TCFT AND CALABI-YAU CATEGORIES

FIX Λ = SET OF D-BRANES

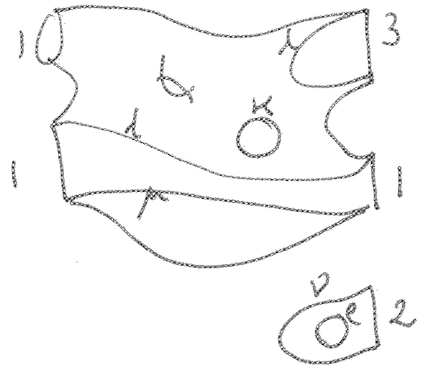
$\mathcal{M}_\Lambda$ = TOPOLOGICAL CATEGORY FOR "OPEN-CLOSED TCFT"

$$Obj(\mathcal{M}_\Lambda) = \{(c, o, s, t) \mid c, o \in \mathbb{Z}_{\geq 0},\ s, t : o \to \Lambda\}$$

c: CLOSED BOUNDARIES / CIRCLES / STRINGS
o: OPEN BOUNDARIES / INTERVALS / STRINGS FROM $s(i)$ TO $t(i)$

MORPHISM SPACES = MODULI SPACES OF RIEMANN SURFACES WITH OPEN, CLOSED AND FREE BOUNDARIES
(OPEN, CLOSED: INCOMING AND OUTGOING; FREE BOUNDARIES: LABELED BY ELEMENTS OF Λ)

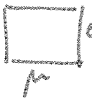
EX:



$$(1, [\lambda, \mu]) \longrightarrow ([\mu, \lambda], [\nu, \nu], [\lambda, \lambda])$$

SPECIAL CONDITIONS:

- Σ HAS AT LEAST ONE INCOMING CLOSED BOUNDARY OR ONE FREE BDRY
-  AND  ARE ALLOWED BUT WITH MODULI REPLACED BY $*$ (BECAUSE OF UNSTABILITY)
-  CLOSED IN, CLOSED OUT: MODULI REPLACED BY $\mathrm{Diff}^+ S^1$ TO HAVE IDENTITIES. "INFINITESIMAL CYLINDER" ↳ REPARAMETRIZATION

 IN, OUT: MODULI REPLACED BY $*$ ACTING AS id.

$\mathcal{M}_\Lambda$ IS SYMMETRIC MONOIDAL UNDER $\amalg$.

$\mathcal{OC}^d_\Lambda = C_*(\mathcal{M}_\Lambda, \det^{\otimes d})$ SYMM. MONOIDAL CAT.

det LOCAL SYSTEM ON $\mathcal{M}_\Lambda(a,b)$

$\det(\Sigma) = \det(H^0\Sigma) \otimes \det^*(H^1(\Sigma)^*) \otimes \det(\mathbb{K}^0)$ — FIX SOME FIELD OF CHAR. 0

SATISFIES $\det(\Sigma_2 \circ \Sigma_1) \cong \det(\Sigma_2) \otimes \det(\Sigma_1)$

$\det(\Sigma_1 \sqcup \Sigma_2) \cong$

$\mathcal{O}^d_\Lambda$ = FULL SUBCATEGORY WITH NO CLOSED BOUNDARIES

$\mathcal{C}^d_\Lambda$ = ———————— NO OPEN OR FREE BOUNDARIES

<u>DEF</u>: AN <u>OPEN TCFT OF dim d</u> IS A PAIR (Λ, Φ) WHERE Λ = SET OF D-BRANES AND $\Phi : \mathcal{O}^d_\Lambda \longrightarrow \mathrm{Comp}_{\mathbb{K}}$ IS h-SPLIT

i.e. $\exists$ q-iso (H_*-iso) $\Phi(a) \otimes \Phi(b) \xrightarrow{\sim} \Phi(a \otimes b)$ $\forall a,b$

<u>THM</u> (1) THE CATEGORY OF OPEN TCFT'S OF dim d, WITH FIXED SET OF D-BRANES Λ, IS HTPY EQUIV. TO THE CATEGORY OF (UNITAL) EXTENDED CALABI-YAU A_∞-CATEGORIES OF dim d WITH SET OF OBJECTS Λ.

<u>DEF</u>: A UNITAL EXTENDED CY A_∞-CATEGORY WITH OBJ. Λ IS AN h-SPLIT FUNCTOR $\Phi : \mathcal{D}^d_{\mathrm{OPEN},\Lambda} \longrightarrow \mathrm{Comp}_{\mathbb{K}}$

<u>THM</u>: $\mathcal{D}^d_{\mathrm{OPEN},\Lambda} \simeq \mathcal{O}^d_\Lambda$

i.e. $\exists$ ZIG-ZAG OF FUNCTORS WHICH ARE q-ISO'S ON THE MORPHISM COMPLEXES (HAVE SAME OBJECTS)

$\forall\ \alpha, \beta \in Obj(\mathcal{M}_\Lambda)$ s.t. α HAS NO CLOSED PART,

CONSTRUCT $\mathcal{G}(\alpha,\beta) \simeq \mathcal{M}_\Lambda(\alpha,\beta)$ CELLULAR CPX

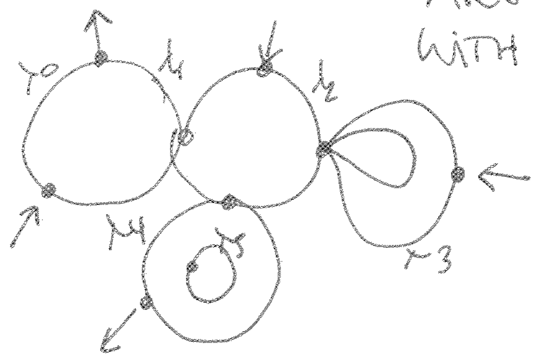
AND $\mathcal{D}^d_{OPEN,\Lambda} = C^{cell}_*\left(\mathcal{G}(\alpha,\beta)_{open}, det^{\otimes d}\right)$

$\overline{N}(\alpha,\beta)$ = MODULI SPACE OF RIEMANN SURFACES WITH

- CLOSED SMOOTH BOUNDARIES
- FREE POSSIBLY SINGULAR BOUNDARIES WITH $O(\alpha)+O(\beta)$ MARKED POINTS REPRESENTING INC. AND OUT. OPEN BDRIES

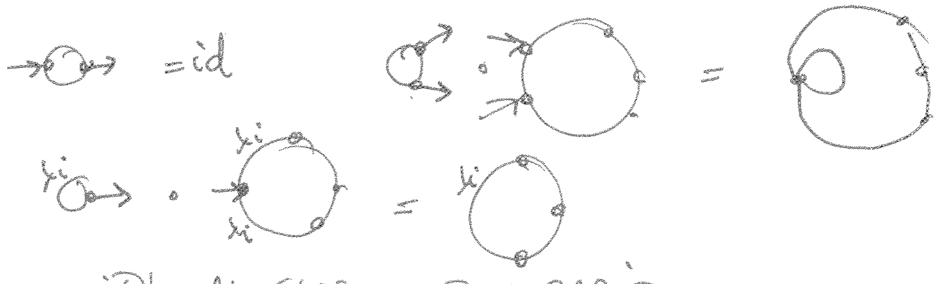
\+ STABILITY RESTRICTIONS AS BEFORE

$\mathcal{G}(\alpha,\beta) \underset{\simeq}{\hookrightarrow} \overline{N}(\alpha,\beta)$ SUBSPACE OF SURFACES WHOSE IRR. COMPONENTS ARE DISCS OR ANNULUS OF MODULUS 1 WITH ONE OUTGOING CLOSED BDRY.



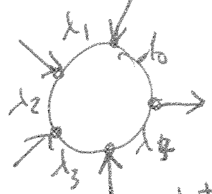
(NO ANNULI → DUAL GRAPH = FAT GRAPH)

(PARTIAL) COMPOSITION BY IDENTIFYING INCOMING AND OUTGOING POINTS, EXCEPT FOR



$\mathcal{G}_{open}$ = CATEGORY WITH NO CLOSED BOUNDARIES

$\mathcal{D}^d_{OPEN} = C^{cell}_*\left(\mathcal{G}_{open}, det^{\otimes d}\right)$

$\mathcal{D}^+_{OPEN} \subseteq \mathcal{D}^d_{OPEN}$ SUBCATEGORY GENERATED BY  1 OUTGOING

UNDER $\circ$ AND $\amalg$

$\mathcal{D}^+(\lambda_0, \dots, \lambda_n)$

PROP: (i) $\left\{ \Phi : \mathcal{D}^{+}_{OPEN,\Lambda} \to \text{Comp SPLIT} \right\} \longleftrightarrow \left\{ \text{UNITAL } A_\infty \text{ CATEGORY WITH OBJ} = \Lambda \right\}$

(ii) $\left\{ \Phi : \mathcal{D}^{d}_{OPEN,\Lambda} \to \text{Comp SPLIT} \right\} \longleftrightarrow \left\{ \text{UNITAL CY } A_\infty \text{ CAT. WITH OBJ.} = \Lambda \right\}$

Φ SPLIT IF $\Phi(a) \otimes \Phi(b) \xrightarrow{\simeq} \Phi(a \otimes b)$

PROOF (i) GIVEN Φ WANT TO CONSTRUCT B

$B(\lambda_0, \lambda_1) := \Phi([\lambda_0, \lambda_1])$

$$B(\lambda_0,\lambda_1) \otimes B(\lambda_1,\lambda_2) \otimes \cdots \otimes B(\lambda_{n-1},\lambda_n) \xrightarrow{m_n} B(\lambda_0,\lambda_n)$$

$$\downarrow \simeq \qquad \nearrow \Phi(\mathcal{D}^{+}(\lambda_0,\ldots,\lambda_n))$$

$$\Phi([\lambda_0,\lambda_1] \amalg \cdots \amalg [\lambda_{n-1},\lambda_n])$$

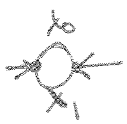
(ii) FOR EACH $\lambda_0, \lambda_1 \in Obj(B)$ $\exists$ NON-DEG. PAIRING

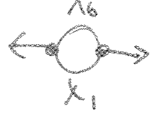
$\langle , \rangle_{\lambda_0,\lambda_1} : Hom(\lambda_0,\lambda_1) \otimes Hom(\lambda_1,\lambda_0) \to \mathbb{K}[d]$

WHICH IS SYMMETRIC $(\langle , \rangle_{\lambda_0\lambda_1} \text{"="} \langle , \rangle_{\lambda_1,\lambda_0})$ AND S.T.

$$\langle m_{n-1}(f_1 \otimes \cdots \otimes f_{n-1}), f_n \rangle = (-1)^{(n+1)+|f_1| \sum_{i=2}^{n} |f_i|} \langle m_{n-1}(f_2 \otimes \cdots \otimes f_n), f_1 \rangle$$

GIVEN BY $\Phi([\lambda_0,\lambda_1]) \otimes \Phi([\lambda_1,\lambda_0]) \longrightarrow \mathbb{K}[d]$

$\Phi($  $)$

WITH INVERSE $\Phi($  $)$

WHICH ARE THE TWO EXTRA GENERATORS IN $\mathcal{D}^{d}_{OPEN}$ ...

THM (2) $\Phi : \mathcal{O}^d \to \mathrm{Comp}$ h-split (OPEN TCFT)

i: $\mathcal{O}^d \hookrightarrow \mathcal{OC}^d$

$\to \mathbb{L}i_* \Phi : \mathcal{OC}^d \to \mathrm{Comp}$ h-split (OPEN-CLOSED TCFT)

$j^* \mathbb{L}i_* \Phi : \mathcal{C}^d \to \mathrm{Comp}$ h-split (CLOSED TCFT)

(3) LET $HH_*(\Phi)$ DENOTE THE HOCHSCHILD HOMOLOGY OF THE A_∞-CATEGORY ASSOCIATED TO Φ. THEN

$$H_*((\mathbb{L}i_* \Phi)(o,c)) = H_*(\Phi(o))^{\otimes o} \otimes HH_*(\Phi)^{\otimes c}$$

COR: H_*(MODULI SPACES) ACT ON HH_*(CY A_∞-CATEGORY)

EX: A FROBENIUS ALGEBRA $\to$ A_∞-CALABIYAU CAT WITH 1 OBJECT

$\to HH_*(A)$ is H_*(CLOSED TCFT)

• $$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & {\scriptstyle F^*\Phi}\searrow & \swarrow {\scriptstyle \Phi} \\ & \mathrm{Comp} & \end{array} \qquad \begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle \Phi}\searrow & & \swarrow {\scriptstyle F_*\Phi = \mathbb{L}F_*\Phi} \\ & \mathrm{Comp} & \end{array}$$

$$F_* \Phi(d) = \bigoplus_{\substack{p \geq 0 \\ c_0, \dots, c_p}} \Phi(c_0) \otimes \mathcal{C}(c_0, c_1) \otimes \dots \otimes \mathcal{C}(c_{p-1}, c_p) \otimes \mathcal{D}(F(c_p), d)$$

[illegible] $\to$ COMPLEX. WITH $d = d_{in} + d_{ext}$

• B dg (A_∞?) CATEGORY. DEFINE

$$C_*(B) = \bigoplus_{\substack{n \geq 0 \\ a_0, \dots, a_n}} (B(a_0, a_1) \otimes \dots \otimes B(a_n, a_0))[-n]$$

WITH
$$d(f_0 \otimes \dots \otimes f_n) = \sum_{i=0}^{n} \pm f_0 \otimes \dots \otimes df_i \otimes \dots \otimes f_n + \sum_{i=0}^{n-1} \pm f_0 \otimes \dots \otimes (f_{i+1} \circ f_i) \otimes \dots \otimes f_n \pm (f_0 \circ f_n) \otimes \dots \otimes f_{n-2}.$$

$$\longrightarrow HH_*(B) = H_*(C_*(B))$$

± PROOF OF THM (3):

$$\Phi : \mathcal{D}^d_{OPEN} \simeq \mathcal{O}^d \longrightarrow Comp$$

$$\mathcal{D}^+_{OPEN} \hookrightarrow \mathcal{D}^d_{OPEN}, \qquad \Phi_2 : \mathcal{D}^+_{OPEN} \longrightarrow Comp$$

⇝ $B(\Phi) = B(\Phi_2)$ A_∞-CATEGORY WITH OBJ. Λ

$1 \in Obj(\mathcal{OC})$ BE THE CLOSED CIRCLE.

$$\{\mathbb{L}i_* \Phi(1)\} = \bigoplus_{p\geq 0} \bigoplus_{\substack{a_0\dots a_p \\ \in Obj(\mathcal{O}^d)}} \Phi(a_0)\otimes \mathcal{O}^d(a_0,a_1)\otimes\dots\otimes\mathcal{O}^d(a_{p-1},a_p)\otimes \mathcal{OC}^d(a_p,1)$$

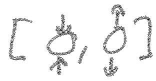
$\simeq$

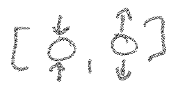
$$= \bigoplus_{p\geq 0}\bigoplus_{a_0\dots a_p} \Phi(a_0)\otimes \mathcal{D}^d_{OPEN}(a_0,a_1)\otimes\dots\otimes \mathcal{D}^d_{OPEN}(a_{p-1},a_p)\otimes\mathcal{D}^d(a_p,1)$$

$\uparrow$

$$\bigoplus_{p\geq 0}\bigoplus_{a_0\dots a_p}\Phi_2(a_0)\otimes\mathcal{D}^+_{OPEN}(a_0,a_1)\otimes\dots\otimes\mathcal{D}^+_{OPEN}(a_{p-1},a_p)\otimes\mathcal{D}^+(a_p,1)$$

CLAIM: THIS IS AN EQUIVALENCE BECAUSE

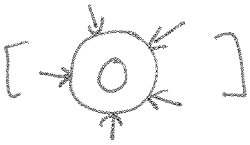
$\mathcal{D}^d_{OPEN} = \mathcal{D}^+_{OPEN}$ 

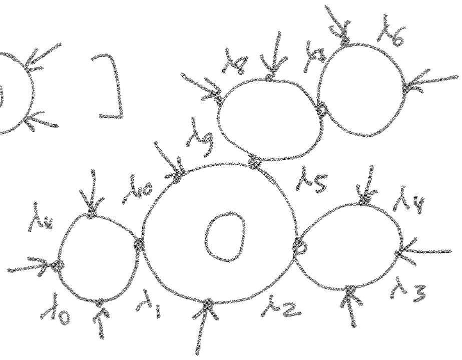
$\mathcal{D}^d = \mathcal{D}^+$ 

OR RATHER BECAUSE

$$\mathcal{D}^d(-,1) \simeq \mathcal{D}^+(-,1) \otimes_{\mathcal{D}^+_{OPEN}} \mathcal{D}^d_{OPEN}$$

AS $\mathcal{D}^d_{OPEN}$-MODULES

$\mathcal{D}^+ = \mathcal{D}^+_{OPEN}$ 

$\mathcal{D}^+(a_p,1) \ni$ 

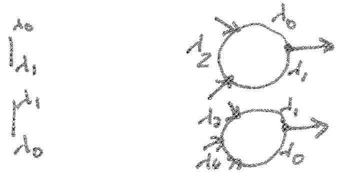
$a_p = \{\lambda_0,\dots,\lambda_{11}\}^c$ CYCLIC

$\mathcal{D}^+(a_p,1) = \emptyset$ IF a_p NOT CYCLIC!

AND $\mathcal{D}^+_{OPEN}(a_0,a_1)\otimes\dots\otimes\mathcal{D}^d_{OPEN}(a_{p-1},a_p)\otimes\mathcal{D}^+(a_p,1)$ NON-EMPTY IF

$a_0 > a_1 > \dots > a_p$ ALL CYCLIC

($\mathcal{D}^d_{OPEN}(a,b) \neq \emptyset$ WHEN b CYCLIC IFF $a > b$ ALSO CYCLIC)

EX: $b = \{\lambda_0,\lambda_1\}^c$  $a = \{\lambda_0,\lambda_2,\lambda_1,\lambda_3,\lambda_4\}^c$

$$C_*(B(\Phi)) \overset{\oplus}{\underset{p\geq 0}{=}} \bigoplus_{\lambda_0,\dots,\lambda_p} \left(B(\lambda_0,\lambda_1) \otimes \dots \otimes B(\lambda_p,\lambda_0) \right) [*-p]$$

$$= \bigoplus_{p\geq 0} \bigoplus_{\lambda_0 \dots \lambda_p} \left(\Phi[\lambda_0,\lambda_1] \otimes \dots \otimes \Phi[\lambda_p,\lambda_0] \right) [-p]$$

$\cong$ | Claim: CAN ASSUME SPLIT

$$\Phi(\{\lambda_0,\dots,\lambda_p\}^c)$$

$\rightsquigarrow$ ESSENTIALLY THE SAME VECTOR SPACES SHOWING UP WITH THE SAME DIFFERENTIALS ...