-disc into itself such that the maps are compositions of positive dilations and translations, and the images are disjoint. The framed discs $f\mathcal{D}_n$ is defined similarly but one allows rotations for the embeddings. As spaces, $f\mathcal{D}_n(k) = \mathcal{D}_n(k) \times (SO(n))^k$, the ith element of $SO(n)$ encoding the rotation of the ith disc. In fact, $\mathcal{D}_n$ is an $SO(n)$-operad and $f\mathcal{D}_n$ is a semidirect product in the above sense:

$$f\mathcal{D}_n = \mathcal{D}_n \rtimes SO(n).$$

The action of $SO(n)$ on $\mathcal{D}_n(k)$ rotates the little discs around their centre. Note that the whole orthogonal group $O(n)$ acts on $\mathcal{D}_n(k)$ in such a way that $\mathcal{D}_n \rtimes O(n)$ is well defined. We will consider semidirect products of $\mathcal{D}_n$ with any topological group G equipped with a continuous homomorphism $\phi : G \to O(n)$. We will suppress ϕ from the notation and denote the resulting semidirect product simply by $\mathcal{D}_n \rtimes G$.

PROPOSITION 2.3 *Let $\mathcal{A}$ and G be as in Definition* 2.1*. A space X is an $(\mathcal{A} \rtimes G)$-algebra if and only if X is an $\mathcal{A}$-algebra in the category of G-spaces, that is, X admits a G-action and $\mathcal{A}$-algebra structure maps $\theta_{\mathcal{A}} : \mathcal{A}(k) \times X^k \longrightarrow X$ satisfying $g(\theta_{\mathcal{A}}(a, x_1, \ldots, x_k)) = \theta_{\mathcal{A}}(ga, gx_1, \ldots, gx_k)$.*

Moreover, $\theta_{\mathcal{A} \rtimes G}((a, (g_1, \ldots, g_k)), x_1, \ldots, x_k) = \theta_{\mathcal{A}}(a, g_1 x_1, \ldots, g_k x_k)$.

As an immediate consequence we have this corollary.

COROLLARY 2.4 *Let X, Y be two $(\mathcal{A} \rtimes G)$-algebras. A map $f : X \longrightarrow Y$ is a map of $(\mathcal{A} \rtimes G)$-algebras if and only if it is an $\mathcal{A}$-algebra map and a G-map.*

We will use the following examples of framed algebras.

EXAMPLE 2.5 Let Y be a pointed G-space and let D_n denote the monad associated to the operad $\mathcal{D}_n$ [**19**, Const. 2.4]; $D_n Y$ is the free $\mathcal{D}_n$-algebra on the pointed space Y. Let $\Omega^n Y$ denote the based n-fold loop space on Y, seen as the space of maps from the unit n-disc D^n to Y sending the boundary to the base point. The space $\Omega^n Y$ carries a natural $\mathcal{D}_n$-algebra structure [**19**, Theorem 5.1].

Let $\phi : G \longrightarrow O(n)$ be a continuous group homomorphism. The spaces $D_n Y$ and $\Omega^n Y$ are $\mathcal{D}_n \rtimes G$-algebras, with the action of $g \in G$ on $[c; y_1, \ldots, y_k] \in D_n Y$, where $c \in \mathcal{D}_n(k)$, $y_i \in Y$, given by

$$g[c; y_1, \ldots, y_k] = [\phi(g)c; gy_1, \ldots, gy_k],$$

and on $[y(t)] \in \Omega^n Y$, where $t \in D^n$ and $[y(t)]$ denotes the n-fold loop $t \mapsto y(t)$, given by

$$g[y(t)] = [gy(\phi(g)^{-1}(t))].$$

3. Recognition principle

Let $\phi : G \longrightarrow O(n)$ be as above and let X be a grouplike $\mathcal{D}_n \rtimes G$-algebra, that is, the components of X form a group by the product induced by any element in $\mathcal{D}_n(2)$. As X is a $\mathcal{D}_n \rtimes G$-algebra, it is in particular a $\mathcal{D}_n$-algebra.

May introduced a deloop functor B_n from $\mathcal{D}_n$-algebras to pointed spaces defined by $B_n X := B(\Sigma^n, D_n, X)$, where B is the two-sided bar construction [**19**, Const. 9.6], Σ the suspension and D_n is the free $\mathcal{D}_n$-algebra functor, as above. May's recognition principle says that X is weakly equivalent to $\Omega^n B_n X$ as $\mathcal{D}_n$-algebra. May also showed that, conversely, B_n applied to an n-fold loop space $\Omega^n Y$ produces a space weakly homotopy equivalent to Y when Y is $(n-1)$-connected; see [**19**, Theorem 13.1] and its improvement in [**3**, p. 487 (21)].

In what follows, we consider the behaviour of Ω^n and B_n with respect to G-actions, and provide a recognition principle for algebras over $\mathcal{D}_n \rtimes G$.

Let $\mathcal{D}_n \rtimes G\text{-}\mathsf{Top}_{gl}$, $\mathcal{D}_n \rtimes G\text{-}\mathsf{Top}_0$ and $G\text{-}\mathsf{Top}_n^*$ be respectively the categories of grouplike, connected $\mathcal{D}_n \rtimes G$-algebras and n-connected pointed G-spaces. The categories $G\text{-}\mathsf{Top}$ and $\mathcal{D}_n \rtimes G\text{-}\mathsf{Top}$ are closed model categories with weak homotopy equivalences as weak equivalences [**5**, 3.1; **1**, Remark 4.2]. For a model category $\mathcal{C}$, we will denote by $\mathrm{Ho}(\mathcal{C})$ its associated homotopy category, obtained by inverting the weak equivalences. We define the homotopy categories $\mathrm{Ho}(\mathcal{D}_n \rtimes G\text{-}\mathsf{Top}_{gl})$, $\mathrm{Ho}(\mathcal{D}_n \rtimes G\text{-}\mathsf{Top}_0)$ and $\mathrm{Ho}(G\text{-}\mathsf{Top}_n^*)$ to be the full subcategories of the categories $\mathrm{Ho}(\mathcal{D}_n \rtimes G\text{-}\mathsf{Top})$, $\mathrm{Ho}(\mathcal{D}_n \rtimes G\text{-}\mathsf{Top})$ and $\mathrm{Ho}(G\text{-}\mathsf{Top}^*)$ containing the grouplike, connected and n-connected objects respectively.

For any G-space Y, we have seen in Example 2.5 that $\Omega^n Y$ has a $\mathcal{D}_n \rtimes G$-algebra structure induced by the diagonal action of G. On the other hand, we will define a G-action on $B_n X$ for any

$\mathcal{D}_n \rtimes G$-algebra X. Hence, Ω^n and B_n will be functors between the categories of pointed G-spaces and of $\mathcal{D}_n \rtimes G$-algebras.

THEOREM 3.1 *For each continuous homomorphism $\phi : G \longrightarrow O(n)$, we have functors*

$$\Omega^n_\phi = \Omega^n : G\text{-}\mathsf{Top}^*_{n-1} \longrightarrow \mathcal{D}_n \rtimes G\text{-}\mathsf{Top}_{gl},$$

$$B_n^\phi = B_n : \mathcal{D}_n \rtimes G\text{-}\mathsf{Top}_{gl} \longrightarrow G\text{-}\mathsf{Top}^*_{n-1}$$

which induce an equivalence of categories

$$\mathrm{Ho}(G\text{-}\mathsf{Top}^*_{n-1}) \simeq \mathrm{Ho}(\mathcal{D}_n \rtimes G\text{-}\mathsf{Top}_{gl}).$$

For $k \geqslant n$, this equivalence restricts to

$$\mathrm{Ho}(G\text{-}\mathsf{Top}^*_k) \simeq \mathrm{Ho}(\mathcal{D}_n \rtimes G\text{-}\mathsf{Top}_{k-n}).$$

Proof. May's recognition principle [**19**, Theorem 13.1] is obtained through the following maps:

$$X \longleftarrow B(D_n, D_n, X) \xrightarrow{\alpha} B(\Omega^n\Sigma^n, D_n, X) \longrightarrow \Omega^n B(\Sigma^n, D_n, X) = \Omega^n B_n X,$$

where all maps are $\mathcal{D}_n$-maps between $\mathcal{D}_n$-spaces. When X is a $\mathcal{D}_n \rtimes G$-algebra, we want to define G-actions on the spaces involved which induce $\mathcal{D}_n \rtimes G$-algebra structures and such that all maps are G-maps.

The functors D_n, Σ^n and Ω^n restrict to functors in the category of G-spaces where, for any G-space Y, we define the action on D_nY, Σ^nY and Ω^nY diagonally as in Example 2.5. Hence for any G-space Y the G-action on $\Omega^n\Sigma^nY$ is given by

$$g[\sigma(t), y(t)] = [\phi(g)\sigma(\phi(g)^{-1}t), gy(\phi(g)^{-1}t)],$$

where $g \in G$, $t, \sigma(t) \in D^n$ and $y(t) \in Y$. This produces a $\mathcal{D}_n \rtimes G$-algebra structure on $\Omega^n\Sigma^nY$ such that May's map $\alpha : D_nY \longrightarrow \Omega^n\Sigma^nY$ is a G-map, and thus a $\mathcal{D}_n \rtimes G$-map.

We extend now these actions on the simplicial spaces $B(D_n, D_n, X)$, $B(\Omega^n\Sigma^n, D_n, X)$ and $\Omega^n B(\Sigma^n, D_n, X)$.

Recall that the double bar construction $B(F, C, X)$ is defined simplicially, for a monad C, a left C-functor F and a C-algebra X by $B(F, C, X) = |B_*(F, C, X)|$, where $B_p(F, C, X) = FC^pX$, with boundary and degeneracy maps using the left functor, monad and algebra structure maps. The group G acts then on $B_p(F, C, X)$ through its action on the functors F and C, which comes to 'rotate everything'. For example, the action of $g \in G$ on a 1-simplex of $B(\Omega^n\Sigma^n, D_n, X)$ is given by

$$\begin{aligned} &g[\sigma(t), c(t), x_1(t), \dots, x_k(t)] \\ &\qquad = [\phi(g)\sigma(\phi(g)^{-1}t), \phi(g)c(\phi(g)^{-1}t), gx_1(\phi(g)^{-1}t), \dots, gx_k(\phi(g)^{-1}t)]. \end{aligned}$$

With these actions, all maps above are G-maps between $\mathcal{D}_n \rtimes G$-spaces and B_nX is equipped with an explicit G-action.

On the other hand, we have a weak homotopy equivalence [**19**, Theorem 13.1; **3**, p. 487 (21)]

$$B_n\Omega^nY = B(\Sigma^n, D_n, \Omega^nY) \xrightarrow{|\delta_0^*|} \Sigma^n\Omega^nY \xrightarrow{e} Y$$

for any $(n-1)$-connected space Y. If Y is a G-space, then this composite is a G-map with the actions on $B_n\Omega^nY$ and $\Sigma^n\Omega^nY$ defined as above.

Note that we have constructed equivariant deloopings in the weak sense: our weak equivalences are G-equivariant maps that are weak equivalences, but do not necessarily induce weak equivalences of all fixed point sets. It would of course be nice if one could construct deloopings in the strong sense.

4. Equivariant algebraic operads

For the next three sections, we work in the category of chain complexes over a field k, considering only operads having trivial differential. We call these operads *graded operads*. For an element x of a chain complex, we denote by $|x|$ its degree.

Let H be a graded associative cocommutative Hopf algebra over a field k. The tensor product of two H-modules inherits an H-structure which is induced by the coproduct of H. As H is cocommutative, the category of differential graded H-modules, denoted H-Mod, is a symmetric monoidal category with product the ordinary tensor product. Hence it makes sense to consider operads and their algebras in this category. We call such operads (*graded*) *operads of H-modules*.

As in the topological case, we can construct semidirect products.

PROPOSITION 4.1 *Let P be a graded operad of H-modules. There exists a graded operad, the* semidirect product $P \rtimes H$, *such that algebras over P in the category of H-modules are exactly $P \rtimes H$-algebras.*

The operad is defined by $(P \rtimes H)(n) = P(n) \otimes H^{\otimes n}$. The structure maps are defined similarly to the topological case, using the comultiplication c of H and using interchanging homomorphisms with appropriate signs.

Taking homology with coefficients in the field k provides a bridge from the topological to the algebraic setting.

PROPOSITION 4.2 *Let G be a topological group acting on a topological operad $\mathcal{A}$. There is a natural isomorphism of operads $H(\mathcal{A} \rtimes G) \cong H(\mathcal{A}) \rtimes H(G)$.*

Suppose now that P is a quadratic (graded) operad, namely P has binary generators and trinary relations [**12**, 2.1.7]. We will restrict ourselves to the case where $P(1) = k$, concentrated in dimension 0. Explicitly $P = F(V)/(R)$, where $F(V)$ is the free operad generated by a $k[\Sigma_2]$-module of binary operations V and (R) is the ideal generated by a $k[\Sigma_3]$-submodule $R \subset F(V)(3)$.

PROPOSITION 4.3 *Let H be a cocommutative Hopf algebra and $P = F(V)/(R)$ a quadratic operad. Then P is an operad of H-modules if and only if*

(i) *V is an $(H, k[\Sigma_2])$-bimodule;*

(ii) *$R \subseteq F(V)(3)$ is an $(H, k[\Sigma_3])$-sub-bimodule.*

In this case, we will call P a quadratic operad of H-modules.

Proof. An element of the free operad on V is described by a tree with vertices labeled by V [**18**, section 1.9]. We define the action of H on such an element by acting on the labels of the vertices, using the comultiplication of H. This is well defined as H is cocommutative. It induces an H-module structure on $F(V)$ which induces one on $P(n)$ for all n by condition (ii). The operad structure maps are then H-equivariant by construction.

Let $c : H \to H \otimes H$ be the comultiplication. For $g \in H$ we write informally $(c \otimes id)(c(g)) = \sum_i g_i' \otimes g_i'' \otimes g_i'''$.

PROPOSITION 4.4 *Let $P = F(V)/(R)$ be a quadratic operad of H-modules as above. A chain complex X is an algebra over $P \rtimes H$ if and only if*

(i) *X is an H-module,*

(ii) *X is a P-algebra,*

(iii) *for each $g \in H$, $v \in V$ and $x, y \in X$,*

$$g(v(x, y)) = \sum_i (-1)^{|g_i''||v|+|g_i'''|(|v|+|x|)} g_i'(v)(g_i''(x), g_i'''(y)).$$

Proof. The H-equivariance of the algebra map $\theta_2 : P(2) \otimes X \otimes X \longrightarrow X$ is given by the commutativity of the following diagram:

$$\begin{array}{ccc}
H \otimes P(2) \otimes X \otimes X & \xrightarrow{H \otimes \theta_2} & H \otimes X \\
\downarrow{\scriptstyle T \circ (c \otimes \mathrm{id}) \circ c} & & \\
H \otimes P(2) \otimes H \otimes X \otimes H \otimes X & & \downarrow{\scriptstyle \phi} \\
\downarrow{\scriptstyle \psi \otimes \phi \otimes \phi} & & \\
P(2) \otimes X \otimes X & \xrightarrow{\theta_2} & X
\end{array}$$

where ϕ and ψ give the action of H on X and $P(2)$ respectively, and T is the interchange. This diagram translates, for the generators of $P(2)$, into condition (iii) of the proposition. The H-equivariance of the structure maps θ_k for $k > 2$ is a consequence of the fact that V generates $P(k)$, that the operadic composition is H-equivariant and that the structure maps θ satisfy the associativity axiom.

5. Batalin–Vilkovisky algebras

In this section we study the homology of the framed n-discs with coefficients in a field k of characteristic 0. Our result generalizes Getzler's result in the case $n = 2$ which says that the homology of the framed 2-discs is the Batalin–Vilkovisky operad [**10**, section 4]. Our proof uses semidirect products.

We will actually work without units, which means that our operads P satisfy $P(0) = 0$.

DEFINITION 5.1 Let e_n denote the operad

$$e_n(k) := H(\mathcal{D}_n(k)), \quad k \geqslant 1,$$
$$e_n(0) = 0.$$

Algebras over e_n, $n \geqslant 2$, are called *n-algebras*.

Let fe_n be the operad

$$fe_n(k) := H(f\mathcal{D}_n(k)), \quad k \geqslant 1,$$
$$fe_n(0) = 0.$$

So $fe_n = e_n \rtimes H(SO(n))$.

We want to describe algebras over fe_n, using what is known for e_n.

Cohen's study of $H(\mathcal{D}_n)$ in [**3**, Theorem 1.2] implies that an n-algebra X is a differential graded commutative algebra with a Lie bracket of degree $n-1$, that is, the bracket satisfies the following relations:

(L1) $[x, y] + (-1)^{(|x|+n-1)(|y|+n-1)}[y, x] = 0$,

(L2) $\partial[x, y] = [\partial x, y] + (-1)^{|x|+n-1}[x, \partial y]$,

(L3) $[x, [y, z]] = [[x, y], z] + (-1)^{(|x|+n-1)(|y|+n-1)}[y, [x, z]]$,

and the Poisson relation

(P1)] $[x, y * z] = [x, y] * z + (-1)^{|y|(|x|+n-1)} y * [x, z]$

holds.

Gerstenhaber algebras correspond to the case $n = 2$. The operad e_n is thus quadratic.

Note that $\mathcal{D}_n(2)$ is $SO(n)$-equivariantly homotopic to S^{n-1}. In an n-algebra, the product comes from the generating class $* \in H_0(\mathcal{D}_n(2)) \cong k$ and the bracket from the fundamental class $b \in H_{n-1}(\mathcal{D}_n(2)) \cong k$, if we define $[x, y] = (-1)^{(n-1)|x|} b(x, y)$ [**3**, Definition 5.7 and Theorem 1.2].

For n even the homology of the framed n-discs will include the following structure.

DEFINITION 5.2 A *Batalin–Vilkovisky n-algebra X* is a graded commutative algebra with a linear endomorphism $\Delta : X \to X$ of degree $n-1$ such that $\Delta^2 = 0$ and for each $x, y, z \in X$ the following BV-axiom holds:

$$\begin{aligned}\Delta(xyz) = {} & \Delta(xy)z + (-1)^{|x|} x\Delta(yz) + (-1)^{(|x|+1)|y|} y\Delta(xz) - \Delta(x)yz \\ & -(-1)^{|x|} x\Delta(y)z - (-1)^{|x|+|y|} xy\Delta(z).\end{aligned} \tag{1}$$

We will denote by BV_n the operad describing Batalin–Vilkovisky n-algebras.

So a Batalin–Vilkovisky algebra (or BV-algebra), in the sense of [**10**], is a BV_2-algebra.

In order to determine the homology operad $H(f\mathcal{D}_n)$, we need to know the Hopf algebra structure of $H(SO(n))$ and the effect in homology of the action of $SO(n)$ on $\mathcal{D}_n(2)$. For dimensional reasons, one always has $\delta(b) = 0$ for each $\delta \in \tilde{H}(SO(n))$. On the other hand $\delta(*) = \pi_*(\delta)$, where π_* is induced in homology by the evaluation map $\pi : SO(n) \to S^{n-1}$, via $\mathcal{D}_n(2) \simeq S^{n-1}$.

LEMMA 5.3 [**20**, Corollary 3.15] *For $n \geqslant 1$, over a field of characteristic 0, the Hopf algebra $H(SO(2n)) = \bigwedge(\beta_1, \ldots, \beta_{n-1}, \alpha_{2n-1})$ is the free exterior algebra on primitive generators $\beta_i \in H_{4i-1}(SO(2n))$ and $\alpha_{2n-1} \in H_{2n-1}(SO(2n))$. Moreover, $\pi_*(\beta_i) = 0$ for all i and $\pi_*(\alpha_{2n-1}) = b \in H_{2n-1}(S^{2n-1})$ is the fundamental class.*

The Hopf algebra $H(SO(2n+1)) = \bigwedge(\beta_1, \ldots, \beta_n)$ is the free exterior algebra on primitive generators $\beta_i \in H_{4i-1}(SO(2n+1))$, and $\pi_*(\beta_i) = 0$ for all i.

If a Hopf algebra H acts trivially, via the co-unit, on an operad P, we call the semidirect product just the *direct product* and denote it by $P \times H$. Note that a $P \times H$-algebra is an H-module X with a P-algebra structure satisfying an H-equivariance condition which is trivial only if H acts trivially on X. In particular, any P-algebra is a $P \times H$-algebra with the trivial H-module structure.

THEOREM 5.4 *For $n \geqslant 1$ there are isomorphisms of operads*

$$fe_{2n+1} \cong e_{2n+1} \times H(SO(2n+1))$$

and

$$fe_{2n} \cong BV_{2n} \times H(SO(2n-1)).$$

Hence an fe_{2n+1}-algebra is a $(2n+1)$-algebra together with endomorphisms β_i of degree $4i-1$ for $i = 1, \dots, n$ such that $\beta_i^2 = 0$, $\beta_i\beta_j = -\beta_j\beta_i$ for each i, j, and each β_i is a $(2n+1)$-algebra derivation, that is, a derivation with respect to both the product and the bracket.

On the other hand, an fe_{2n}-algebra is a BV$_{2n}$*-algebra together with endomorphisms β_i of degree $4i-1$ for $i = 1, \dots, n-1$ squaring to 0 and anti-commuting as in the odd case, which moreover anti-commute with the* BV *operator Δ and are derivations with respect to the product.*

Proof. By Proposition 4.4, X is an fe_n-algebra if and only if X is a module over $H(SO(n))$ which is an n-algebra equivariantly with respect to the action of $H(SO(n))$. If δ is an element of $H(SO(n))$, the equivariance condition requires that

$$\delta(x * y) = \delta(*)(x, y) + \delta x * y + (-1)^{|x|} x * \delta y, \tag{2}$$

$$\delta b(x, y) = \delta(b)(x, y) + (-1)^{n-1} b(\delta x, y) + (-1)^{|x|+n-1} b(x, \delta y)\,, \tag{3}$$

where equations (2), (3) are obtained by setting $g = \delta,\ v = *$ and $g = \delta,\ v = b$ in turn in condition (iii) of Proposition 4.4.

Recall that $\delta(b) = 0$ for any $\delta \in H(SO(n))$. Also, recall that $[x, y] = (-1)^{(n-1)|x|} b(x, y)$.

In the odd case, $H(SO(2n+1))$ is generated by operator $\beta_1, \dots, \beta_n$ which satisfy $\beta_i(b) = 0$. So equations (2) and (3) become

$$\beta_i(x * y) = \beta_i x * y + (-1)^{|x|} x * \beta_i y, \tag{4}$$

$$\beta_i[x, y] = [\beta_i x, y] + (-1)^{|x|}[x, \beta_i y]. \tag{5}$$

Hence all the operators are derivations of the product and the bracket and X is an n-algebra with additional operators $\beta_1, \dots, \beta_n$.

In the even case, $H(SO(2n))$ is generated by operators $\beta_1, \dots, \beta_{n-1}$ and α_{2n-1}, with $\beta_i(*) = 0$ and $\alpha_{2n-1}(*) = b$. So the β_i satisfy equations (4) and (5). Hence they are derivations of the product and the bracket. On the other hand, $\alpha = \alpha_{2n-1}$ satisfies the equations

$$\alpha(x * y) = (-1)^{|x|}[x, y] + \alpha x * y + (-1)^{|x|} x * \alpha y, \tag{6}$$

$$\alpha[x, y] = [\alpha x, y] + (-1)^{|x|+1}[x, \alpha y]; \tag{7}$$

equation (6) expresses the bracket in terms of the product and α :

$$[x, y] = (-1)^{|x|}\alpha(x * y) - (-1)^{|x|}\alpha(x) * y - x * \alpha(y). \tag{8}$$

If we substitute this expression into the Poisson relation of the $2n$-algebra structure, we get exactly the BV_{2n}-axiom (equation (1)). Hence α and the product produce a Batalin–Vilkovisky $2n$-algebra structure on X. Proposition 1.2 of [**10**] shows that equation (7) and the Lie algebra axioms of an n-algebra follow from the BV_n-axiom in the case $n = 2$. The general case follows from the same calculation. Finally, using equation (6), one can rewrite equation (5) in terms of α, the β_i and the product. It becomes a redundant equation. Hence all the relations involving the bracket are redundant and an fe_{2n}-algebra is a BV_{2n}-algebra with additional operators $\beta_1, \dots, \beta_{n-1}$ (α and the bracket being included in the BV-structure).

Note that there is no non-trivial Σ_2-equivariant map from $H_0(\mathcal{D}_{2n+1}(2))$ to $H_{2n}(\mathcal{D}_{2n+1}(2))$. So $(2n + 1)$-algebras cannot give rise to 'odd' BV-structures as in the even case.

We have already seen that iterated loop spaces are algebras over the framed discs operad. We deduce the following example.

EXAMPLE 5.5 The homology of an n-fold loop space on a pointed $SO(n)$-space is an algebra over fe_n.

Another interesting class of algebras over the homology of $f\mathcal{D}_n$ comes from the space $\Lambda^{n-1}M$ of unbased maps from S^{n-1} to a manifold M. For a complex V, define the *suspension* ΣV of V by $(\Sigma V)_i = V_{i-1}$.

EXAMPLE 5.6 [**23**] Let M be a d-dimensional oriented manifold. Then the d-fold desuspended homology $\Sigma^{-d} H(\Lambda^{n-1}M)$ is an algebra over fe_n.

We recall that the *braid group* β_k on k strings has generators $r_1, \dots, r_{k-1}$ and relations $r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1}$ for $1 \leqslant i \leqslant k-2$ and $r_i r_j = r_j r_i$ for $1 \leqslant i < j-1 \leqslant k-2$. The *ribbon braid group* $R\beta_k$ has an extra generator t_k (see Fig. 2), and an extra relation $r_{k-1} t_k r_{k-1} t_k = t_k r_{k-1} t_k r_{k-1}$. There is a surjection $R\beta_k \twoheadrightarrow \Sigma_k$, sending a ribbon to the permutation induced by its ends. The *pure ribbon braid group* $PR\beta_k$ is the kernel of this map, and the *pure braid group* is $P\beta_k = PR\beta_k \cap \beta_k$. It turns out that $R\beta_k = \beta_k \rtimes \mathbb{Z}^k$ and $PR\beta_k = P\beta_k \times \mathbb{Z}^k$, where $\mathbb{Z}^k$ encodes the number of twists on each ribbon. Note that $\pi_1(f\mathcal{D}_2(k)) = PR\beta_k$. Indeed $f\mathcal{D}_2(k) = \mathcal{D}_2(k) \times (S^1)^k$, the factor $\mathcal{D}_2(k)$ is homotopy equivalent to the configuration space $F(\mathbb{R}^2, k)$ of k ordered points in $\mathbb{R}^2$, and it is well known that $\pi_1(F(\mathbb{R}^2, k)) = P\beta_k$ [**15**, Proposition X.6.14].

REMARK 5.7 The lantern relation was introduced in [**13**] by Johnson for its relevance to the mapping class group of surfaces. It is defined by the following equation in the mapping class group of a sphere with four holes: $T_{E_4} = T_{E_1} T_{E_2} T_{E_3} T_{C_1} T_{C_2} T_{C_3}$, where T_C denotes the Dehn twist along the curve C. See Fig. 3 for the relevant curves on a sphere with four holes, or equivalently on a disc with three holes. The mapping class group is the group of path components of orientation preserving diffeomorphisms which fix the boundary pointwise. For a sphere with four holes, this group is isomorphic to the pure ribbon braid group $PR\beta_3$. The lantern relation is thus a relation in $PR\beta_3$ and gives rise to a relation in $H_1(f\mathcal{D}_2(3))$ which is the abelianization of $PR\beta_3$. It was noted by Tillmann that, with this interpretation, one gets precisely the BV-axiom (equation 1). Indeed, up to signs, the curve E_1 represents the operation $(x, y, z) \mapsto \Delta x * y * z$. Moreover E_2 corresponds to $x * \Delta y * z$, E_3 to $x * y * \Delta z$, E_4 to $\Delta(x * y * z)$, C_1 to $\Delta(x * y) * z$, C_2 to $x * \Delta(y * z)$ and C_3 to $y * \Delta(x * z)$.

This geometric interpretation gives a direct proof of the fact that any $H(f\mathcal{D}_2)$-algebra is a BV-algebra.

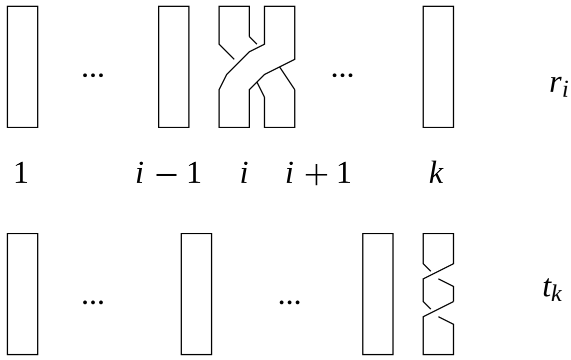


Fig. 2 Generators of the ribbon braid group $R\beta_k$

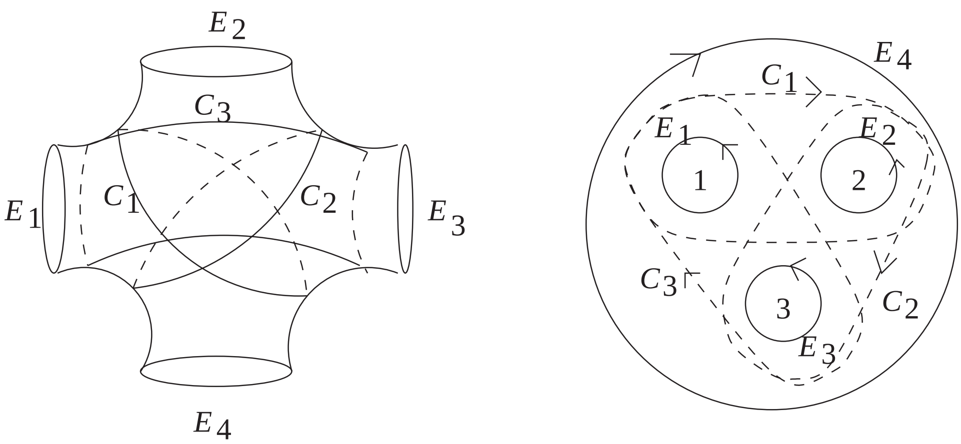


Fig. 3 Lantern relation

6. Quadratic duality for semidirect products

Recall that we work in the category of differential graded vector spaces, or chain complexes, over a field k of characteristic 0, and with operads having trivial differential. In this section, we assume moreover that the vector spaces have finite type.

Let P be a quadratic operad of H-modules. We assume that $P(0) = 0$ and $P(1) = k$ is concentrated in degree 0.

Recall that if $P = F(V)/(R)$ is the quadratic operad generated by V with relations R, then its *quadratic dual* is the operad $P^! := F(\check{V})/(R^{\perp})$, where $\check{V} = V^* \otimes \mathrm{sgn}_2$, V^* is the linear dual, sgn_2 is the sign representation of the symmetric group Σ_2, and $R^{\perp}$ is the annihilator of R in $F(\check{V})(3) = F(V)(3)^*$ [**12**, 2.1.7]. The dual $(P \rtimes H)^!$ of $P \rtimes H$ in this sense is not naturally a semidirect product operad. Thinking of $P \rtimes H$ as the operad P in H-Mod, we consider instead the following duality.

DEFINITION 6.1 The *dual of a semidirect product* $P \rtimes H$ is the operad $(P \rtimes H)^\dagger := P^! \rtimes H^{op}$, where $P^!$ is the quadratic dual of P and H^{op} denotes H with the opposite multiplication.

This makes sense because $P^!$ is an operad of H^{op}-modules (using Proposition 4.3).

The *suspension* of an operad P is the operad ΛP defined by

$$\Lambda P(n) = \Sigma^{n-1}(P(n)) \otimes \mathrm{sgn}_n,$$

with the operad structure induced by P (where $\Sigma^{n-1}(P(n))$ is the $n-1$ suspension of the graded vector space $P(n)$). This operad has the property that a chain complex A is a ΛP-algebra if and only if ΣA is a P-algebra.

In the following example, we show that the operad of Batalin–Vilkovisky algebras, as a semidirect product, is self-dual up to suspension.

EXAMPLE 6.2 1. $\mathrm{BV}_{2n}^\dagger := e_{2n}^! \rtimes H(S^{2n-1}) = \Lambda^{1-2n}\mathrm{BV}_{2n}$;

2. $fe_{2n}^\dagger := e_{2n}^! \rtimes H(SO(2n))^{op} = \Lambda^{1-2n}\mathrm{BV}_{2n} \times H(SO(2n-1))$;

3. $fe_{2n+1}^\dagger = \Lambda^{-2n}e_{2n+1} \times H(SO(2n+1))$.

Proof. We give a proof of (1). It is known that the quadratic dual of the operad e_{2n} is its own $(2n-1)$-fold desuspension $\Lambda^{1-2n}e_{2n}$, with product $p' = b^*$ dual to the original bracket b, and bracket $b' = p^*$ dual to the original product p ([**11**, Theorem 3.1] or [**9**, Proposition 9.5]). Since $\alpha_{2n-1}(p) = b$, it follows that $\alpha_{2n-1}(b^*) = p^*$. So the class α_{2n-1} gives an operator Δ of degree $2n-1$ with $\Delta(p') = b'$ and $\Delta(b') = 0$, which thus induces a BV structure as in Theorem 5.4 but this time with product in degree $1-2n$.

Let the complex A be a P-algebra in H-Mod. Define the complex $C_P(A)$ to be the free $P^!$-algebra on the desuspension of A^*:

$$C_P(A) = \bigoplus_{k\geqslant 0}(P^!(k+1) \otimes_{\Sigma_{k+1}} (\Sigma^{-1}A^*)^{\otimes k+1})$$

with differential $d_1 + d_2$, where d_1 is induced by the differential ∂ of A and d_2 is induced by the P-algebra structure of A. (See [**16**, sections 1.1.9, 1.4] for more details. Note that Livernet works with the dual complex, the free $P^!$-coalgebra on ΣA.) Then $C_P(A)$ has an H^{op}-module structure induced by the action of H on A and on $(P^!(n))^*$. This action commutes with d_1 as A is an H-module, and it commutes with d_2 as the P-algebra structure of A is H-equivariant. Finally, the $P^!$-algebra structure of $C_P(A)$ is H^{op}-equivariant because P is an operad of H-modules.

For a P-algebra A, the *P-homology of A*, denoted $H_P(A)$, is defined to be the homology of the complex $C_P(A)$ [**18**, Definition 3.93]. As $P \rtimes H$ has trivial differential, we have the following proposition.

PROPOSITION 6.3 *Let A be a $P \rtimes H$-algebra. Then $C_P(A)$ and $H_P(A)$ are $P^! \rtimes H^{op}$-algebras.*

REMARK 6.4 When the operad P is *Koszul* [**12**, Definition 4.1.3], the complex $C_P(A)$ induces an equivalence of homotopy categories between appropriate categories of P-algebras and $P^!$-(co)algebras. (See [**16**, section 1.4] for P in degree 0, or [**11**, Theorem 2.25]. Note that in [**11**] Getzler and Jones work with cooperads as well as coalgebras, using a dual cooperad $P^\perp$ instead of $P^!$, where $P^! = \Lambda^{-1}(P^\perp)^*$. This comes to working with the complex dual to $C_P(A)$ and placing the suspension in the operad rather than in the algebra.) The proposition can be used to extend these equivalences to the corresponding categories of $P \rtimes H$-algebras and $P^! \rtimes H^{op}$-(co)algebras.

We apply operadic homology in order to compute explicitly the BV-algebra structure of the homology of the double loop space on a manifold. This improves [**11**, Theorem 6.1], where the Gerstenhaber algebra structure was computed.

Let M be a 2-connected manifold with a smooth S^1-action $f : S^1 \times M \to M$ preserving a base point x_0. Let $\mathcal{A}^*(M)$ be the commutative algebra of differential forms on M vanishing on x_0 in degree 0, with non-negative grading, and let $d\theta$ denote the standard volume form on S^1. The S^1-action induces a derivation Δ on $\mathcal{A}^*(M)$ of degree -1. To define Δ on a form ω, write uniquely $f^*(\omega) = d\theta \wedge \omega_1 + \omega_2$, so that $d\theta$ does not appear in the expression of ω_2 in local coordinates $(\theta, x_1, \ldots, x_m)$. Then set $\Delta(\omega) := i^*(\omega_1)$, where i is the inclusion $i : M \cong \{1\} \times M \hookrightarrow S^1 \times M$. A simple computation shows that Δ is a derivation with respect to the wedge product and commutes with the exterior derivative.

We want to show that $\Delta^2 = 0$. Write uniquely $(S^1 \times f)^* f^*(\omega) = d\theta' \wedge d\theta'' \wedge \omega_a + \omega_b$ so that $d\theta' \wedge d\theta''$ does not appear in the expression of ω_b in local coordinates $(\theta', \theta'', x_1, \ldots, x_m)$. Then $\Delta^2(\omega) = i^*(\omega_a)$, where i now denotes the inclusion $M \hookrightarrow S^1 \times S^1 \times M$. On the other hand, let $m : S^1 \times S^1 \to S^1$ denote the multiplication. As we have an action, $(S^1 \times f)^* f^*(\omega) = (m \times M)^* f^*(\omega) = (d\theta' + d\theta'') \wedge \omega_1 + \omega_2$. Hence $\omega_a = 0$, which shows that $\Delta^2(\omega) = 0$.

Thus, by reversing the signs of the grading, $\mathcal{A}^*(M)$ is a BV-algebra with trivial bracket and, by Example 6.2, the suspension $\Sigma\mathcal{A}^*(M)$ is a $\mathrm{BV}^\dagger$-algebra.

Let $\tilde{H}$ denote the reduced homology with coefficients in k, and let $G = e_2$ denote the Gerstenhaber operad.

THEOREM 6.5 *The $G^!$-homology of the* $\mathrm{BV}^\dagger$*-algebra* $\Sigma\mathcal{A}^*(M)$ *is isomorphic to* $\tilde{H}(\Omega^2 M)$ *as a* BV*-algebra.*

Proof. By the Milnor–Moore theorem, $\tilde{H}(\Omega^2 M)$ is the free commutative algebra on the homotopy groups $\pi_*(\Omega^2 M) \otimes k = \Sigma^{-2}\pi_*(M) \otimes k$, which are embedded in $\tilde{H}(\Omega^2 M)$ via the Hurewicz homomorphisms [**6**, Theorem 21.5]. Cohen showed in [**3**, Remark 1.2] that the bracket on $\tilde{H}(\Omega^2 M)$ restricts to the Whitehead product on $\pi_*(\Omega^2 M) \otimes k$. On the other hand, since $\mathcal{A}^*(M)$ has a trivial bracket, an easy computation (see [**11**, section 6.3]) shows that the operadic homology $H_{G^!}(\Sigma\mathcal{A}^*(M))$ is the free commutative algebra on $H_{\Lambda^{-1}(Com)}(\Sigma\mathcal{A}^*(M))$, and this, by the deRham–Sullivan theory, is naturally isomorphic to the Λ(Lie)-algebra $\Sigma^{-2}\pi_*(M) \otimes k$, where the bracket is the Whitehead product, up to a sign [**24**, I.3.(11); **4**, Theorem 5(b)].

Thus $H_{G^!}(\Sigma\mathcal{A}^*(M))$ is naturally isomorphic to $\tilde{H}(\Omega^2 M)$ as a Gerstenhaber algebra. As $\mathrm{BV}^\dagger = G^! \rtimes H(S^1)$, all we are left to prove is that the geometric operator Δ_g on $\tilde{H}(\Omega^2 M)$ coincides with the algebraic operator Δ_a on $H_{G^!}(\Sigma\mathcal{A}^*(M))$ under this isomorphism. It is sufficient to check it on the generators $\Sigma^{-2}\pi_*(M) \otimes k \subset \tilde{H}(\Omega^2 M)$. It is also enough to prove this in the universal case. Indeed, take an element $c \in \pi_n(M)$, with $n > 2$, and a smooth representative $\gamma : S^n \to M$. The universal (based) S^1-space is

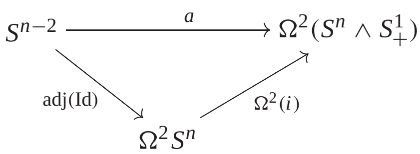
$$(S^n \times S^1)/(* \times S^1) \simeq S^n \wedge S^1_+,$$

with S^1-action given by multiplication on the second factor. As this is not a manifold, we have to work in the relative case. Let $\mathcal{A}^*(S^n \times S^1, S^1)$ be the kernel of the restriction $\mathcal{A}^*(S^n \times S^1) \to \mathcal{A}^*(* \times S^1)$. As before, the S^1-action induces an operator Δ on $\mathcal{A}^*(S^n \times S^1, S^1)$. With non-positive grading this makes $\mathcal{A}^*(S^n \times S^1, S^1)$ into a BV-algebra. The map of pairs $z : (S^n \times S^1, * \times S^1) \to (M, x_0)$ defined by $z(x, \theta) = \theta(\gamma(x))$ induces a BV-algebra map $\mathcal{A}^*(M) \to \mathcal{A}^*(S^n \times S^1, S^1)$, and thus a BV-algebra map on $G^!$-homology. Let ω be the standard volume form on S^n; $H(\mathcal{A}^*(S^n \times S^1, S^1))$ is generated by the class $[\omega]$ in dimension n and $[\omega \wedge d\theta]$ in dimension $n+1$. The vector subspace V of $\mathcal{A}^*(S^n \times S^1, S^1)$ generated by $d\theta \wedge \omega$ and ω is a sub-BV-algebra, with trivial product, bracket and differential. By definition, $\Delta(d\theta \wedge \omega) = \omega$. The inclusion $V \subset \mathcal{A}^*(S^n \times S^1, S^1)$ is a weak equivalence and thus induces an isomorphism of BV-algebras on $G^!$-homology. So $H_{G^!}(\Sigma V) \cong H_{G^!}(\Sigma \mathcal{A}^*(S^n \times S^1, S^1))$ as BV-algebra, and the latter is isomorphic to $\tilde{H}(\Omega^2(S^n \wedge S^1_+))$ as Gerstenhaber algebra, by the procedure described above applied to the relative case.

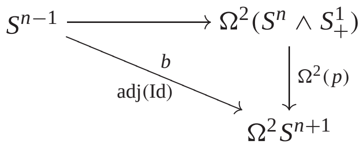
By definition, the operadic homology $H_{G^!}(\Sigma V)$ is the free Gerstenhaber algebra on $\Sigma^{-2}(V^*)$, with dual basis $\{e_{n-2}, e_{n-1}\}$ (where $e_{n-2} = \Sigma^{-2}\omega^*$ and $e_{n-1} = \Sigma^{-2}(\omega \wedge d\theta)^*$), and (algebraic) operator $\Delta = \Delta_a$ induced by $\Delta_a(e_{n-2}) = e_{n-1}$.

Let us compute the geometric bracket Δ_g.

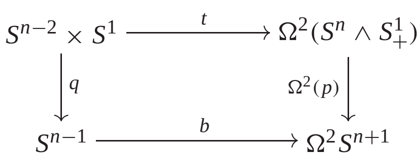
The class e_{n-2} corresponds under the isomorphism to the spherical class induced by the map



where i is the inclusion of S^n in $S^n \wedge S^1_+$. Let $p : S^n \wedge S^1_+ \to S^n \wedge S^1 = S^{n+1}$ be the projection. Then $(\Omega^2 p)_*(e_{n-1}) \in H_{n-1}(\Omega^2 S^{n+1})$ is represented by the map $b : S^{n-1} \to \Omega^2 S^{n+1}$ adjoint to the identity.



Now consider the diagram



where q is the projection $q : S^{n-2} \times S^1 \to S^{n-2} \wedge S^1 = S^{n-1}$ and $t(x, \theta)$ is the result of the action of θ on $a(x)$. Because the respective adjoints $S^2 \times S^{n-2} \times S^1 \to S^{n+1}$ have both degree one this diagram homotopy commutes. As $S^n \wedge S^1_+ \simeq S^n \vee S^{n+1}$, we have $\Omega^2(S^n \wedge S^1_+) \simeq \Omega^2\Sigma^2(S^{n-2} \vee S^{n-1})$. By computations of Fred Cohen, $H(\Omega^2\Sigma^2(S^{n-2} \vee S^{n-1}))$ is the free Gerstenhaber algebra on generators e_{n-1} and e_{n-2} of degree $n-1$ and $n-2$ respectively [**3**, Theorem 3.2]. Similarly, $H(\Omega^2 S^{n+1}) = H(\Omega^2\Sigma^2(S^{n-1}))$ is the free Gerstenhaber algebra on a single generator e_{n-1}. The map $(\Omega^2 p)_*$ kills e_{n-2}. As e_{n-2} does not generate anything in degree $n-1$, $(\Omega^2 p)_*$ is an isomorphism in that degree.

Since q has degree one and $t_*([S^{n-2} \times S^1]) = \Delta_g(e_{n-2})$, we conclude that $\Delta_g(e_{n-2}) = e_{n-1}$. The general case follows by naturality.

7. Application to ribbon braided categories

We construct in this section a categorical operad out of the ribbon braid groups. Taking the nerve of the categories provides a topological operad equivalent to the framed 2-discs operad. We exhibit this equivalence using a characterization of operads equivalent to $f\mathcal{D}_2$. We also describe the algebras over the categorical ribbon operad. This extends work of Fiedorowicz in the braid case [**7**,**8**]. Details and proofs about the ribbon case can be found in [**26**, sections 1.4 and 1.5].

In order to characterize topological operads equivalent to the framed 2-discs, we consider the following notion of equivalence.

DEFINITION 7.1 A map of topological operads $\mathcal{A} \to \mathcal{B}$ is an *equivalence* if each map $\mathcal{A}(k) \to \mathcal{B}(k)$ is a Σ_k-equivariant homotopy equivalence.

An operad $\mathcal{A}$ is an E_n-operad (resp. fE_n-operad) if there is a chain of equivalences connecting $\mathcal{A}$ to $\mathcal{D}_n$ (resp. $f\mathcal{D}_n$).

We will need to consider operads having ribbon braid groups actions instead of the usual symmetric groups actions.

DEFINITION 7.2 A collection of spaces $\mathcal{A} = \{\mathcal{A}(k)\}$ is a *ribbon operad* if there is a right action of $R\beta_k$ on $\mathcal{A}(k)$ for each k and if there are associative structure maps

$$\gamma : \mathcal{A}(k) \times \mathcal{A}(n_1) \times \cdots \times \mathcal{A}(n_k) \to \mathcal{A}(n_1 + \cdots + n_k)$$

with two-sided unit $e \in A(1)$, satisfying the equivariance conditions

$$\gamma(a.\sigma, b_1, \ldots, b_k) = \gamma(a, b_{[\sigma]^{-1}(1)}, \ldots, b_{[\sigma]^{-1}(k)}).\sigma(n_1, \ldots, n_k)$$

and

$$\gamma(a, b_1.\tau_1, \ldots, b_k.\tau_k) = \gamma(a, b_1, \ldots, b_k).(\tau_1 \oplus \cdots \oplus \tau_k),$$

for all $a \in \mathcal{A}(k), b_i \in \mathcal{A}(n_i), \sigma \in R\beta_k, \tau_i \in R\beta_{n_i}$, where $[\sigma]$ is the permutation induced by σ, the ribbon $\sigma(n_1, \ldots, n_k) \in R\beta_{n_1+\cdots+n_k}$ is obtained from σ by replacing the ith ribbon by n_i ribbons, and $(\tau_1 \oplus \cdots \oplus \tau_k)$ is the block sum of the ribbons $\tau_1, \ldots, \tau_k$.

A ribbon operad $\mathcal{A}$ is called an R_∞ *operad* if each $\mathcal{A}(k)$ is a contractible numerable principal $R\beta_k$-space.

Adapting Fiedorowicz's proof in the case of braid groups and E_2-operads, we obtain the following.

THEOREM 7.3 *A topological operad $\mathcal{A}$ is an fE_2 operad if and only if its operad structure lifts to an R_∞ operad structure on its universal cover $\tilde{\mathcal{A}}$.*

There is a general method for constructing categorical operads from certain families of groups [**25**; **26**, section 1.2]. Our main example of fE_2 operads comes from this construction, after taking the nerve. The ribbon braid groups give rise in this way to a categorical operad R, where $R(k)$ is the *translation* category with set of objects $R\beta_k/PR\beta_k = \Sigma_k$, and the morphisms $\mathrm{Hom}_R(\sigma PR\beta_k, \tau PR\beta_k) \cong PR\beta_k$ can be thought as multiplications on the left by elements of $R\beta_k$:

$$\tau PR\beta_k \xleftarrow{\tau h \sigma^{-1}} \sigma PR\beta_k,$$

where $h \in PR\beta_k$ and $\sigma, \tau \in R\beta_k$. The operad structure maps are defined on objects as in the associative operad, and on morphisms by

$$\gamma(\sigma_1 \xleftarrow{\tau} \sigma_0, \rho_1, \ldots, \rho_k) = \tau(n_{\sigma_0^{-1}(1)}, \ldots, n_{\sigma_0^{-1}(k)})(\rho_{\sigma_0^{-1}(1)} \oplus \cdots \oplus \rho_{\sigma_0^{-1}(k)}),$$

where the right-hand side is defined as in Definition 7.2.

Let $|R|$ denote the topological operad obtained by applying the nerve construction to R. So $|R|(k) = |R(k)|$ is the nerve of the category $R(k)$.

PROPOSITION 7.4 *The operad $|R|$ is an fE_2-operad.*

Proof. The universal cover of $|R(k)|$ is the realization of the simplicial set $ER\beta_k$, which is a contractible space with free $R\beta_k$-action. In order to use Theorem 7.3, one has to show that the operad structure of $|R|$ lifts to its universal cover. The ribbon operad structure one obtains on $ER\beta_k$ is a ribbon version of the well-known E_∞ operad structure on classifying spaces of the categories $E\Sigma_k$.

We next describe R-algebras.

DEFINITION 7.5 A *braided monoidal category* is a monoidal category $(\mathcal{A}, \otimes)$ equipped with a *braiding*, that is, a natural family of isomorphisms

$$c = c_{A,B} : A \otimes B \longrightarrow B \otimes A$$

satisfying the 'braid relations':

$$(\mathrm{id} \otimes c_{A,C}) \circ a \circ (c_{A,B} \otimes \mathrm{id}) = a \circ c \circ a : (A \otimes B) \otimes C \to B \otimes (C \otimes A),$$

and

$$(c_{A,C} \otimes \mathrm{id}) \circ a^{-1} \circ (\mathrm{id} \otimes c_{B,C}) = a^{-1} \circ c \circ a^{-1} : A \otimes (B \otimes C) \to (C \otimes A) \otimes B,$$

where a denotes the associativity isomorphism.

The category is braided *strict* monoidal if the monoidal structure is strict, that is, if the associativity and unit isomorphisms are the identity.

A *ribbon braided* (*strict*) *monoidal category* $(\mathcal{A}, \otimes, c, \tau)$ is a braided (strict) monoidal category $(\mathcal{A}, \otimes, c)$ equipped with a *twist*, that is, a natural family of isomorphisms

$$\tau = \tau_A : A \longrightarrow A$$

such that $\tau_1 = \mathrm{id}_1$, where 1 is the unit object of $\mathcal{A}$, and satisfying the following compatibility with the braiding: $\tau_{A\otimes B} = c_{B,A} \circ \tau_B \otimes \tau_A \circ c_{A,B} : A \otimes B \to A \otimes B$ (see Fig. 4).

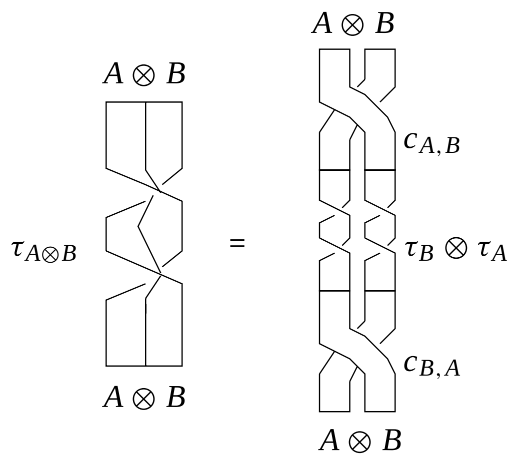


Fig. 4 Compatibility between the twist and the braiding

Braided monoidal categories arise in the theory of quantum groups and their associated link invariants [**21**; **15**, Chapter XIII]. Shum shows in [**22**] that a ribbon braided structure provides a natural solution to coherence questions for braided monoidal categories equipped with a duality. Note that ribbon braided categories with a left duality are called *ribbon categories* in [**15**].

Symmetric monoidal and braided monoidal categories are the algebras over categorical operads constructed out of the symmetric and braid groups respectively. In the ribbon case we have the following.

PROPOSITION 7.6 *A category is an R-algebra if and only if it is a ribbon braided strict monoidal category.*

If $\mathcal{A}$ is an R-algebra, there are functors $\theta_k : R(k) \times \mathcal{A}^k \to \mathcal{A}$. The product in $\mathcal{A}$ is defined on objects by $A \otimes B = \theta_2(\mathrm{id}_{\Sigma_2}, A, B)$ and on morphisms by $f \otimes g = \theta_2(\mathrm{id}_{R\beta_2}, f, g)$. The braiding is given by $c_{A,B} = \theta_2(r_1, \mathrm{id}_A, \mathrm{id}_B)$, where $r_1 \in R\beta_2$ is is given in Fig. 2 and the twist is defined by $\tau_A = \theta_1(t_1, \mathrm{id}_A)$, where t_1 is the generator of $R\beta_1$.

Consider the monoid $\mathbb{R} \times_{\mathbb{Z}} E\mathbb{Z}$. There are monoid maps

$$S^1 \cong (\mathbb{R} \times_{\mathbb{Z}} *) \overset{\simeq}{\longleftarrow} \mathbb{R} \times_{\mathbb{Z}} E\mathbb{Z} \overset{\simeq}{\longrightarrow} (* \times_{\mathbb{Z}} E\mathbb{Z}) \cong B\mathbb{Z}.$$

So any S^1-space or $B\mathbb{Z}$-space is canonically an $\mathbb{R} \times_{\mathbb{Z}} E\mathbb{Z}$-space. If one denotes by $f\tilde{\mathcal{D}}_2$ and $|\tilde{R}|$ the universal cover operads (R_∞-operads) of $f\mathcal{D}_2$ and $|R|$ respectively, the above maps are restrictions of the operad maps $f\mathcal{D}_2 \overset{\simeq}{\longleftarrow} f\tilde{\mathcal{D}}_2 \times_{PR\beta} |\tilde{R}| \overset{\simeq}{\longrightarrow} |R|$ in arity 1. Using our recognition principle (Theorem 3.1) and Theorem 7.3, we obtain the following:

THEOREM 7.7 *The nerve of a ribbon braided monoidal category $\mathcal{C}$, after group completion, is weakly homotopy equivalent to a double loop space $\Omega^2 Y$ on an S^1-space Y. The S^1-action on Y is induced by the twist on $\mathcal{C}$ and the equivalence is given by $\mathbb{R} \times_{\mathbb{Z}} E\mathbb{Z}$-equivariant maps.*

Proof. Let $\mathcal{C}$ be a ribbon braided monoidal category and let $\mathcal{C}'$ be the strictification of $\mathcal{C}$ as a monoidal category. The category $\mathcal{C}'$ then inherits a ribbon braided structure from the one existing on $\mathcal{C}$ (see [**14**, Example 2.4] for the braid case). Indeed, let $F : (\mathcal{C}', \otimes') \to (\mathcal{C}, \otimes)$ be an equivalence of monoidal categories. Define the braiding $c'_{A,B} : A \otimes' B \to B \otimes' A$ on $\mathcal{C}'$ to be the unique morphism in $\mathcal{C}'$ whose image under F is the composite $F(A \otimes' B) \overset{\simeq}{\to} FA \otimes FB \overset{c}{\to} FB \otimes FA \overset{\simeq}{\to} F(B \otimes' A)$. Define the twist similarly. The relations are satisfied by the faithfulness of F.

The nerve $|\mathcal{C}'|$ is an $|R|$-algebra. The space $|\mathcal{C}|$ is not necessarily an $|R|$-algebra, but it admits a $B\mathbb{Z}$-action induced by the twist on $\mathcal{C}$, and the equivalence $|\mathcal{C}| \overset{\simeq}{\longrightarrow} |\mathcal{C}'|$ is $B\mathbb{Z}$-equivariant.

Now the space $X = B(fD_2, f\tilde{D}_2 \times_{PR\beta} |\tilde{R}|, |\mathcal{C}'|)$ is weakly homotopy equivalent to $|\mathcal{C}'|$ and is an $f\mathcal{D}_2$-algebra. The equivalence is obtained through the following diagram of weak equivalences in $\mathbb{R} \times_{\mathbb{Z}} E\mathbb{Z}$-Top:

$$\begin{array}{ccc} B(fD_2, f\tilde{D}_2 \times_{PR\beta} |\tilde{R}|, |\mathcal{C}'|) & \longleftarrow & B(f\tilde{D}_2 \times_{PR\beta} |\tilde{R}|, f\tilde{D}_2 \times_{PR\beta} |\tilde{R}|, |\mathcal{C}'|) \\ & & \downarrow \\ & & B(|R|, |R|, |\mathcal{C}'|) \longrightarrow |\mathcal{C}'|. \end{array}$$

The group completion of X is then equivalent to a double loop space $\Omega^2 Y$, where $Y = B(\Sigma^2, D_2, X)$ and the S^1-action on X now induces one on Y, as explained in Theorem 3.1.

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References

1. C. Berger and I. Moerdijk, *Axiomatic homotopy theory for operads*, preprint math.AT/0206094 at http//arxiv.org (2002).
2. J. M. Boardman and R. M. Vogt, *Homotopy Invariant Algebraic Structures on Topological Spaces*, Lecture Notes in Mathematics 347, Springer, Berlin, 1973.
3. F. R. Cohen, T. J. Lada, and J. P. May, *The Homology of Iterated Loop Spaces*, Lecture Notes in Mathematics 533, Springer, Berlin, 1976.
4. N. Dupont, Problems and conjectures in rational homotopy theory: a survey, *Expositio Math.* **12** (1994), 323–352.
5. W. G. Dwyer and D. M. Kan, Equivariant homotopy classification, *J. Pure Appl. Algebra* **35** (1985), 269–285.
6. Y. Felix, S. Halperin, and J.-C. Thomas, *Rational Homotopy Theory*, Graduate Texts in Mathematics 205, Springer, New York, 2001.
7. Z. Fiedorowicz, *The symmetric bar construction*, preprint.
8. Z. Fiedorowicz, *Constructions of E_n operads*, preprint math.AT/9808089 at http//arxiv.org (1999).
9. F. T. Fox and M. Markl, Distributive laws, bialgebras, and cohomology, *Operads: Proceedings of Renaissance Conferences*, *Contempory Mathematics* 202, (Eds J.-L. Loday, J. D. Stasheff, and A. A. Voronov), American Mathematical Society, Providence, RI, (1997), 167–205.
10. E. Getzler, Batalin–Vilkovisky algebras and two-dimensional topological field theories, *Comm. Math. Phys.* **159** (1994), 265–285.

11. E. Getzler and J. D. S. Jones, *Operads, homotopy algebra, and iterated integrals for double loop spaces*, preprint hep-th/9403055 at http//arxiv.org (1994).

12. V. Ginzburg and M. Kapranov, Koszul duality for operads, *Duke Math. J.* **76** (1994), 203–272.

13. D. L. Johnson, Homeomorphisms of a surface which act trivially on homology, *Proc. Amer. Math. Soc.* **75** (1979), 119–125.

14. A. Joyal and R. Street, Braided tensor categories, *Adv. Math.* **102** (1993), 20–78.

15. C. Kassel, *Quantum Groups*, Springer, Berlin, 1995.

16. M. Livernet, Homotopie rationnelle des algèbres sur une opérade, Ph.D. Thesis, Strasbourg, 1998. (http://irmasrv1.u-strasbg.fr/irma/publications/1998/98032.shtml)

17. M. Markl, A compactification of the real configuration space as an operadic completion, *J. Algebra* **215** (1999), 185–204.

18. M. Markl, S. Shnider, and J. Stasheff, *Operads in Algebra, Topology and Physics*, American Mathematical Society, Providence, 2002.

19. J. P. May, *The Geometry of Iterated Loop Spaces*, Lecture Notes in Mathematics 271, Springer, Berlin, 1972.

20. M. Mimura and H. Toda, *Topology of Lie Groups I and II*, Translations AMS 91, American Mathematical Society, Providence, 1991.

21. S. Sawin, Links, quantum groups and TQFTS, *Bull. Amer. Math. Soc.* **33** (1996), 413–445.

22. M. C. Shum, Torile tensor categories, *J. Pure Appl. Algebra* **93** (1994), 57–110.

23. D. Sullivan and A. A. Voronov, paper in progress.

24. D. Tanré, *Homotopie Rationnelle: Modèles de Chen, Quillen, Sullivan*, Lecture Notes in Mathematics 1025, Springer, Berlin, 1983.

25. U. Tillmann, Higher genus surface operad detects infinite loop spaces, *Math. Ann.* **317** (2000), 613–628.

26. N. Wahl, Ribbon braids and related operads. D.Phil. Thesis, Oxford, 2001. (http://www.maths.ox.ac.uk/research/theses/).